



HWA CHONG INSTITUTION  
2026 JC2 Preliminary Examination  
Higher 2

**FURTHER MATHEMATICS**  
**Question Paper**

9649/01  
17 September 2026  
3 hours  
Total marks: 100

|                |  |
|----------------|--|
| Candidate Name |  |
|----------------|--|

|          |    |
|----------|----|
| CT Group | 25 |
|----------|----|

**READ THESE INSTRUCTIONS FIRST**

Do not write anything on the List of Formulae (MF27).

Answer **all** the questions.

Write your answer to each question in the space provided in the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part of question.

Remarks for Candidates:

- a) **INSTR**: Did not follow **instructions** stated in the question (e.g. correct significant figures, exact values, coordinates, required form, etc.).
- b) **NOT**: Incorrect or inappropriate use of mathematical **notation**.
- c) **ACC**: **Inaccurate** answer (e.g. due to premature rounding, omission of “+C” in indefinite integrals, etc.).

This document consists of 7 printed pages and 1 blank page.

- 1 It is given that  $z_1 = 2ie^{i\frac{\pi}{12}}$  and  $z_2 = \cos\left(\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)$ . Let  $w = z_1(z_2)^*$ .
- (a) Find the modulus and argument of  $w$ . [3]
- (b) Find the set of integers  $n$  in the range of  $1 \leq n \leq 20$  such that the point representing  $w^n$  lies strictly in the second quadrant of the Argand diagram. [3]
- 2 The surface  $S$  has equation  $z = 1 - x^2 - y^2$ . Using calculus, show that the point on  $S$  that is nearest to the point  $(0, 0, L)$ , for some  $L > 1$ , is  $(0, 0, 1)$ . [7]
- 3 The sequence of numbers  $x_1, x_2, x_3, \dots$  is such that  $x_{k+1} = F(x_k)$ , where  $F(x) = x + \lambda[f(x) - x]$  with  $\lambda$  being a non-zero constant.
- (a) Show that, if the sequence converges, its limit is a root of the equation  $x = f(x)$ . [2]
- The first derivatives of the functions  $f(x)$  and  $F(x)$  are denoted by  $f'(x)$  and  $F'(x)$  respectively. It is known that the speed and likelihood of convergence of the sequence depends on the value of  $F'(x)$  near the root. Ideally,  $F'(x)$  should be as close to zero as possible.
- (b) It is given that the first term  $x_1$  of the sequence is near a root of the equation  $x = f(x)$ , and  $f'(x) \neq 1$  for all  $x$ . Find  $\lambda$  in terms of  $f'(x_1)$  such that  $F'(x_1) = 0$ . [2]
- (c) Given that the root  $\alpha$  of the equation  $x = \cos^{-1} x$  is near 0.7, use  $x_1 = 0.7$  as an initial approximation to find an appropriate value of  $\lambda$  such that the iteration  $x_{k+1} = x_k + \lambda(\cos^{-1} x_k - x_k)$  is expected to converge to  $\alpha$ . [2]
- (d) Use the iterative formula to calculate the value of  $\alpha$  correct to 3 decimal places. [2]
- 4 (a) Solve the equation  $z^{11} - 1 = 0$ . [2]
- Let  $z_1, z_2, \dots, z_{10}$  and 1 be the roots found in part (a) where  $z_1$  is the root with the smallest positive argument.
- (b) Find the modulus of  $1 - z_1$ , leaving your answer in a single trigonometric ratio. [3]
- (c) Show that  $z_1$  is a root to the equation  $1 + z + z^2 + \dots + z^{10} = 0$ . Hence show  $(1 - z_1)(1 - z_2) \dots (1 - z_{10}) = 11$ . [3]
- (d) Deduce the value of  $\sin \frac{\pi}{11} \sin \frac{2\pi}{11} \sin \frac{3\pi}{11} \dots \sin \frac{10\pi}{11}$ . [2]

- 5 The planes  $\pi_1$  and  $\pi_2$  are given by the equations  $\pi_1 : ax + y + z = b$  and  $\pi_2 : x + y + z = c$ .
- (a) Determine the necessary and sufficient conditions on  $b$  and  $c$  for which  $\pi_1 \cap \pi_2$  is a subspace of  $\mathbb{R}^3$ . [3]
- (b) Assume these necessary and sufficient conditions on  $b$  and  $c$  hold. Find the necessary and sufficient condition on  $a$  for which  $\pi_1 \cap \pi_2$  is a subspace of dimension 1. [2]

The conditions found in part (b) are to be used for parts (c) and (d).

A third plane  $\pi_3$  is given by the equation  $\pi_3 : x + y + z = 2$ .

- (c) Is  $(\pi_1 \cap \pi_2) \cap \pi_3$  a subspace of  $\mathbb{R}^3$ ? Justify your answer. [2]
- (d) Is  $(\pi_1 \cap \pi_2) \cup \pi_3$  a subspace of  $\mathbb{R}^3$ ? Justify your answer. [2]

- 6 For a cubic equation  $ax^3 + bx^2 + cx + d = 0$  with real coefficients, the discriminant  $D$  is given by

$$D = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

If  $D > 0$ , then  $ax^3 + bx^2 + cx + d = 0$  has three distinct real roots.

If  $D = 0$ , then  $ax^3 + bx^2 + cx + d = 0$  has a repeated root and all its roots are real.

If  $D < 0$ , then  $ax^3 + bx^2 + cx + d = 0$  has a real root and a pair of complex roots.

Let  $\mathbf{A}$  be a square matrix in  $\mathbb{R}^{3 \times 3}$  with a characteristic equation  $f(\lambda) = 0$ , where

$$f(\lambda) = \lambda^3 - \lambda^2(k+5) + \lambda(5k+6) - 6k \text{ and } k \in \mathbb{R}.$$

- (a) By considering the discriminant of  $f(\lambda) = 0$  or otherwise, determine whether  $\mathbf{A}$  is always diagonalizable when  $k \in \mathbb{R}, k \neq 2, 3$ . [3]
- (b) By considering the eigenvalues of  $\mathbf{A}$ , determine the range of values of  $k$  such that  $\mathbf{A}$  is invertible. [2]

For the remaining part of the question, let  $k = 4$  and  $\mathbf{A} = \begin{pmatrix} -1 & -13 & 19 \\ -2 & 0 & 6 \\ -2 & -4 & 10 \end{pmatrix}$ .

- (c) Evaluate  $\sum_{r=1}^{\infty} (\mathbf{A}^{-1})^r$ , leaving your answer as a single matrix in exact form. [6]

- 7 The equation  $x^3 - 2x - 2 = 0$  has a root  $\alpha$  in the interval  $[1, 2]$ . To find the root, three rearrangements are suggested for an iteration of the form  $x_{n+1} = g(x_n)$ :

- $g(x) = \frac{x^3 - 2}{2}$ ;
- $g(x) = (2x + 2)^{\frac{1}{3}}$ ;
- $g(x) = \frac{2x^3 + 2}{3x^2 - 2}$ .

- (a) Identify one of the above rearrangements that will fail to converge to  $\alpha$  with the initial approximation close to  $\alpha$ . Justify your answer with the aid of diagram(s). [3]
- (b) Use the other two rearrangements, with initial approximation  $x_0 = 1.5$ , to find the value of  $\alpha$  correct to 4 decimal places. For each iterative method, state the least value of  $n$  such that  $x_n = \alpha$  correct to 4 decimal places. [4]
- (c) Suppose an iterative method using  $x_{n+1} = g(x_n)$  produces a sequence  $\{x_n\}$  that converges to the root  $\alpha$ .

The error term for each round of iteration  $e_n$  is defined as  $x_n - \alpha$ .

If there exists a constant  $p > 0$  and a number  $\lambda > 0$  such that  $\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^p} = \lambda$ , then  $p$  is

called the order of convergence for the iterative method.

- (i) Show that  $e_{n+1} = g(x_n) - g(\alpha)$ . [1]
- (ii) It is given that

$$g(x_n) - g(\alpha) = g'(\alpha)(x_n - \alpha) + \frac{g''(\alpha)}{2!}(x_n - \alpha)^2 + \dots + \frac{g^{(r)}(\alpha)}{r!}(x_n - \alpha)^r + \dots$$

Show that, for the iterative methods used in part (b), exactly one of them has first order of convergence. [4]

8 Let

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

be a function of two variables.

(a) Find the directional derivative in the direction of the unit vector  $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$  at  $(x_0, y_0) \neq (0, 0)$ . Express your answer in terms of  $x_0, y_0, a$  and  $b$ . [3]

(b) Find the vector equation of the tangent line at  $(x_0, y_0, f(x_0, y_0))$  in the direction of  $\mathbf{u}$ , where  $(x_0, y_0) \neq (0, 0)$ . Express your answer in terms of  $x_0, y_0, a$  and  $b$ . [2]

A plane  $\pi$  is defined by  $\pi: z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ .

(c) Let  $P(x_0, y_0, z_0)$  be the point  $\left(1, 2, \frac{4}{5}\right)$  on the surface of  $f$ . Show that  $\pi$  at  $P$  contains tangent lines at  $P$  in all directions. [3]

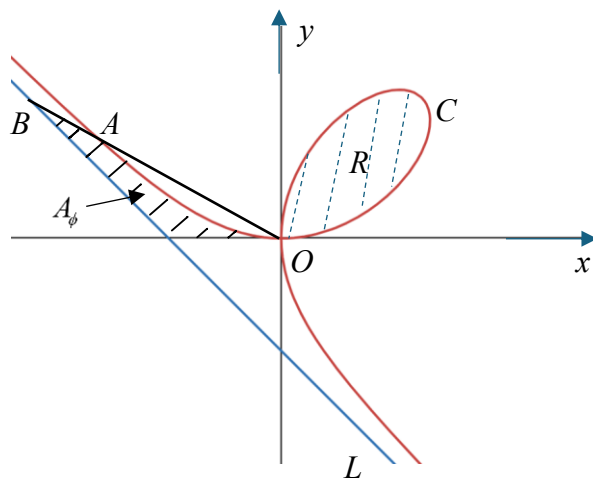
It is defined that

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \quad \text{and} \quad f_y(x, y) = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

(d) Find the equation of the plane  $\pi$  at  $(0, 0, 0)$  using the above definition. By considering a particular straight path along the  $xy$ -plane, is this plane a tangent plane to  $f$  at  $(0, 0, 0)$ ? Justify your answer. [3]

- 9 The curve  $C$ , known as the Folium of Descartes, has the cartesian equation  $x^3 + y^3 = 3axy$ , where  $a > 0$ . The line  $L$  is the asymptote of  $C$ .

[You may use the identity  $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$  for this question.]



- (a) Evaluate  $\lim_{x \rightarrow \infty} \frac{y}{x}$ . [You may assume that  $\lim_{x \rightarrow \infty} \frac{y}{x}$  exists].  
Hence, or otherwise, show that the cartesian equation of  $L$  is  $x + y + a = 0$ . [4]
- (b) Find the polar equations of  $C$  and  $L$ . [2]
- (c) (i) Prove that  $\frac{\cos^2 \theta \sin^2 \theta}{(\cos^3 \theta + \sin^3 \theta)^2} = \frac{\tan^2 \theta \sec^2 \theta}{(1 + \tan^3 \theta)^2}$ . [1]
- (ii) The region  $R$  lies in the first quadrant and is bounded by the curve  $C$ . Find the exact area of  $R$ , giving your answer in terms of  $a$ . [3]
- (d) A straight line through  $O$  cuts  $C$  at  $A$  and  $L$  at  $B$  in the second quadrant.  
Let  $\phi$  be the angle between  $OB$  and the positive  $x$ -axis.  
Let  $A_\phi$  be the area of the region bounded by the negative  $x$ -axis, curve  $C$ , asymptote  $L$  and line  $AB$ .  
Find  $A_\phi$  when  $\phi = \frac{5\pi}{6}$  and  $a = 1$ . [3]

- 10** A drug-eluting stent (DES) is a small mesh tube that doctors place inside blocked heart arteries to keep them open and improve blood flow. The stent is coated with a medical drug that slowly releases into the artery wall to prevent the artery from becoming blocked again. The drug concentration in the stent coating,  $y$  mg/cm<sup>2</sup>, and the drug concentration in the arterial wall tissue,  $z$  mg/cm<sup>2</sup>, can be modelled by the following differential equations.

$$\frac{dy}{dt} = -\frac{1}{2}(y - z) \text{ and } \frac{dz}{dt} = -\frac{dy}{dt} - \frac{3}{4}z,$$

where  $t$  is time measured in months.

For a patient undergoing such treatment, it is known that  $y = I$  and  $z = 0$  initially.

- (a) Show  $\frac{d^2y}{dt^2} + \frac{7}{4}\frac{dy}{dt} + \frac{3}{8}y = 0$ . [2]
- (b) Hence find  $y$  and  $z$  in terms of  $I$  and  $t$ . [5]
- (c) For the safety of the patient, the drug concentration in arterial wall tissue must not exceed 0.1 mg/cm<sup>2</sup>. Determine the maximum value of  $I$ . [3]
- (d) It is known that the drug concentration in the arterial wall tissue must be kept at a level of at least 0.03 mg/cm<sup>2</sup> over a continuous period of 2 months for the treatment to be considered effective. Is the treatment considered safe and effective if a patient is undergoing the treatment where  $I = 0.2$ ? [3]

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**FURTHER MATHEMATICS**  
**Mark Scheme**

9649/01  
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3 hours  
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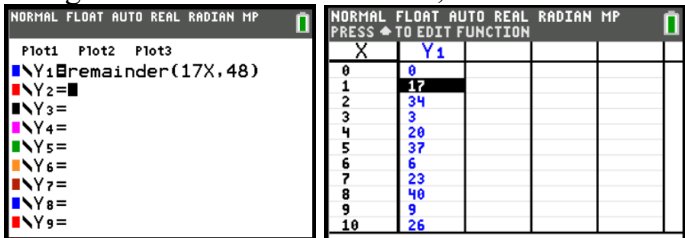
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This document consists of **21** printed pages.

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(a) Find the modulus and argument of  $w$ . [3]

(b) Find the set of integers  $n$  in the range of  $1 \leq n \leq 20$  such that the point representing  $w^n$  lies strictly in the second quadrant of the Argand diagram. [3]

|     | Suggested Solution                                                                                                                                                                                                                                                                                                                                                |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (a) | $z_1 = 2ie^{i\frac{\pi}{12}} = 2e^{i\frac{\pi}{2}}e^{i\frac{\pi}{12}} = 2e^{i\frac{7\pi}{12}}$ $z_2 = \cos\left(\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)$ $= \cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)$ $ w  =  z_1 z_2^*  =  z_1  z_2  = 2$ $\arg(w) = \frac{7\pi}{12} + \left(-\frac{\pi}{8}\right) = \frac{17\pi}{24}$ |
| (b) | <p>Since <math>\frac{\pi}{2} + 2k\pi &lt; \arg(w^n) &lt; \pi + 2k\pi</math>,</p> $\frac{1}{2} + 2k < \frac{17n}{24} < 1 + 2k$ $12 + 48k < 17n < 24 + 48k$ <p>Using Remainder Function in GC,</p>  <p>Set of integers <math>n = \{1, 4, 7, 15, 18\}</math>.</p>                 |

- 2 The surface  $S$  has equation  $z = 1 - x^2 - y^2$ . Using calculus, show that the point on  $S$  that is nearest to the point  $(0, 0, L)$ , for some  $L > 1$ , is  $(0, 0, 1)$  [7]

|  | Suggested Solution                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|--|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|  | <p>Let <math>d</math> be the distance from <math>(0, 0, L)</math> to the surface of <math>S</math>. Then,</p> $d = \sqrt{(x-0)^2 + (y-0)^2 + (1-x^2-y^2-L)^2}$ $\Leftrightarrow d^2 = x^2 + y^2 + (1-x^2-y^2-L)^2$ <p>Let <math>g(x, y) = x^2 + y^2 + (1-x^2-y^2-L)^2</math>.</p> <p>Now,</p> $g_x(x, y) = 0 \text{ and } g_y(x, y) = 0$ $\Leftrightarrow 2x + 2(1-x^2-y^2-L)(-2x) = 0 \text{ and}$ $2y + 2(1-x^2-y^2-L)(-2y) = 0$ $\Leftrightarrow x - 2x(1-x^2-y^2-L) = 0 \text{ and}$ $y - 2y(1-x^2-y^2-L) = 0$ $\Leftrightarrow (x = 0 \text{ or } 1 - 2(1-x^2-y^2-L) = 0) \text{ and}$ $(y = 0 \text{ or } 1 - 2(1-x^2-y^2-L) = 0)$ $\Leftrightarrow (x = 0 \text{ or } x^2 + y^2 = 0.5 - L) \text{ and}$ $(y = 0 \text{ or } x^2 + y^2 = 0.5 - L)$ $\Leftrightarrow (x, y) = (0, 0) \text{ since } (x^2 + y^2 = 0.5 - L)$ <p>has no real solutions because <math>L &gt; 1</math>.</p> <p>Next,</p> $g_{xx}(x, y)$ $= 2 + 2(1-x^2-y^2-L)(-2) + 2(-2x)^2,$ $g_{yy}(x, y)$ $= 2 + 2(1-x^2-y^2-L)(-2) + 2(-2y)^2,$ $g_{xy}(x, y) = 2(-2x)(-2y) = 8xy$ <p>Let <math>D(x, y) = g_{xx}(x, y)g_{yy}(x, y) - [g_{xy}(x, y)]^2</math>.</p> <p>Then,</p> $D(0, 0) = g_{xx}(0, 0)g_{yy}(0, 0) - [g_{xy}(0, 0)]^2$ $= (2 + 4(L-1))(2 + 4(L-1)) - 0^2$ $> 0$ |

|  |                                                                                                                                                                                                                                                                                                                                         |
|--|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|  | <p>and <math>g_{xx}(0,0) = 2 + 4(L-1) &gt; 0</math>.</p> <p>Hence the distance between <math>f</math> and <math>(0,0,L)</math> is the minimum at <math>(x,y) = (0,0)</math>, which verifies the point on <math>S</math> that is nearest to the point <math>(0,0,L)</math>, for some <math>L &gt; 1</math>, is <math>(0,0,1)</math>.</p> |
|--|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

- 3** The sequence of numbers  $x_1, x_2, x_3, \dots$  is such that  $x_{k+1} = F(x_k)$ , where  $F(x) = x + \lambda[f(x) - x]$  with  $\lambda$  being a non-zero constant.

**(a)** Show that, if the sequence converges, its limit is a root of the equation  $x = f(x)$ . [2]

The first derivatives of the functions  $f(x)$  and  $F(x)$  are denoted by  $f'(x)$  and  $F'(x)$  respectively. It is known that the speed and likelihood of convergence of the sequence depends on the value of  $F'(x)$  near the root. Ideally,  $F'(x)$  should be as close to zero as possible.

**(b)** It is given that the first term  $x_1$  of the sequence is near a root of the equation  $x = f(x)$ , and  $f'(x) \neq 1$  for all  $x$ . Find  $\lambda$  in terms of  $f'(x_1)$  such that  $F'(x_1) = 0$ . [2]

**(c)** Given that the root  $\alpha$  of the equation  $x = \cos^{-1} x$  is near 0.7, use  $x_1 = 0.7$  as an initial approximation to find an appropriate value of  $\lambda$  such that the iteration  $x_{k+1} = x_k + \lambda(\cos^{-1} x_k - x_k)$  is expected to converge to  $\alpha$ . [2]

**(d)** Use the iterative formula to calculate the value of  $\alpha$  correct to 3 decimal places. [2]

|            | Suggested Solution                                                                                                                                                                                                                                                                             |
|------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>(a)</b> | $x_{k+1} = F(x_k) = x_k + \lambda[f(x_k) - x_k]$ <p>If the sequence converges to <math>l</math> as <math>k \rightarrow \infty</math>, we have</p> $l = l + \lambda[f(l) - l]$ $\lambda[f(l) - l] = 0$ $f(l) = l$ <p>Hence <math>l</math> is a root for the equation <math>x = f(x)</math>.</p> |
| <b>(b)</b> | $F'(x) = 1 + \lambda[f'(x) - 1]$ $F'(x_1) = 0$ $\Rightarrow 1 + \lambda[f'(x_1) - 1] = 0$ $\Rightarrow \lambda = \frac{1}{1 - f'(x_1)}$                                                                                                                                                        |
| <b>(c)</b> | Let $x_1 = 0.7$ , since $f(x) = \cos^{-1} x$ , $F(x) = x + \lambda[f(x) - x]$                                                                                                                                                                                                                  |

|     |                                                                                                                                                                                                                                                                                                                                                                                                                             |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|     | $\lambda = \frac{1}{1-f'(x_1)} = \frac{1}{1+\frac{1}{\sqrt{1-0.7^2}}} = 0.416618$ <p>Choose <math>\lambda = 0.417</math>. For the chosen value of <math>\lambda</math>, <math>F'(x_1)</math> will be very small and it is likely that the sequence <math>\{x_k\}</math> produced by <math>x_{k+1} = F(x_k)</math> will converge.</p>                                                                                        |
| (d) | <p>By G.C., <math>x_1 = 0.7, x_2 = 0.7398, x_3 = 0.7391, x_4 = 0.7391</math><br/>Hence, <math>\alpha = 0.739</math></p> <p>Let <math>g(x) = x - \cos^{-1} x</math>,<br/>since <math>g</math> is continuous in <math>[0.7385, 0.7395]</math> and<br/><math>g(0.7385)g(0.7395) = (-0.00145)(0.00103) &lt; 0</math>,<br/><math>\therefore 0.7385 &lt; \alpha &lt; 0.7395</math><br/><math>\therefore \alpha = 0.739</math></p> |

4 (a) Solve the equation  $z^{11} - 1 = 0$ . [2]

Let  $z_1, z_2, \dots, z_{10}$  and 1 be the roots found in part (a) where  $z_1$  is the root with the smallest positive argument.

(b) Find the modulus of  $1 - z_1$ , leaving your answer in a single trigonometric ratio. [3]

(c) Show that  $z_1$  is a root to the equation  $1 + z + z^2 + \dots + z^{10} = 0$ . Hence show

$$(1 - z_1)(1 - z_2) \dots (1 - z_{10}) = 11. \quad [3]$$

(d) Deduce the value of  $\sin \frac{\pi}{11} \sin \frac{2\pi}{11} \sin \frac{3\pi}{11} \dots \sin \frac{10\pi}{11}$ . [2]

|     | Suggested Solution                                                                                               |
|-----|------------------------------------------------------------------------------------------------------------------|
| (a) | $z^{11} - 1 = 0$ $z^{11} = 1$ $z^{11} = e^{i(0+2k\pi)}$ $z = e^{i\frac{2k\pi}{11}} \quad k = 0, 1, 2, \dots, 10$ |
| (b) | $z_1 = e^{i\frac{2\pi}{11}}$                                                                                     |

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|     | $1 - z_1 = 1 - e^{i\frac{2\pi}{11}}$ $= e^{i\frac{\pi}{11}} \left( e^{-i\frac{\pi}{11}} - e^{i\frac{\pi}{11}} \right)$ $= e^{i\frac{\pi}{11}} \left[ \cos\left(-\frac{\pi}{11}\right) + i \sin\left(-\frac{\pi}{11}\right) - \cos\left(\frac{\pi}{11}\right) - i \sin\left(\frac{\pi}{11}\right) \right]$ $= \left( -2i \sin \frac{\pi}{11} \right) e^{i\frac{\pi}{11}}$ $\therefore  1 - z_1  = 2 \sin \frac{\pi}{11}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
| (c) | $z^{11} - 1 = (z - 1)(z - z_1)(z - z_2) \dots (z - z_{10})$ $(z - 1)(1 + z + z^2 + \dots + z^{10}) = (z - 1)(z - z_1)(z - z_2) \dots (z - z_{10})$ <p>Consider <math>z_1</math> is a root to RHS = 0.</p> <p><math>\Rightarrow z_1</math> is a root to LHS = 0</p> <p>Since <math>z_1 \neq 1</math>, <math>z_1 - 1 \neq 0</math></p> <p><math>\Rightarrow z_1</math> is a root to <math>1 + z + z^2 + \dots + z^{10} = 0</math> (shown)</p> <p><math>\therefore (z - z_1)</math> is a factor of <math>1 + z + z^2 + \dots + z^{10}</math></p> <p>Similarly, <math>(z - z_k)</math> is a factor of <math>1 + z + z^2 + \dots + z^{10}</math></p> <p>for <math>k = 1, 2, \dots, 10</math></p> <p><math>\therefore</math> coefficient of <math>z^{10} = 1</math> and we have 10 factors.</p> <p><math>\therefore 1 + z + z^2 + \dots + z^{10} = (z - z_1)(z - z_2) \dots (z - z_{10})</math></p> <p>Let <math>z = 1</math>, <math>11 = (1 - z_1)(1 - z_2) \dots (1 - z_{10})</math> (Shown)</p> |
| (d) | $(1 - z_1)(1 - z_2) \dots (1 - z_{10}) = 11$ $ (1 - z_1)(1 - z_2) \dots (1 - z_{10})  = 11$ $ 1 - z_1   1 - z_2  \dots  1 - z_{10}  = 11$ <p>Using (b),</p> $ 1 - z_1   1 - z_2  \dots  1 - z_{10}  = 11$ <p><math>\therefore \sin \theta \geq 0</math> for <math>0 \leq \theta \leq \pi</math></p> $\left( 2 \sin \frac{\pi}{11} \right) \left( 2 \sin \frac{2\pi}{11} \right) \dots \left( 2 \sin \frac{10\pi}{11} \right) = 11$ $\sin \frac{\pi}{11} \sin \frac{2\pi}{11} \sin \frac{3\pi}{11} \dots \sin \frac{10\pi}{11} = \frac{11}{2^{10}} = \frac{11}{1024}.$                                                                                                                                                                                                                                                                                                                                                                                                                        |

5 The planes  $\pi_1$  and  $\pi_2$  are given by the equations  $\pi_1 : ax + y + z = b$  and  $\pi_2 : x + y + z = c$ .

- (a) Determine the necessary and sufficient conditions on  $b$  and  $c$  for which  $\pi_1 \cap \pi_2$  is a subspace of  $\mathbb{R}^3$ . [3]
- (b) Assume these necessary and sufficient conditions on  $b$  and  $c$  hold. Find the necessary and sufficient condition on  $a$  for which  $\pi_1 \cap \pi_2$  is a subspace of dimension 1. [2]

The conditions found in part (b) are to be used for parts (c) and (d).

A third plane  $\pi_3$  is given by the equation  $\pi_3 : x + y + z = 2$ .

- (c) Is  $(\pi_1 \cap \pi_2) \cap \pi_3$  a subspace of  $\mathbb{R}^3$ ? Justify your answer. [2]
- (d) Is  $(\pi_1 \cap \pi_2) \cup \pi_3$  a subspace of  $\mathbb{R}^3$ ? Justify your answer. [2]

| Suggested Solution |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|--------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (a)                | <p>If <math>\pi_1 \cap \pi_2</math> is a subspace of <math>\mathbb{R}^3</math>, then <math>\mathbf{0} \in \pi_1 \cap \pi_2</math>.<br/> Hence <math>\mathbf{0} \in \pi_1</math> and <math>\mathbf{0} \in \pi_2</math>.<br/> <math>b = 0, c = 0</math>.</p> <p>If <math>b = 0</math> and <math>c = 0</math>, <math>\pi_1 : ax + y + z = 0</math> and <math>\pi_2 : x + y + z = 0</math> are subspaces in <math>\mathbb{R}^3</math>.</p> <p>Therefore <math>\pi_1 \cap \pi_2</math> is a subspace of <math>\mathbb{R}^3</math> since the intersection of subspaces is a subspace.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
| (b)                | <p>In the (governing) environment provided by part (a), i.e., <math>b = 0</math> and <math>c = 0</math>, we examine the values of <math>a</math>.</p> <p>If <math>a \neq 1</math>, then<br/> <math>\pi_1 \cap \pi_2</math><br/> <math>\Leftrightarrow ax + y + z = 0</math> and <math>x + y + z = 0</math><br/> <math>\Leftrightarrow ax + y + z = 0</math> and <math>y + z = -x</math><br/> <math>\Leftrightarrow ax - x = 0</math> and <math>y + z = -x</math><br/> <math>\Leftrightarrow x(a - 1) = 0</math> and <math>y + z = -x</math><br/> <math>\Leftrightarrow x = 0</math> and <math>y + z = -x</math><br/> <math>\Leftrightarrow x = 0</math> and <math>y + z = 0</math><br/> <math>\Leftrightarrow x = 0</math> and <math>y = -z</math><br/> <math>\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -z \\ z \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}</math></p> <p>If <math>a = 1</math>, then<br/> <math>\pi_1 \cap \pi_2</math><br/> <math>\Leftrightarrow (ax + y + z = 0 \text{ and } x + y + z = 0)</math><br/> <math>\Leftrightarrow x + y + z = 0</math></p> |

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|     | <p>Hence <math>\pi_1 : ax + y + z = 0</math> intersects with <math>\pi_2 : x + y + z = 0</math> in a subspace of dimension 1 if and only if <math>a \neq 1</math>.</p> <p>Hence the required values are <math>a \neq 1</math> and <math>b = 0</math> and <math>c = 0</math>.</p>                                                                                                                                                                        |
| (c) | <p>We note that <math>\pi_1 \cap \pi_2 : \mathbf{r} = \lambda \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}</math> and <math>\pi_3 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2</math>. Since <math>\mathbf{0} \in \pi_1 \cap \pi_2</math> and <math>\mathbf{0} \notin \pi_3</math>, we have <math>\mathbf{0} \notin (\pi_1 \cap \pi_2) \cap \pi_3</math>. Hence <math>(\pi_1 \cap \pi_2) \cap \pi_3</math> is not a vector space.</p> |
| (d) | <p>We note that <math>(1, 1, 0) \in (\pi_1 \cap \pi_2) \cup \pi_3</math> since <math>(1, 1, 0) \in \pi_3</math>. But <math>2 \times (1, 1, 0) = (2, 2, 0) \notin \pi_3</math> and <math>2 \times (1, 1, 0) = (2, 2, 0) \notin \pi_1 \cap \pi_2</math>. Since scalar multiplication is not closed in <math>(\pi_1 \cap \pi_2) \cup \pi_3</math>, <math>(\pi_1 \cap \pi_2) \cup \pi_3</math> is not a vector space.</p>                                   |

- 6 For a cubic equation  $ax^3 + bx^2 + cx + d = 0$  with real coefficients, the discriminant  $D$  is given by

$$D = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

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If  $D > 0$ , then  $ax^3 + bx^2 + cx + d = 0$  has three distinct real roots.

If  $D = 0$ , then  $ax^3 + bx^2 + cx + d = 0$  has a repeated root and all its roots are real.

If  $D < 0$ , then  $ax^3 + bx^2 + cx + d = 0$  has a real root and a pair of complex roots.

Let  $\mathbf{A}$  be a square matrix in  $\mathbb{R}^{3 \times 3}$  with a characteristic equation  $f(\lambda) = 0$ , where

$$f(\lambda) = \lambda^3 - \lambda^2(k+5) + \lambda(5k+6) - 6k \text{ and } k \in \mathbb{R}.$$

- (a) By considering the discriminant of  $f(\lambda) = 0$  or otherwise, determine whether  $\mathbf{A}$  is always diagonalizable when  $k \in \mathbb{R}, k \neq 2, 3$ . [3]

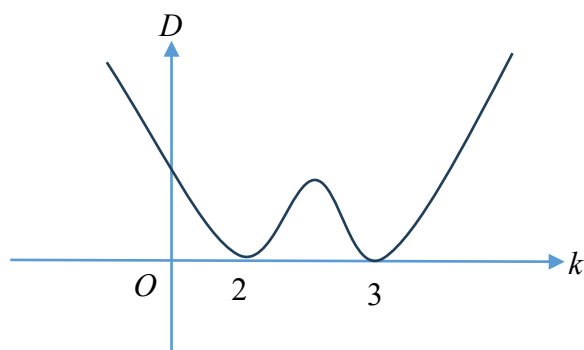
- (b) By considering the eigenvalues of  $\mathbf{A}$ , determine the range of values of  $k$  such that  $\mathbf{A}$  is invertible. [2]

For the remaining part of the question, let  $k = 4$  and  $\mathbf{A} = \begin{pmatrix} -1 & -13 & 19 \\ -2 & 0 & 6 \\ -2 & -4 & 10 \end{pmatrix}$ .

- (c) Evaluate  $\sum_{r=1}^{\infty} (\mathbf{A}^{-1})^r$ , leaving your answer as a single matrix in exact form. [6]

|     | Suggested Solution                                                                      |
|-----|-----------------------------------------------------------------------------------------|
| (a) | $D = (-k-5)^2(5k+6)^2 - 4(5k+6)^3$ $- 4(-k-5)^3(-6k) - 27(-6k)^2 + 18(-k-5)(5k+6)(-6k)$ |

Using G.C,



Observe that the curve is a quartic curve such that  $D > 0$  when  $k \in \mathbb{R}, k \neq 2, 3$ , so  $f(\lambda) = 0$  always has three distinct real roots, hence  $\mathbf{A}$  has three distinct real eigenvalues and is always diagonalizable.

$$\begin{aligned} \text{(b)} \quad \lambda^3 - \lambda^2(k+5) + \lambda(5k+6) - 6k &= (\lambda - k)(\lambda^2 - 5\lambda + 6) \\ &= (\lambda - k)(\lambda - 2)(\lambda - 3) \end{aligned}$$

Note that  $k$  is an eigenvalue. The other eigenvalues are  $\lambda = 2$  and  $\lambda = 3$ .

$\mathbf{A}$  is not invertible  $\Leftrightarrow 0$  is an eigenvalue.

Therefore, for  $\mathbf{A}$  to be invertible,  $k \neq 0$ .

(c) Since the eigenvalues of  $\mathbf{A}$  are 2, 3 and 4, using G.C,  
Solving

$$\begin{pmatrix} -3 & -13 & 19 \\ -2 & -2 & 6 \\ -2 & -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}, \quad \begin{pmatrix} -4 & -13 & 19 \\ -2 & -3 & 6 \\ -2 & -4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \text{ and}$$

$$\begin{pmatrix} -5 & -13 & 19 \\ -2 & -4 & 6 \\ -2 & -4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

The corresponding eigenvectors are  $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ .

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}^{-1}$$

|  |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
|--|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|  | $ \begin{aligned} \sum_{r=1}^{\infty} (\mathbf{A}^{-1})^r &= \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix} \left[ \sum_{r=1}^{\infty} \begin{pmatrix} \frac{1}{2^r} & 0 & 0 \\ 0 & \frac{1}{3^r} & 0 \\ 0 & 0 & \frac{1}{4^r} \end{pmatrix} \right] \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{5}{2} & \frac{41}{6} & -\frac{59}{6} \\ 1 & \frac{10}{3} & -\frac{13}{3} \\ 1 & 3 & -4 \end{pmatrix} \end{aligned} $ |
|--|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

- 7 The equation  $x^3 - 2x - 2 = 0$  has a root  $\alpha$  in the interval  $[1, 2]$ . To find the root, three rearrangements are suggested for an iteration of the form  $x_{n+1} = g(x_n)$ :

- $g(x) = \frac{x^3 - 2}{2}$ ;
- $g(x) = (2x + 2)^{\frac{1}{3}}$ ;
- $g(x) = \frac{2x^3 + 2}{3x^2 - 2}$ .

- (a) Identify one of the above rearrangements that will fail to converge to  $\alpha$  with the initial approximation close to  $\alpha$ . Justify your answer with the aid of diagram(s). [3]
- (b) Use the other two rearrangements, with initial approximation  $x_0 = 1.5$ , to find the value of  $\alpha$  correct to 4 decimal places. For each iterative method, state the least value of  $n$  such that  $x_n = \alpha$  correct to 4 decimal places. [4]
- (c) Suppose an iterative method using  $x_{n+1} = g(x_n)$  produces a sequence  $\{x_n\}$  that converges to the root  $\alpha$ .

The error term for each round of iteration  $e_n$  is defined as  $x_n - \alpha$ .

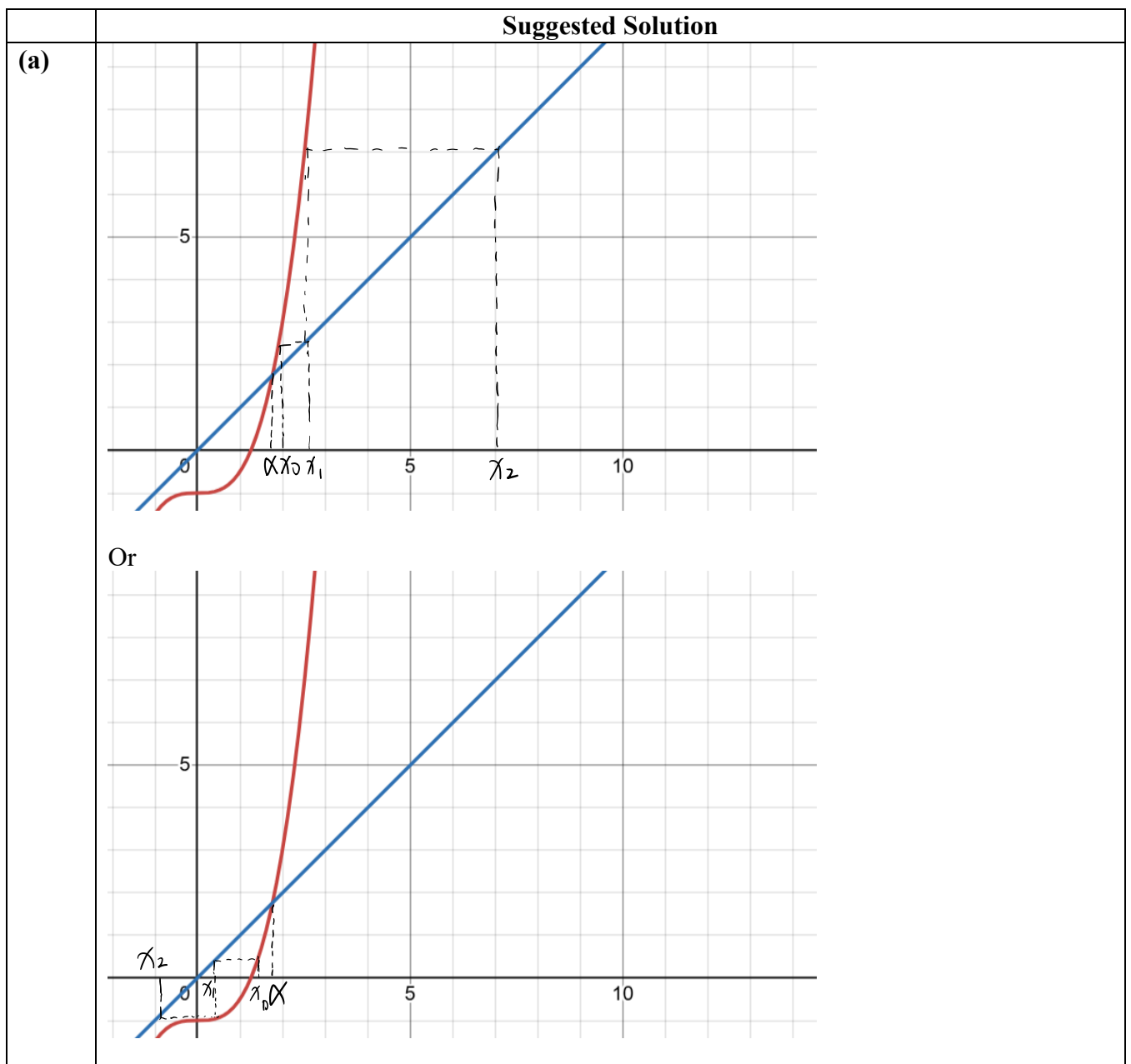
If there exists a constant  $p > 0$  and a number  $\lambda > 0$  such that  $\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^p} = \lambda$ , then  $p$  is called the order of convergence for the iterative method.

(i) Show that  $e_{n+1} = g(x_n) - g(\alpha)$ . [1]

(ii) It is given that

$$g(x_n) - g(\alpha) = g'(\alpha)(x_n - \alpha) + \frac{g''(\alpha)}{2!}(x_n - \alpha)^2 + \dots + \frac{g^{(r)}(\alpha)}{r!}(x_n - \alpha)^r + \dots$$

Show that, for the iterative methods used in part (b), exactly one of them has first order of convergence. [4]



|         |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
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|         | <p>We can see from the sketches that the rearrangement of <math>g(x) = \frac{x^3 - 2}{2}</math> will not give convergence to <math>\alpha</math> for the iterative method.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
| (b)     | <p>For <math>x = (2x + 2)^{\frac{1}{3}}</math>,</p> <p>By G.C., <math>\alpha = 1.7693</math>.<br/>the least value of <math>n</math> is 6.</p> <p>For <math>x = \frac{2x^3 + 2}{3x^2 - 2}</math>,</p> <p>By GC, <math>\alpha = 1.7693</math>.</p> <p>the least value of <math>n</math> is 3.</p> <p>Let <math>f(x) = x^3 - 2x - 2</math><br/> <math>f(1.76925) = -3.13 \times 10^{-4}</math><br/> <math>f(1.76935) = 4.26 \times 10^{-4}</math><br/>         Since <math>f</math> is continuous in <math>[1.76925, 1.76935]</math> and <math>f(1.76925)f(1.76935) &lt; 0</math>, <math>\alpha = 1.7693</math></p>                                                                                                                                                                                                                                                                                                                                                                                            |
| (c)(i)  | <p><math>e_{n+1} = x_{n+1} - \alpha</math><br/> <math>= g(x_n) - \alpha</math></p> <p>Since <math>\alpha</math> is a root of the equation, we have <math>\alpha = g(\alpha)</math>. Hence, we have <math>e_{n+1} = g(x_n) - g(\alpha)</math>.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
| (c)(ii) | <p><math>e_{n+1} = g(x_n) - g(\alpha)</math><br/> <math>= g'(\alpha)(x_n - \alpha) + \frac{g''(\alpha)}{2}(x_n - \alpha)^2 + \dots</math><br/> <math>= g'(\alpha)e_n + \frac{g''(\alpha)}{2}e_n^2 + \frac{g'''(\alpha)}{3!}e_n^3 + \dots</math></p> <p>For <math>g(x) = (2x + 2)^{\frac{1}{3}}</math>, <math>g'(x) = \frac{2}{3}(2x + 2)^{-\frac{2}{3}}</math>.</p> $\lim_{n \rightarrow \infty} \frac{ e_{n+1} }{ e_n } = \lim_{n \rightarrow \infty} \left  \frac{\frac{2}{3}(2\alpha + 2)^{-\frac{2}{3}}e_n + \frac{g''(\alpha)}{2}e_n^2 + \frac{g'''(\alpha)}{3!}e_n^3 + \dots}{e_n} \right $ $= \lim_{n \rightarrow \infty} \left  \frac{2}{3}(2\alpha + 2)^{-\frac{2}{3}} + \frac{g''(\alpha)}{2}e_n + \frac{g'''(\alpha)}{3!}e_n^2 + \dots \right $ $= \frac{2}{3}(2\alpha + 2)^{-\frac{2}{3}} > 0 \quad \text{for } \alpha \in [1, 2] \quad (\because e_n \rightarrow 0 \text{ as } n \rightarrow \infty)$ <p>Hence, <math>g(x) = (2x + 2)^{\frac{1}{3}}</math> has first order of convergence.</p> |

For  $g(x) = \frac{2x^3 + 2}{3x^2 - 2}$ ,

$$g'(x) = \frac{(3x^2 - 2)(6x^2) - (2x^3 + 2)(6x)}{(3x^2 - 2)^2}$$

$$= \frac{6x^4 - 12x^2 - 12x}{(3x^2 - 2)^2}$$

$$= \frac{6x(x^3 - 2x - 2)}{(3x^2 - 2)^2}$$

$$g'(\alpha) = 0$$

$$\lim_{n \rightarrow \infty} \frac{|e_{n+1}|}{|e_n|^2} = \lim_{n \rightarrow \infty} \left| \frac{\frac{g''(\alpha)}{2} e_n^2 + \frac{g'''(\alpha)}{3!} e_n^3 + \dots}{e_n^2} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{g''(\alpha)}{2} + \frac{g'''(\alpha)}{3!} e_n + \dots \right|$$

$$= \left| \frac{g''(\alpha)}{2} \right| \geq 0 \quad \text{for } \alpha \in [1, 2] \quad (\because e_n \rightarrow 0 \text{ as } n \rightarrow \infty)$$

Hence,  $g(x) = \frac{2x^3 + 2}{3x^2 - 2}$  has an order of convergence higher than 1.

8 Let

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

be a function of two variables.

(a) Find the directional derivative in the direction of the unit vector  $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$  at  $(x_0, y_0) \neq (0, 0)$ . Express your answer in terms of  $x_0, y_0, a$  and  $b$ . [3]

(b) Find the vector equation of the tangent line at  $(x_0, y_0, f(x_0, y_0))$  in the direction of  $\mathbf{u}$ , where  $(x_0, y_0) \neq (0, 0)$ . Express your answer in terms of  $x_0, y_0, a$  and  $b$ . [2]

A plane  $\pi$  is defined by  $\pi: z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ .

(c) Let  $P(x_0, y_0, z_0)$  be the point  $\left(1, 2, \frac{4}{5}\right)$  on the surface of  $f$ . Show that  $\pi$  at  $P$  contains tangent lines at  $P$  in all directions. [3]

It is defined that

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \text{ and } f_y(x, y) = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

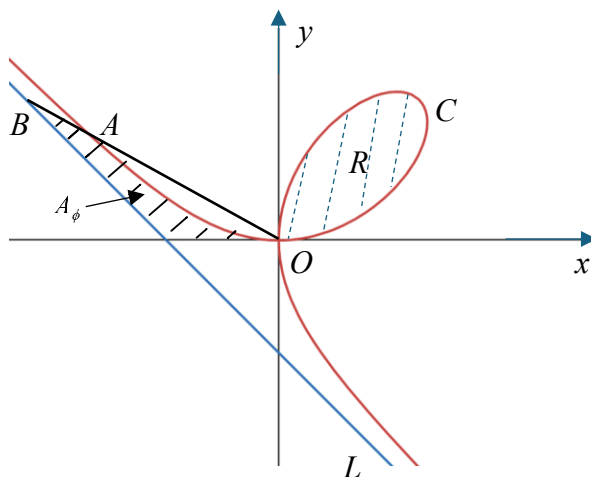
- (d) Find the equation of the plane  $\pi$  at  $(0, 0, 0)$  using the above definition. By considering a particular straight path along the  $xy$ -plane, is this plane a tangent plane to  $f$  at  $(0, 0, 0)$ ? Justify your answer. [3]

| Suggested Solution |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
|--------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (a)                | <p>For <math>(x_0, y_0) \neq (0, 0)</math>, we have</p> $f_x(x_0, y_0) = \frac{(x_0^2 + y_0^2)2y_0 - 2x_0y_0(2x_0)}{(x_0^2 + y_0^2)^2}$ $= \frac{2y_0(y_0^2 - x_0^2)}{(x_0^2 + y_0^2)^2}$ $f_y(x_0, y_0) = \frac{(x_0^2 + y_0^2)2x_0 - 2x_0y_0(2y_0)}{(x_0^2 + y_0^2)^2}$ $= \frac{2x_0(x_0^2 - y_0^2)}{(x_0^2 + y_0^2)^2}$ $D_u(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$ $= \frac{2y_0(y_0^2 - x_0^2)}{(x_0^2 + y_0^2)^2}a + \frac{2x_0(x_0^2 - y_0^2)}{(x_0^2 + y_0^2)^2}b$                                                                                                                                                                                         |
| (b)                | <p>We note that <math>\mathbf{u} = a\mathbf{i} + b\mathbf{j}</math> is a unit vector, which means the length of <math>\mathbf{u}</math> is 1. Hence over vector <math>\mathbf{u}</math> (length and direction included), the height of the surface will rise/fall exactly in accordance to the value of <math>D_u(x_0, y_0)</math>. This means that the tangent line at <math>(x_0, y_0, f(x_0, y_0))</math> in the direction of <math>\mathbf{u}</math> is <math>\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ f(x_0, y_0) \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ D_u(x_0, y_0) \end{pmatrix}, \lambda \in \mathbb{R}.</math></p> |
| (c)                | <p>We express <math>\pi</math> in scalar product form:</p> $z - \frac{4}{5} = f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2)$ $\Leftrightarrow z - \frac{4}{5} = \frac{12}{25}x - \frac{12}{25} - \frac{6}{25}y + \frac{6}{25}(2)$ $\Leftrightarrow 25z - 20 = 12x - 12 - 6y + 12$ $\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 12 \\ -6 \\ -25 \end{pmatrix} = -20$                                                                                                                                                                                                                                                                             |

|            |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
|------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|            | <p>By <b>(b)</b>, the tangent line at <math>\left(1, 2, \frac{4}{5}\right)</math> in the direction of <b>u</b> is</p> $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \frac{4}{5} \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ \frac{12}{25}a - \frac{6}{25}b \end{pmatrix}, \lambda \in \mathbb{R}$ <p>Now,</p> $\left[ \begin{pmatrix} 1 \\ 2 \\ 0.8 \end{pmatrix} + \frac{\lambda}{25} \begin{pmatrix} 25a \\ 25b \\ 12a - 6b \end{pmatrix} \right] \cdot \begin{pmatrix} 12 \\ -6 \\ -25 \end{pmatrix}$ $= \begin{pmatrix} 1 + \lambda a \\ 2 + \lambda b \\ 0.8 + \frac{12}{25}\lambda a - \frac{6}{25}\lambda b \end{pmatrix} \cdot \begin{pmatrix} 12 \\ -6 \\ -25 \end{pmatrix}$ $= 12 + 12\lambda a - 12 - 6\lambda b - 20 - 12\lambda a + 6\lambda b$ $= -20$ |
| <b>(d)</b> | $f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2h \cdot 0}{h^2 + 0^2} - 0}{h} = 0$ $f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \rightarrow 0} \frac{\frac{2 \cdot 0 \cdot k}{0^2 + k^2} - 0}{k} = 0$ <p>Hence <math>z - 0 = f_x(0,0)(x - 0) + f_y(0,0)(y - 0) \Leftrightarrow z = 0</math>.</p> <p>As <math>(x, y) \rightarrow (0,0)</math> along the line <math>y = x</math>, <math>\frac{2xy}{x^2 + y^2} = \frac{2x^2}{x^2 + x^2} \rightarrow 1</math>, which is not well-approximated by <math>z = 0</math>, i.e., <math>z = 1</math> is far from <math>z = 0</math>.</p> <p>Hence <math>z = 0</math> is not the tangent plane to <math>f</math> at <math>(0,0,0)</math>.</p>                                  |

- 9 The curve  $C$ , known as the Folium of Descartes, has the cartesian equation  $x^3 + y^3 = 3axy$ , where  $a > 0$ . The line  $L$  is the asymptote of  $C$ .

[You may use the identity  $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$  for this question.]



- (a) Evaluate  $\lim_{x \rightarrow \infty} \frac{y}{x}$ . [You may assume that  $\lim_{x \rightarrow \infty} \frac{y}{x}$  exists].  
Hence, or otherwise, show that the cartesian equation of  $L$  is  $x + y + a = 0$ . [4]
- (b) Find the polar equations of  $C$  and  $L$ . [2]
- (c) (i) Prove that  $\frac{\cos^2 \theta \sin^2 \theta}{(\cos^3 \theta + \sin^3 \theta)^2} = \frac{\tan^2 \theta \sec^2 \theta}{(1 + \tan^3 \theta)^2}$ . [1]
- (ii) The region  $R$  lies in the first quadrant and is bounded by the curve  $C$ . Find the exact area of  $R$ , giving your answer in terms of  $a$ . [3]
- (d) A straight line through  $O$  cuts  $C$  at  $A$  and  $L$  at  $B$  in the second quadrant.  
Let  $\phi$  be the angle between  $OB$  and the positive  $x$ -axis.  
Let  $A_\phi$  be the area of the region bounded by the negative  $x$ -axis, curve  $C$ , asymptote  $L$  and line  $AB$ .  
Find  $A_\phi$  when  $\phi = \frac{5\pi}{6}$  and  $a = 1$ . [3]

|     | Suggested Solution                                                                                      |
|-----|---------------------------------------------------------------------------------------------------------|
| (a) | $x^3 + y^3 = 3axy$ $1 + \left(\frac{y}{x}\right)^3 = 3\left(\frac{a}{x}\right)\left(\frac{y}{x}\right)$ |

|        |                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
|--------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|        | <p>Let <math>\lim_{x \rightarrow \infty} \frac{y}{x} = m</math></p> <p>As <math>x \rightarrow \infty</math>, we have</p> $1 + m^3 = 3(0)(m)$ $m = -1$ <p>Since</p> $x^3 + y^3 = 3axy$ $x + y = \frac{3axy}{x^2 - xy + y^2} = \frac{3a\left(\frac{y}{x}\right)}{1 - \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2}$ <p>As <math>x \rightarrow \infty</math>, we have</p> $x + y = \frac{3a(-1)}{1 - (-1) + (-1)^2}$ $x + y + a = 0 \text{ (Shown)}$ |
| (b)    | <p>Let <math>x = r \cos \theta</math>, <math>y = r \sin \theta</math></p> <p>For curve <math>C</math>: <math>x^3 + y^3 = 3axy</math></p> $r^3 \cos^3 \theta + r^3 \sin^3 \theta = 3ar^2 \sin \theta \cos \theta$ $r = \frac{3a \sin \theta \cos \theta}{\cos^3 \theta + \sin^3 \theta}$ <p>For Asymptote <math>L</math>: <math>x + y + a = 0</math></p> $r \cos \theta + r \sin \theta + a = 0$ $r = \frac{-a}{\cos \theta + \sin \theta}$                 |
| (c)(i) | $\frac{\cos^2 \theta \sin^2 \theta}{(\cos^3 \theta + \sin^3 \theta)^2} = \frac{\cos^2 \theta \sin^2 \theta}{(\cos^3 \theta + \sin^3 \theta)^2} \times \frac{1/\cos^6 \theta}{1/\cos^6 \theta}$ $= \frac{\sin^2 \theta}{\cos^2 \theta} \times \frac{1}{\cos^2 \theta}$ $= \frac{1}{(1 + \tan^3 \theta)^2}$ $= \frac{\tan^2 \theta \sec^2 \theta}{(1 + \tan^3 \theta)^2}$                                                                                    |

|         |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
|---------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (c)(ii) | $\begin{aligned} \text{Area of } R &= \frac{1}{2} \int_0^{\pi/2} r^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \frac{9a^2 \sin^2 \theta \cos^2 \theta}{(\cos^3 \theta + \sin^3 \theta)^2} d\theta \\ &= \frac{9a^2}{2} \int_0^{\pi/2} \frac{\tan^2 \theta \sec^2 \theta}{(1 + \tan^3 \theta)^2} d\theta \\ &= \frac{3a^2}{2} \int_0^{\pi/2} \frac{3 \tan^2 \theta \sec^2 \theta}{(1 + \tan^3 \theta)^2} d\theta \\ &= \frac{3a^2}{2} \left[ -\frac{1}{1 + \tan^3 \theta} \right]_0^{\pi/2} \\ &= \frac{3a^2}{2} \left[ -0 - \left( -\frac{1}{1+0} \right) \right] \\ &= \frac{3a^2}{2} \end{aligned}$ |
| (d)     | $A_\phi = \frac{1}{2} \int_\phi^\pi r_1^2 - r_2^2 d\theta$ <p>where <math>r_1 = \frac{-1}{\cos \theta + \sin \theta}</math> and <math>r_2 = \frac{3 \sin \theta \cos \theta}{\cos^3 \theta + \sin^3 \theta}</math></p> <p>Therefore</p> $\begin{aligned} A_\phi &= \frac{1}{2} \int_{\frac{5\pi}{6}}^\pi \frac{1}{(\sin \theta + \cos \theta)^2} - \frac{9 \sin^2 \theta \cos^2 \theta}{(\cos^3 \theta + \sin^3 \theta)^2} d\theta \\ &= 0.32554 \\ &= 0.326 \text{ (to 3 sf)} \end{aligned}$                                                                                                        |

- 10** A drug-eluting stent (DES) is a small mesh tube that doctors place inside blocked heart arteries to keep them open and improve blood flow. The stent is coated with a medical drug that slowly releases into the artery wall to prevent the artery from becoming blocked again. The drug concentration in the stent coating,  $y$  mg/cm<sup>2</sup>, and the drug concentration in the arterial wall tissue,  $z$  mg/cm<sup>2</sup>, can be modelled by the following differential equations.

$$\frac{dy}{dt} = -\frac{1}{2}(y - z) \text{ and } \frac{dz}{dt} = -\frac{dy}{dt} - \frac{3}{4}z,$$

where  $t$  is time measured in months.

For a patient undergoing such treatment, it is known that  $y = I$  and  $z = 0$  initially.

- (a) Show  $\frac{d^2y}{dt^2} + \frac{7}{4} \frac{dy}{dt} + \frac{3}{8}y = 0$ . [2]
- (b) Hence find  $y$  and  $z$  in terms of  $I$  and  $t$ . [5]
- (c) For the safety of the patient, the drug concentration in arterial wall tissue must not exceed  $0.1 \text{ mg/cm}^2$ . Determine the maximum value of  $I$ . [3]
- (d) It is known that the drug concentration in the arterial wall tissue must be kept at a level of at least  $0.03 \text{ mg/cm}^2$  over a continuous period of 2 months for the treatment to be considered effective. Is the treatment considered safe and effective if a patient is undergoing the treatment where  $I = 0.2$ ? [3]

| Suggested Solution |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|--------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (a)                | $\frac{dy}{dt} = -\frac{1}{2}(y - z)$ $\frac{d^2y}{dt^2} = -\frac{1}{2} \frac{dy}{dt} + \frac{1}{2} \frac{dz}{dt}$ $\frac{d^2y}{dt^2} = -\frac{1}{2} \frac{dy}{dt} + \frac{1}{2} \left( -\frac{dy}{dt} - \frac{3}{4}z \right)$ $\frac{d^2y}{dt^2} = -\frac{dy}{dt} - \frac{3}{4} \left( \frac{dy}{dt} + \frac{1}{2}y \right)$ $\frac{d^2y}{dt^2} = -\frac{dy}{dt} - \frac{3}{4} \frac{dy}{dt} - \frac{3}{8}y$ $\frac{d^2y}{dt^2} + \frac{7}{4} \frac{dy}{dt} + \frac{3}{8}y = 0$                                                                                                            |
| (b)                | <p>Characteristic equation:</p> $r^2 + \frac{7}{4}r + \frac{3}{8} = 0 \Rightarrow r = -\frac{1}{4}, -\frac{3}{2}$ <p>Therefore <math>y = Ae^{-\frac{1}{4}t} + Be^{-\frac{3}{2}t}</math>, where <math>A</math> and <math>B</math> are arbitrary constants</p> $z = y + 2 \frac{dy}{dt}$ $= Ae^{-\frac{1}{4}t} + Be^{-\frac{3}{2}t} + 2 \left( -\frac{1}{4}Ae^{-\frac{1}{4}t} - \frac{3}{2}Be^{-\frac{3}{2}t} \right)$ $= \frac{1}{2}Ae^{-\frac{1}{4}t} - 2Be^{-\frac{3}{2}t}$ <p>When <math>t = 0</math>, <math>y = I</math> and <math>z = 0</math>:</p> $A + B = I$ $\frac{1}{2}A - 2B = 0$ |

Solving,

$$\frac{1}{2}(I - B) - 2B = 0$$

$$\Rightarrow I - B - 4B = 0$$

$$\Rightarrow B = \frac{1}{5}I$$

$$\therefore A = \frac{4}{5}I$$

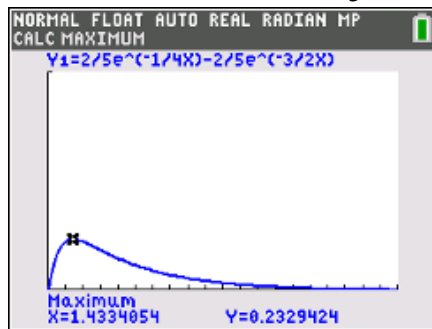
So we have

$$y = \frac{4I}{5}e^{-\frac{1}{4}t} + \frac{I}{5}e^{-\frac{3}{2}t} = I\left(\frac{4}{5}e^{-\frac{1}{4}t} + \frac{1}{5}e^{-\frac{3}{2}t}\right),$$

$$z = \frac{2I}{5}e^{-\frac{1}{4}t} - \frac{2I}{5}e^{-\frac{3}{2}t} = I\left(\frac{2}{5}e^{-\frac{1}{4}t} - \frac{2}{5}e^{-\frac{3}{2}t}\right).$$

(c)

Consider the graph of  $c = \frac{2}{5}e^{-\frac{1}{4}d} - \frac{2}{5}e^{-\frac{3}{2}d}$ :



Maximum of  $z$  is  $0.2329424I$

$$\Rightarrow 0.2329424I \leq 0.1$$

$$\Rightarrow I \leq 0.42929$$

Hence maximum value of  $I$  is  $0.429$  mg per  $\text{cm}^2$  (to 3sf)

Alternatively, at maximum  $z$ ,  $\frac{dz}{dt} = 0$

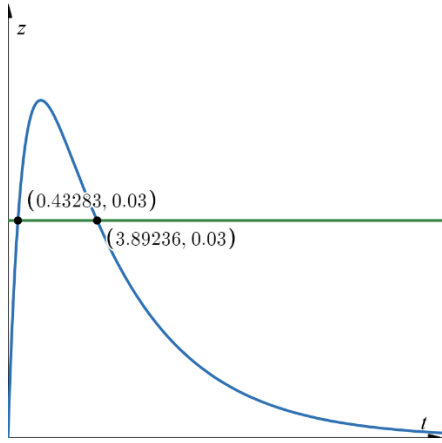
$$\text{i.e. } \frac{2I}{5}\left(-\frac{1}{4}e^{-\frac{1}{4}t} + \frac{3}{2}e^{-\frac{3}{2}t}\right) = 0$$

$$\text{Since } I > 0, -\frac{1}{4}e^{-\frac{1}{4}t} + \frac{3}{2}e^{-\frac{3}{2}t} = 0$$

$$e^{-\frac{5}{4}t} = \frac{1}{6}$$

$$t = \frac{4}{5}\ln 6$$

$$\text{When } t = \frac{4}{5}\ln 6, z = \frac{2I}{5}(6^{-0.2} - 6^{-1.2}) \leq 0.1$$

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|     | $0.2329424I \leq 0.1$ $I \leq 0.42929$ <p>Hence maximum value of <math>I</math> is 0.429 mg per <math>\text{cm}^2</math> (to 3sf)</p>                                                                                                                                                                                                                                                                                                                                                                           |
| (d) | <p>Since <math>I = 0.2</math>, <math>z = 0.08 \left( e^{-\frac{1}{4}t} - e^{-\frac{3}{2}t} \right)</math></p>  <p><math>t_2 - t_1 = 3.89236 - 0.43283 = 3.46</math> (to 3 sf) <math>&gt; 2</math></p> <p>Since the time interval where <math>z = 0.03</math> is more than 2, then <math>z</math> would be <math>&gt; 0.03</math> when the time interval is 2 months. Thus, the treatment is considered safe and effective.</p> |