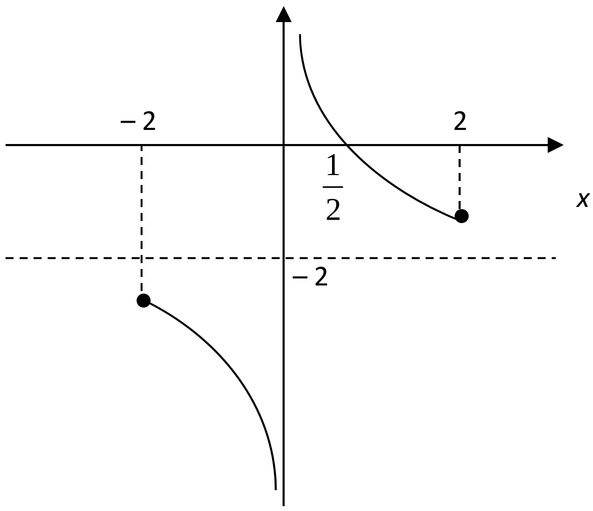
Topic: Relations and functions

Question 1 [2008 EOY, Q13]	marks
A function *f* is defined by $f : x \mapsto \frac{1}{x} - 2$, where $-2 \le x \le 2$, $x \neq 0$.	
(i) Sketch the graph of $y = f(x)$, showing clearly any asymptote(s) and intersection(s) with the axes.	[3]

Solution:



[G1] – correct shape

[G1] – correct asymptotes

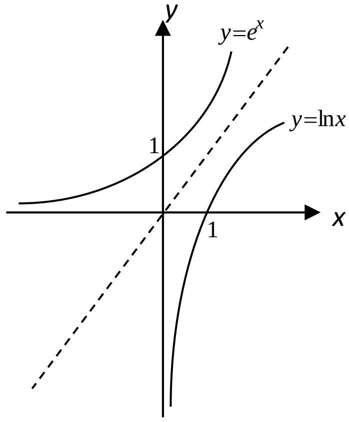
[G1] – correct intercepts

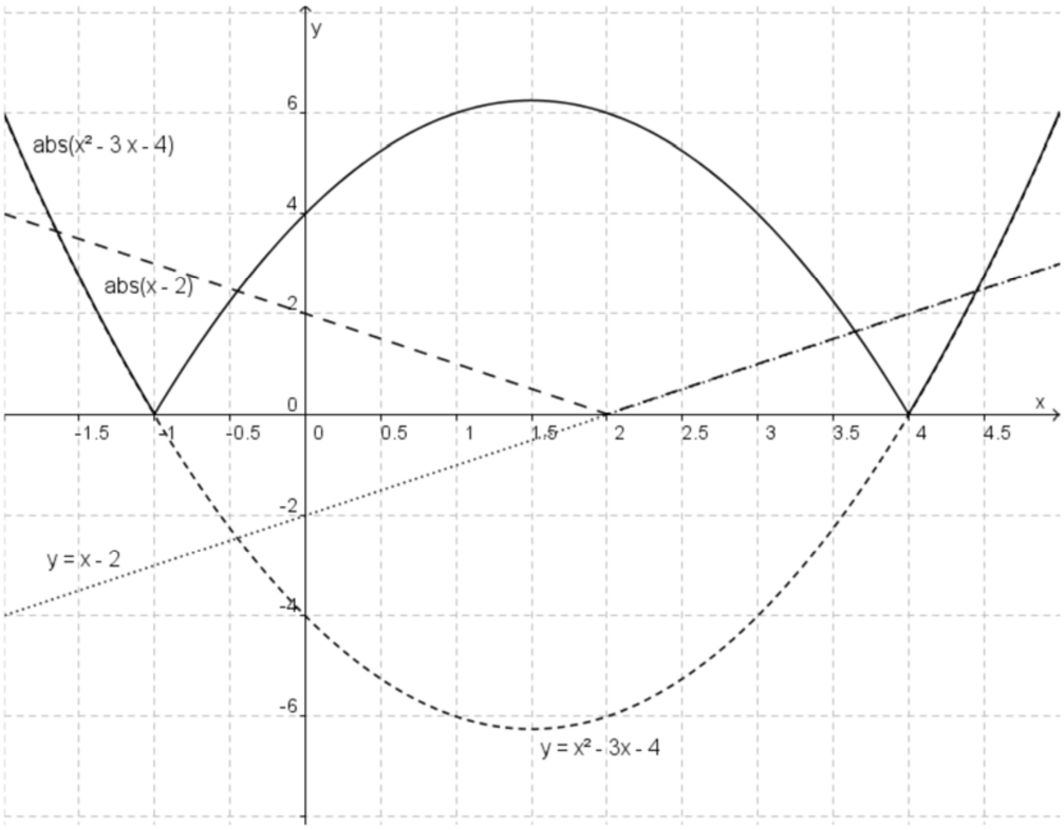
(ii) Find an expression for f^{-1}, the inverse function of *f*, and state its domain.	[3]

Solution:

Let $y = \frac{1}{x} - 2$ then $x = \frac{1}{y+2}$ hence $f^{-1}(x) = \frac{1}{x+2}$

$f(-2) = -\frac{5}{2}$, $f(2) = -\frac{3}{2}$, so domain of f^{-1} is $x \le -\frac{5}{2} \cup x \ge -\frac{3}{2}$

Question 2 [2008 EOY, Q14]	marks
Sketch the graphs of $y=\ln x$ and $y=e^x$ **on the same axes**. **<u>Solution:</u>**  **[G1]** – correct $y=\ln x$ **[G1]** – correct $y=e^x$	[2]
Describe using geometrical transformations how you would obtain the graphs of (a) $y=\ln x^2$, and **<u>Solution:</u>** $y=\ln x^2 \Rightarrow y=2\ln x$. Stretch graph by a factor of 2 about *x*-axis parallel to the *y*-axis and reflect a copy of the graph about *y*-axis	[2]
(b) $y=2+e^{-x}$ from the graphs you have drawn above. You are **not** required to draw them. **<u>Solution:</u>** Reflect the graph about *y*-axis and translate the graph upwards along the *y*-axis by 2 units	[2]

Question 3 [2009 EOY, Q4]	marks
Sketch the graphs of $y=\|x-2\|$ and $y=\|x^2-3x-4\|$ for the domain $-2\le x\le 5$ on the same axes and hence state the number of solutions of the equation $\|x-2\|=\|x^2-3x-4\|$. **Solution:**  	[5]
Question 4 [2009 EOY, Q8]	marks
Function f is defined by $f : x \mapsto 3x+1$. (i) Write down the inverse function of f in similar form. **Solution:** $\frac{y-1}{3}=x$ $f^{-1} : x \mapsto \frac{x-1}{3}$	[2]

(ii)

Describe using geometrical transformation(s) how you would obtain the graph of $y = -f(x)$ from the graph of $y = f(x)$. [1]

You are not required to draw them.

Solution:

$y = -f(x)$ is the reflection of $y = f(x)$ in the *x*-axis.

Question 5 [2010 EOY, Q15]	marks

A function **f** is defined by $\mathbf{f} : x \mapsto \dfrac{2x+k}{3+x}$, where $x \neq -3$ and *k* is a real constant.

(i)

If $\mathbf{f}(-1) = -1\dfrac{1}{2}$, find the value of *k*. [2]

Solution:

$$\frac{2(-1)+k}{3+(-1)} = -1\frac{1}{2} \Rightarrow k-2=-3$$

$$\Rightarrow k = -1$$

(ii)

Find an expression for $\mathbf{f}^{-1}(x)$, where $\mathbf{f}^{-1}$ is the inverse function of **f**, and state its domain. [3]

Solution:

Let $y = \dfrac{2x-1}{3+x}$, then $2x - xy = 3y+1 \Rightarrow x = \dfrac{3y+1}{2-y}$

Hence $\mathbf{f}^{-1}(x) = \dfrac{3x+1}{2-x}$ and the domain is $x \neq 2$

(iii)

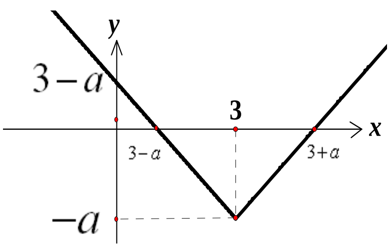
Find the *x*-coordinates of the points of intersection, if they exist, when $\mathbf{f}(x) = \mathbf{f}^{-1}(x)$. [2]

Solution:

Solve for $\dfrac{2x-1}{3+x} = \dfrac{3x+1}{2-x} \Rightarrow x^2 + x + 1 = 0$ [M1]

But this equation has no real roots, so they do not intersect [A1]

Question 6 [2011 EOY, Q8]	marks
(i) Solve the inequality $\|2x-3\|<5$. Hence state the solution of $-4<\|2x-3\|<5$. **<u>Solution:</u>** (a) $-5<2x-3<5$, $-1<x<4$ $-1<x<4$	[3]
(ii) By sketching appropriate graphs or otherwise determine the number of root(s) for the equation $\frac{2}{x}=2^x$. **<u>Solution:</u>** (a) Graph of $y=\frac{2}{x}$ Graph of $y=2^x$ One intersection means one root	[3]

Question 7 [2012 EOY, Q4]	marks
(a) Sketch the graph $y=\|x-3\|-a$ where $0<a<3$, showing clearly the intercepts with the axes. **<u>Solution:</u>**  G1 – correct V-shaped G1 – V-shape shifted to be right by 3 units. G1 – V-shape shifted downwards by *a* units.	[3]

(b) Find the range of values of *k*, in terms of *a*, for which the equation $$\|x-3\|-a=k$$ (i) has no solution, (ii) has two solutions of opposite signs. **Solution:** $(i)\ \ k<-a \qquad (ii)\ \ k>3-a$	[2]
Question 8 [2012 EOY, Q14]	marks
Function f is defined by $\mathrm{f}: x \mapsto \dfrac{2x+13}{x+3}$. (a) (i) State the largest possible domain for f. **Solution:** $x \in \Re, x \neq -3$	[1]
(ii) *A* and *B* are such that $\dfrac{2x+13}{x+3} = A + \dfrac{B}{x+3}$ for all values of *x* except $x=-3$. By long division or otherwise, find the value of *A* and of *B*. **Solution:** $$\frac{2x+13}{x+3} = \frac{2(x+3)+7}{x+3} = 2 + \frac{7}{x+3}$$ $A=2, \qquad B=7$	[2]

(a)(iii)

State precisely a sequence of transformations by which the graph of

$y=\frac{2x+13}{x+3}$ may be obtained from the graph of $y=\frac{1}{x}$. [3]

Solution:

$$y=\frac{1}{x}$$

$$\xrightarrow{\text{(I)}} y=\frac{1}{x+3}$$

Translation of 3 unit in the negative x-direction.

$$\xrightarrow{\text{(II)}} y=\frac{7}{x+3}$$

Stretching/Scaling parallel to the y-axis by a factor of 7

$$\xrightarrow{\text{(III)}} y=2+\frac{7}{x+3}$$

Translation of 2 unit in the positive y-direction

(b) The function f has an inverse if its domain is restricted to $x>k$, state the least value of k and find an expression for $\mathrm{f}^{-1}(x)$ corresponding to this domain for f. [4]

Solution:

f is one-to-one as any horizontal line

$y=k$ where $k\in R_{\mathrm{f}}$ cuts the graph of f exactly once.

$$Let\ \ y=\mathrm{f}^{-1}(x)$$

$$Peform\ \ a\ \ function\ operation\ on\ \ y$$

$$\mathrm{f}(y)=\mathrm{f}\left[\mathrm{f}^{-1}(x)\right]$$

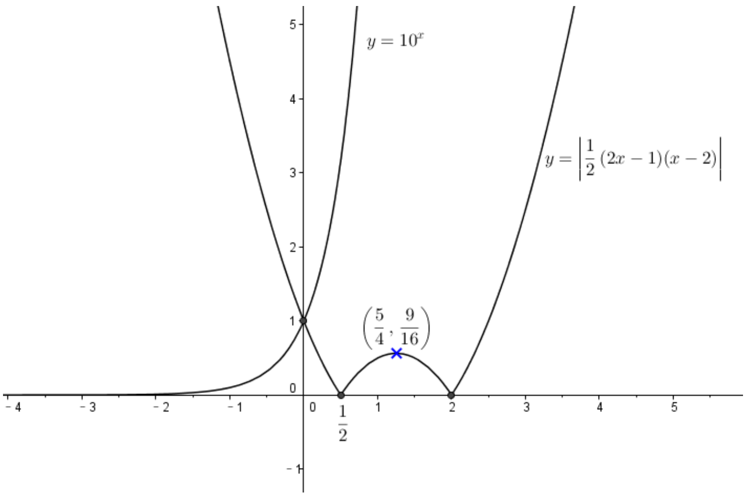
$$\Rightarrow \mathrm{f}(y)=x$$

$$\Rightarrow \frac{2y+13}{y+3}=x$$

$$\Rightarrow y=\frac{13-3x}{x-2}$$

$$\therefore \underbrace{\mathrm{f}^{-1}(x)=\frac{13-3x}{x-2}},\ \ \underbrace{x\in\Re, x\neq 2}$$

Question 9 [2014 EOY, Q11]	marks
A function f is defined, for $x \neq 0$, by $\mathrm{f} : x \mapsto \dfrac{a}{x} + 1$, where a is a real constant. Given that $2\,\mathrm{f}(-1) + \mathrm{f}^{-1}(2) = 4$, calculate the value of a. **<u>Solution:</u>** $\mathrm{f}(x) = \dfrac{a}{x} + 1$ Let $y = \mathrm{f}^{-1}(x)$ $\Rightarrow \mathrm{f}(y) = x$ $\dfrac{a}{y} + 1 = x$ $y = \dfrac{a}{x-1}$ $\Rightarrow \mathrm{f}^{-1}(x) = \dfrac{a}{x-1},\ x \neq 1$ $2\,\mathrm{f}(-1) + \mathrm{f}^{-1}(2) = 4$ $2\left(\dfrac{a}{-1} + 1\right) + \dfrac{a}{2-1} = 4$ $-2a + 2 + a = 4$ $a = -2$	[4]

Question 10 [2014 EOY, Q13]	marks
(i) Sketch the graphs of $y = 10^x$ and $y = \left\|\frac{1}{2}(2x-1)(x-2)\right\|$ on the same axes, showing clearly the x-intercepts, y-intercepts and turning points, if any. Hence, state the range of values of x such that $10^x < \left\|\frac{1}{2}(2x-1)(x-2)\right\|$. **<u>Solution:</u>**  Hence, the range of values of x such that $10^x < \left\|\frac{1}{2}(2x-1)(x-2)\right\|$ is $x < 0$.	[5]
(ii) Describe the sequence of transformations to obtain a graph of $y = 2\ln(x-1)$ from the graph of $y = e^x + 1$. **<u>Solution:</u>** $y = e^x + 1 \Rightarrow e^x = y - 1 \Rightarrow x = \ln(y-1)$ First transformation: reflection in the line $y = x$. Second transformation: stretching / scaling parallel to the y-axis by a factor of 2.	[2]

Question 11 [2015 EOY, Q5]	marks
It is given that f is a function such that $\mathrm{f}: x \mapsto 2-\sqrt{4-x},\ x \le k$. (a) Write down the largest possible value of *k*. **<u>Solution:</u>** Largest possible value of *k* is 4.	[1]
(b) If $k=3$, (i) find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}, **<u>Solution:</u>** $\mathrm{f}^{-1}(x)=4-(2-x)^2$ or $x(4-x)$, domain of f^{-1} is $x \le 1$	[3]
(b) (ii) solve for *x* when $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. **<u>Solution:</u>** $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ $\Rightarrow \mathrm{f}^{-1}(x)=x$ $\Rightarrow 4x-x^2=x$ $\Rightarrow x(x-3)=0$ hence $x=0$ since $x=3 \notin$ domain of f^{-1}	[3]

<table>
<tr><th>Question 12 [2015 EOY, Q12]</th><th>marks</th></tr>
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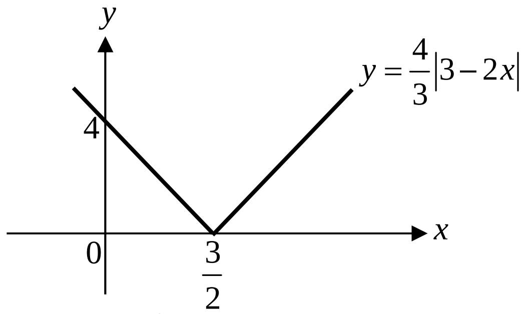
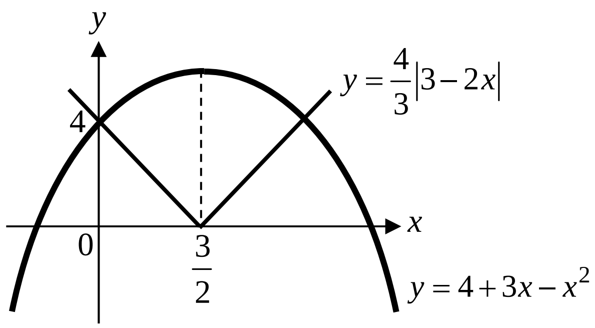
Sketch the graph of $y=\frac{4}{3}\left|3-2x\right|$, showing clearly any intercepts with the axes. By sketching the graph of $y=4+3x-x^2$, solve for the range of x when $12+9x-3x^2>4\left|3-2x\right|$.

<u>Solution:</u>



$y=4+3x-x^2 \Rightarrow y=-(x+1)(x-4)$



$12+9x-3x^2>4\left|3-2x\right| \Rightarrow 4+3x-x^2>\frac{4}{3}\left|3-2x\right|$

By symmetry, $0<x<3$.

</td><td>[5]</td></tr>
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