Oscillations

1 **(a)** What are the conditions for simple harmonic motion? [2]

(b) A point mass moves along the line AB (Fig. 1.1) with simple harmonic motion of amplitude 0.040 m and period 1.6 s.

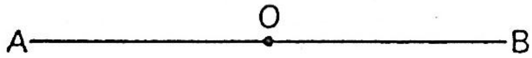


Fig. 1.1

At time t = 0, the mass is at the equilibrium position O, and is moving in the direction OA. What is the displacement of the mass at time t = 1.3 s? In which direction is it moving at this instant? [6]

(c) An ideal spring of unstretched length L and force constant k is fastened between two fixed points X and Y a distance L apart just above a horizontal, frictionless table. A mass M is then clamped to the mid-point of the spring so that it rests on the table. The mass is drawn aside through a small distance x, perpendicular to the original line of the spring, and is then released, as shown in plan view in Fig. 1.2.

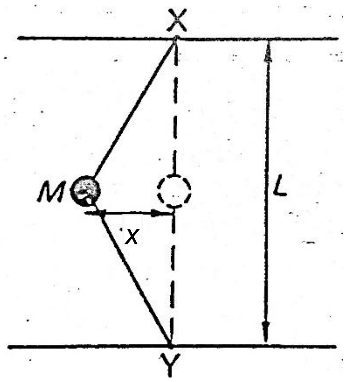


Fig. 1.2

Find the initial acceleration of the mass, and hence determine whether the resulting oscillations are simple harmonic. [8]

(Hint: for $z \leq 1, (1+z)^n = 1 + nz$)

(d) Why is simple harmonic motion so important in Physics? [4]

[N93/P1/Q4]

2 **(a)** A straight wire has unstretched length L and area of cross-section A. Find the relation between the force constant k of the wire and the Young modulus E of the metal of which it is made.
Why does this relation not apply to a spring in the form of a coil of the same wire? [4]

(b) A mass m is supported by a spring of force constant k. A simple derivation shows that the period T of vertical oscillations of the mass is given by $T = 2\pi\left(m/k\right)^{1/2}$. What assumptions must be made if this expression is to hold? (You are *not* asked to derive the expression.) [3]

(c) A mass M is supported by two springs, of force constants k_1 and k_2 respectively, connected in series as shown in Fig. 2.1.

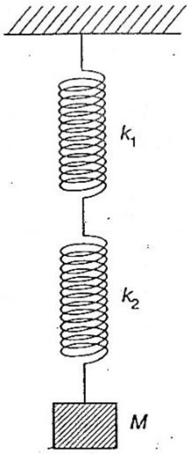


Fig. 2.1

Find the effective force constant of the spring combination, and hence deduce the period of vertical oscillation of the mass. [4]

(d) The mass M is now set between the same two springs on a horizontal, frictionless surface, as shown in Fig. 2.2. Each spring is of unstretched length l. The distance between the clamps at each side is D, where D is greater than $2l$.

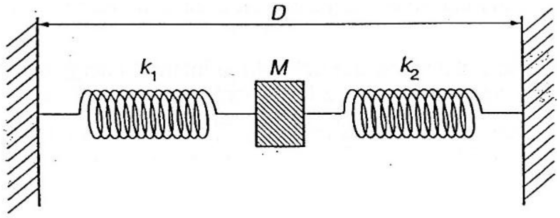


Fig. 2.2

(i) Find the distance d of the equilibrium position of the mass from the left-hand clamp. [3]

(ii) The mass is then displaced a distance x to the right of the equilibrium position, along the axis of the springs. The distance x is sufficiently short for both springs to remain in tension. Show that, when released, the mass moves with simple harmonic motion. Find the period of the oscillations. [6]

[N95/P1/Q7]

3 **(a)** A particle of mass m moves in a straight line with simple harmonic motion. Its displacement x at time t is given by the expression

$$x = A\sin\omega t\ .$$

(i) Write down an expression for the velocity v of the particle at time t, in terms of A, ω and t. [1]

(ii) Sketch a graph showing the variation with time t of the kinetic energy E_k of the particle. Label important features of your graph with corresponding values of E_k and t. [4]

(iii) Use your graph in **(ii)** to show that the average value of the kinetic energy over one complete cycle of the motion is

$$\langle E_k \rangle = \frac{1}{4} m A^2 \omega^2\ .$$

Write down an expression for the average value $\langle E_p \rangle$ of the potential energy of the particle due to its motion over one cycle. [2]

(b) An electron of mass m_e and charge $-e$ moves in a straight line with damped harmonic motion of angular frequency ω . According to classical electromagnetic theory, the power P radiated by the electron at any instant is given by

$$P = \frac{e^2 a^2}{6\pi\varepsilon_0 c^3}\ ,$$

where a is the acceleration of the electron at that instant, ε_0 is the permittivity of free space, and c is the speed of electromagnetic waves in free space. It is the continuous radiation of energy by the electron that causes the motion to be damped. Because of the damping, the amplitude of the oscillation of the electron decreases exponentially with time. At time t, the amplitude A is given by

$$A = A_0 e^{-\gamma t}$$

where A_0 and γ are constants.

(i) Write down, in terms of ω and A, the average value of a^2 (the square of the acceleration of the electron) over one complete cycle. Hence show that the average power $\langle P \rangle$ radiated over one cycle is given by

$$\langle P \rangle = \frac{e^2 A^2 \omega^4}{12\pi\varepsilon_0 c^3}\ .$$ [2]

(ii) Obtain an expression, in terms of A_0, γ , m_e and ω, for the total energy E_{tot} (kinetic energy plus potential energy) of the electron at time t. [2]

(iii) Hence find the rate of change of E_{tot} with time. Use your result to show that

$$\gamma = \frac{e^2 \omega^2}{12\pi\varepsilon_0 m_e c^3}\ .$$ [3]

(iv) Suppose that an oscillating electron were radiating light of wavelength 500 nm. Use the classical theory to calculate how long it would take for the intensity of the light to fall to half of its initial value. [6]

Wave Motion

4 **(a)** Sound is propagated in air as a longitudinal progressive wave, in which there is a repeated sequence of displacements of the air particles. These displacements give rise to pressure variations; hence, the wave may also be described in terms of a repeated sequence of changes in pressure.

(i) Describe the motion of the particles in a longitudinal wave. [2]

(ii) Fig. 4.1 illustrates nine particles, equally spaced along the line AB, in still air.

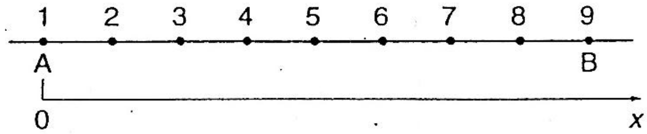


Fig. 4.1

A sound wave of wavelength equal to AB is sent through the air in the direction Ox. Copy Fig. 4.1 and underneath it draw a second diagram showing possible positions of the nine particles in the wave relative to the undisturbed positions they occupy in Fig. 4.1. Use information from this diagram to draw a graph to show how the displacement *d* of the particles varies with distance *x* along the direction of propagation. [2]

(iii) Explain how the points of high and low pressure in the air can be deduced from your diagram of the positions of the particles in the wave. Hence draw a second graph to show how the pressure *p* in the air varies with *x*. State the phase difference, if any, between the displacement wave and the pressure wave. [5]

(b) A characteristic of a progressive wave is that it transmits energy.

(i) Define the *intensity* of a wave. [1]

(ii) The intensity I of a sound wave in air is related to the amplitude d_o of the displacement of the air particles by

$$I = Ad_o^{\,2},$$

where A is a constant. Find A in terms of the speed of propagation c, the angular frequency ω of the wave and the mean density ρ_o of the air. Assume the expression

$$E_{tot} = \tfrac{1}{2}ma^2\omega^2$$

for the total energy of a mass m moving in simple harmonic motion of amplitude a and angular frequency ω. [4]

(c) When a sound wave passes along a straight pipe of constant cross-section, energy is lost from the wave. The rate of loss of energy in each element δx is proportional to the intensity of the wave at the beginning of that element. The constant of proportionality is *k*.

(i) Obtain an expression for the intensity I of the wave at a distance *x* along the pipe, in terms of the intensity I_o at the source end (when *x* = 0), *k* and *x*. [2]

(ii) The intensity of sound at two points 0.80 m apart in a certain pipe used as a speaking-tube is found to be 4.8×10^{-10} W m^{-2} and 4.2×10^{-10} W m^{-2}. Find the value of *k* for this sound wave. Calculate the length of pipe over which the intensity would be halved. [4]

[N99/P1/Q8]

5 **(a)** In a simplified description of a particular earthquake, shock waves travel radially outwards as though from a point source. The waves are of two types, P-waves and S-waves. These P-waves travel with a speed of 8.4 km s^{-1}, and these S-waves with a speed of 5.6 km s^{-1}.

(i) Following the earthquake, a monitoring station receives P- and S-wave signals directly from the source. The signals are separated by a time interval of 6.5 s. Calculate the distance of the earthquake from the monitoring station. [3]

(ii) The information obtained from this monitoring station is limited to the distance of the source of the earthquake from the station. However, similar information from a number of stations may be combined to locate the source accurately. What minimum number of stations is required? Explain your answer with the aid of a diagram. [2]

(b) The arrival of the P- and S-wave is detected using an instrument called a seismograph, which converts the mechanical vibration due to the earthquake into an electrical output. On the Richter scale of earthquake strength, which was introduced in 1935, earthquakes that were only just detectable at that time were assigned magnitude zero. An increase of one on the Richter scale is defined as corresponding to a ten times increase in seismograph output.

(i) An expression used to calculate the magnitude *M* of an earthquake from seismograph signals is

$$M = \lg\left(\frac{A}{A_0}\right),$$

where *A* is the seismograph output produced by the earthquake and A_0 is the seismograph output produced by an earthquake of magnitude zero. Show that this expression is consistent with the definition of the Richter scale. [2]

(ii) For a certain seismograph, the output corresponding to an earthquake of magnitude zero is 4.0 units. Calculate the output which would be registered for an earthquake of magnitude 7.2 on the Richter scale. [2]

(iii) In recent years it has become possible to study micro-earthquakes with negative magnitudes. What is the implication of a negative magnitude on the Richter scale? [1]

[N02/P1/Q6]

6 **(a)** **(i)** Explain, in terms of energy transfer, what is meant by the **intensity** of a progressive wave. [2]

(ii) Sketch a graph to show how the intensity of a wave is related to its amplitude. [1]

(b) Some waves can be polarised. State the common feature of such waves. Give one example, other than an electromagnetic wave, of a wave that can be polarised. [2]

(c) Light can be polarised using a polariser, such as a sheet of Polaroid. A polariser has an axis for the "easy" transmission of light (the easy axis). It transmits the component of the electric field (E-field) of light which is parallel to this axis. In a perfect polariser, this component is transmitted without absorption. The component perpendicular to the easy axis is completely absorbed.

(i) Fig. 6.1 shows a perfect polariser with its easy axis vertical.

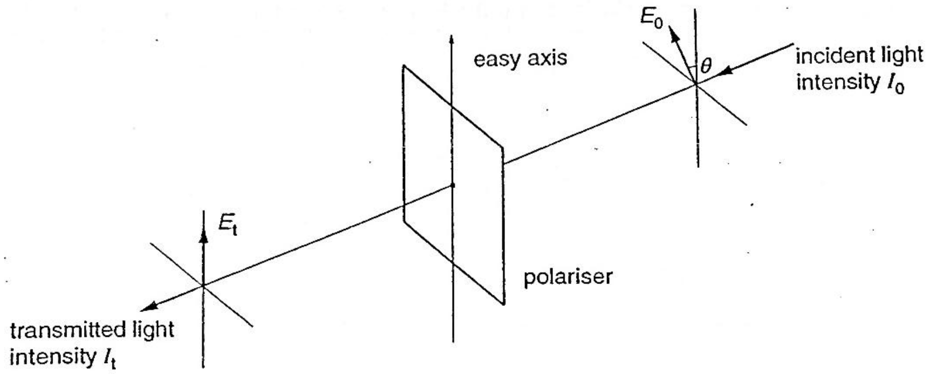


Fig. 6.1

A parallel beam of polarised light of intensity I_0 is incident on the polariser with its E-field, of amplitude E_0, at an angle θ to the vertical. The transmitted light has amplitude E_t. Find an expression, in terms of I_0 and θ, for the intensity I_t of the transmitted light. [2]

(ii) Two perfect polarisers A and B are placed in contact with their easy axes parallel. Polarised light of intensity I_0 is incident on A with its E-field parallel to the easy axis. Keeping the polarisers in contact, polariser B is then rotated so that its easy axis makes an increasing angle ϕ with the easy axis of A. Sketch a graph to show how the intensity I_t of the light transmitted by the polariser combination varies with the angle ϕ, for values of ϕ between 0 and 2π rad. Label the axes with appropriate values. [4]

(iii) A large number $(N+1)$ of perfect polarisers are stacked parallel to each other so that the easy axis of each is at a small angle α to the one before it. This causes the easy axis of the last sheet to make an angle $\psi = N\alpha$ with that of the first. Polarised light of intensity I_0 is incident on the stack with its *E*-field parallel to the easy axis of the first sheet.

Show that the output intensity I_t is given by

$$I_t \approx I_0\left(1-\frac{\psi^2}{N}\right),$$

where ψ is measured in radians. Use the approximations

$$\cos\theta \approx 1-\frac{1}{2}\theta^2 \qquad \text{for small } \theta$$

and $$(1+x)^n \approx 1+nx \qquad \text{for } x \ll 1.$$ [6]

(iv) Compare the result obtained in **(iii)** when $\psi = \pi/2$ rad, i.e. when the first and last polarisers of the stack are crossed, with that obtained in **(ii)** for $\phi = \pi/2$ rad, i.e. when the two polarisers are crossed. [3]

[N04/P1/Q9]

Additional Questions

7 **(a)** A particle of mass *m* moves in a straight line with simple harmonic motion of angular frequency ω and amplitude x_0. Derive expressions for the kinetic energy E_k and the potential energy E_p of the particle as functions of displacement *x*. Hence show that the total vibrational energy of the particle is $\frac{1}{2}m{x_0}^2\omega^2$. [6]

(b) Suppose that a crystalline solid can be represented as an array of regularly spaced atoms of equal mass, all of which are subject to thermal agitation which causes them to vibrate with simple harmonic motion about their mean positions. At a temperature of 500 K, the total vibrational energy of a single atom in a certain solid is 2.0×10^{-20} J and the vibration frequency is 8.0×10^{12} Hz. The mass of the atom is 1.0×10^{-25} kg.

(i) Find the amplitude of vibration of the atom. [3]

(ii) State a typical value for the separation of atoms in a solid, and express your answer to **(i)** as a fraction of this separation. [2]

(iii) Calculate the total vibrational energy of one mole of the solid at the temperature of 500 K. [2]

(iv) If the temperature were increased to 501 K, by how much would the total vibrational energy of one mole of the solid increase? [2]

(v) Calculate the fractional change in the amplitude of atomic vibration if the temperature were increased from 500 K to 501 K. [2]

(c) In the simple crystalline model described in **(b)**, different solids have different atomic masses and vibration frequencies, but their atomic vibrational energies are the same at any given temperature. Show that, at a given temperature, the amplitude x_0 of atomic vibration depends upon the maximum atomic restoring force F_{max} according to the relation

$$x_0 F_{max} = A\ ,$$

where A is a constant. [3]

[N96/P1/Q3]

8 (a) (i) The defining equation for simple harmonic motion (s.h.m.) is

$$a = -\omega^2 x\ .$$

Identify the quantities a, ω and x. [2]

(ii) The vertical oscillations of a mass m suspended from a spring of force constant k are often quoted as an example of simple harmonic motion. For this system,

$$\omega = \sqrt{\frac{k}{m}}\ .$$

Use base units to check the homogeneity of this equation. [2]

(b) Fig. 8.1 shows a box of mass *M* and internal height *h* suspended by a spring of force constant *k*. A blob of plasticine, of mass *m*, sticks to the inside of the top of the box. Initially the system is in equilibrium and at rest.

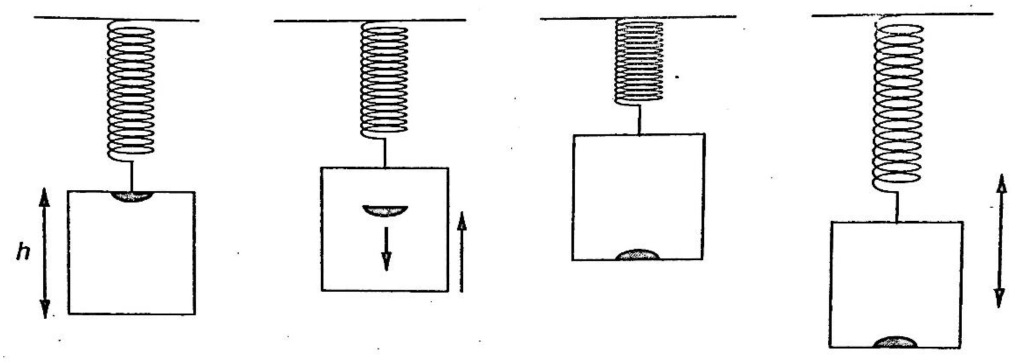


Fig. 8.1 Box and plasticine at rest

Fig. 8.2 Plasticine falls freely; box moves upwards in s.h.m.

Fig. 8.3 Plasticine sticks to box; box at rest at top of first oscillation

Fig. 8.4 Box and plasticine move in s.h.m.

Suddenly the plasticine becomes detached and falls freely towards the bottom of the box. Because the mass suspended from the spring has been reduced, there is a resultant upwards force, and the box starts to move up in simple harmonic motion, as represented in Fig. 8.2. The plasticine reaches the bottom of the box at the instant that the box is momentarily at rest at the top of its first half-oscillation, and sticks to the bottom of the box, as in Fig. 8.3. The system then oscillates in simple harmonic motion (Fig. 8.4).

(i) Obtain expressions, in terms of *m*, *M*, *k* and *g* as appropriate, for

1 the time taken for the box to move from the position shown in Fig. 8.1 to that shown in Fig. 8.3, [2]

2 the distance the top of the box rises from the position shown in Fig. 8.1 to that shown in Fig. 8.3, [2]

3 the distance through which the plasticine falls between the position shown in Fig. 8.1 and that shown in Fig. 8.3, [1]

4 the final angular frequency of oscillation of the system when it is moving as in Fig. 8.4. [1]

(ii) Use your answers in **(i)1** and **(i)3** to show that the height *h* of the box is given by

$$h = \frac{4mg + \pi^2 Mg}{2k}.$$ [2]

(iii) Use the results in **(i)3** and **(ii)** to show that the speed *v* of the plasticine when it hits the bottom of the box, at the instant shown in Fig. 8.3, is given by

$$v = \pi g\sqrt{\frac{M}{k}}.$$ [2]

(iv) Hence find the initial downwards speed of the box and plasticine after the impact. [3]

(v) Use your results for **(i)2**, **(i)4** and **(iv)** to find an expression for the amplitude of the final motion of the box, as illustrated in Fig. 8.4. Give your answer in terms of *m*, *M*, *k* and *g*. [3]

[N04/P1/Q8]

Answers

1 **(b)** –0.0370 m, left **(c)** $a = -\dfrac{8k}{ML^2}x^3$

2 **(a)** $k = \dfrac{EA}{L}$ **(c)** $k = \dfrac{k_1 k_2}{k_1 + k_2}$, $T = 2\pi\left(\dfrac{M(k_1 + k_2)}{k_1 k_2}\right)^{1/2}$

(d)(i) $d = \dfrac{k_1 l + k_2(D - l)}{k_1 + k_2}$ **(d)(ii)** $T = 2\pi\sqrt{\dfrac{M}{k_1 + k_2}}$

3 **(a)(i)** $v = A\omega\cos\omega t$ **(a)(iii)** $\langle E_p \rangle = \dfrac{1}{4}mA^2\omega^2$ **(b)(i)** $\langle a^2 \rangle = \dfrac{1}{2}A^2\omega^4$

(b)(ii) $E_{tot} = \dfrac{1}{2}m_e {A_0}^2\omega^2 e^{-2\gamma t}$ **(b)(iii)** $\dfrac{dE_{tot}}{dt} = -m_e {A_0}^2\omega^2\gamma e^{-2\gamma t}$ **(b)(iv)** 7.82×10^{-9} s

4 **(a)(iii)** phase diff = $\pi/2$ **(b)(ii)** $A = \dfrac{1}{2}\rho_0 c\omega^2$

(c)(i) $I = I_0 e^{-kx}$ **(c)(ii)** $k = 0.167\ \text{m}^{-1}$, $x = 4.15$ m

5 **(a)(i)** 109 km **(a)(ii)** 3 **(b)(ii)** $A = 6.34 \times 10^7$ units

6 **(c)(i)** $I_t = I_0\cos^2\theta$

7 **(b)(i)** 1.3×10^{-11} m **(b)(ii)** 3×10^{-10} m, $1/23$ **(b)(iii)** 1.2×10^4 J **(b)(iv)** 24 J

(b)(v) $1/1000$

8 **(a)(ii)** homogeneous **(b)(i)1** $\pi\sqrt{\dfrac{M}{k}}$ **(b)(i)2** $\dfrac{2mg}{k}$ **(b)(i)3** $\dfrac{1}{2}g\pi^2\dfrac{M}{k}$

(b)(i)4 $\sqrt{\dfrac{k}{M + m}}$ **(b)(iv)** $\left(\dfrac{m}{M + m}\right)\left(\pi g\sqrt{\dfrac{M}{k}}\right)$ **(b)(v)** $\sqrt{\left(\dfrac{\pi^2 g^2 M m^2}{(M + m)k^2}\right) + \left(\dfrac{4m^2 g^2}{k^2}\right)}$

Suggested Solutions to H3 Tutorial on Topics (10) Oscillations & (11) Wave Motion

Oscillations

1 **(a)** There must be a restoring force acting on the system.
The system must have inertia to move to and fro about a fixed point.
The acceleration of the system is always directly proportional to its displacement from a fixed point and is directed towards that point.

(b) Take the direction OA to be positive

$$x = x_0 \sin \omega t$$

$$x = 0.040 \sin\left[\left(\frac{2\pi}{T}\right)t\right]$$

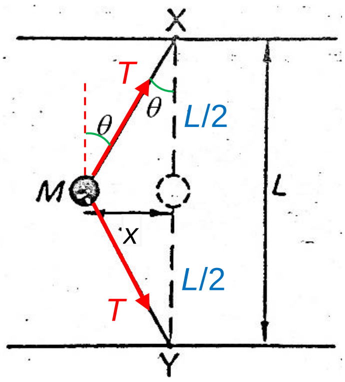
At t = 1.3 s

$$x = 0.040 \sin\left[\left(\frac{2\pi}{1.6}\right)(1.3)\right]$$

$$x = -0.03696 = -0.0370 \text{ m}$$

Mass is on the right side of O and 0.0370 m from O.
It is moving to the left, towards O at this instant.

(c)



By Pythagoras theorem, half length of spring when stretched is

$$\left[\left(\frac{L}{2}\right)^2 + x^2\right]^{1/2}$$

Extension of each half length of spring,

$$e = \left[\left(\frac{L}{2}\right)^2 + x^2\right]^{1/2} - \frac{L}{2}$$

$$= \left(\frac{L}{2}\right)\left[1 + \left(\frac{x}{L/2}\right)^2\right]^{1/2} - \frac{L}{2}$$

$$= \frac{L}{2}\left[1 + \frac{1}{2}\left(\frac{2x}{L}\right)^2\right] - \frac{L}{2} \qquad \text{for small values of } x$$

$$= \frac{L}{2} + \frac{L}{2}\left(\frac{2x^2}{L^2}\right) - \frac{L}{2}$$

$$= \frac{x^2}{L}$$

When the spring is halved, the force constant doubles.
By Hooke's Law, tension in each half of the spring is $T = 2ke$

Consider forces along the x direction:
Let F_R be the resultant force on M and a be its acceleration.
Magnitude of the resultant force,

$$F_R = 2T\sin\theta$$

$$= 2(2ke)\sin\theta$$

$$= 4k\left(\frac{x^2}{L}\right)\left(\frac{x}{L/2}\right) \qquad \left(\text{for small } \theta,\ \sin\theta \approx \tan\theta \approx \frac{x}{L/2}\right)$$

$$= \frac{8kx^3}{L^2}$$

Since F_R is always directed towards the equilibrium position of M and x is directed away from the equilibrium position, F_R and x will always be in opposite directions.

$$Ma = -\frac{8kx^3}{L^2} \qquad \Rightarrow \qquad a = -\frac{8k}{ML^2}x^3$$

Since a is not directly proportional to $-x$, the resulting oscillations are not simple harmonic.

(d) Simple harmonic motion is important in Physics because it is involved in almost every branch of Physics.

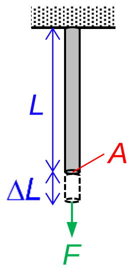
This ranges from mechanical systems such as the motion of air and water molecules in waves; stationary waves in musical instruments (piano strings and organ pipes), pendulums and car suspension systems; electrical systems such as the oscillation of electromagnetic wave vectors, radio tuners and microwave ovens; and quantum Physics (resonant absorption of photons in the absorption line spectra and modern day technology – computers, hand phones, electronic devices).

There are also areas where simple harmonic motion can do damage, such as resonance in suspension bridges and buildings under the effect of strong winds and earth tremours, resulting in damage to the infrastructures.

An understanding of simple harmonic motion is necessary to model the behaviour of all the above-mentioned systems, thus enabling them to be modified and improved.

2 **(a)** Let F be the applied force applied perpendicularly to the cross-sectional area A and ΔL be the change in length of the wire when F is applied

$$E = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta L/L} = \frac{FL}{A\Delta L}$$



By Hooke's Law, $F = k\Delta L$

$$E = \frac{k(\Delta L)L}{A\Delta L}$$

$$k = \frac{EA}{L}$$

Young modulus is a measure of the elasticity of a material. The force applied is perpendicular to the cross-sectional area of the wire and along the length of the wire in the direction of stretch.

When the spring is in the form of a coil of wire, the extension is not due to the elongation of the wire, but rather the gap between adjacent coils. Hence it is inappropriate to bring in the Young modulus of the metal into the equation when considering the extension of a spring.

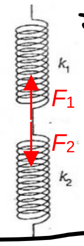
(b) The mass of the spring is negligible.

The spring obeys Hooke's Law thus the restoring force is proportional to the displacement of the mass and performs simple harmonic motion.

There is no damping or energy loss which causes the period to change.

(c) $F_1 = F_2$

$k_1 e_1 = k_2 e_2 = k\left(e_1 + e_2\right)$ where k is the effective force constant



$= k_{eff}\left(\frac{1}{k_1}+\frac{1}{k_2}\right) mg$

$= mg$

$$\frac{k_2}{k_1} = \frac{e_1}{e_2} \quad \text{and} \quad \frac{k_2}{k} = \frac{e_1 + e_2}{e_2} = \frac{k_2}{k_1} + 1 = \frac{k_2 + k_1}{k_1}$$

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

$\boxed{k_{eff} = \frac{1}{\frac{1}{k_1}+\frac{1}{k_2}}}$

$\boxed{F = k_{eff}\,\Delta x_{tot}} = mg$

$\Delta x_{tot} = \Delta x_1 + \Delta x_2$

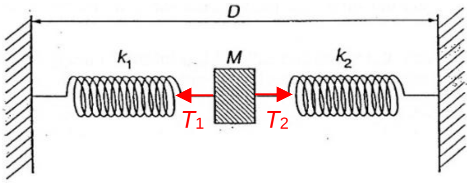
$k_1 \Delta x_1 = mg = k_2 \Delta x_2$

$\Delta x_1 = \frac{mg}{k_1} \quad \Delta x_2 = \frac{mg}{k_2}$

Period of vertical oscillation of the mass,

$$T = 2\pi\left(\frac{M}{k}\right)^{1/2} = 2\pi\left(\frac{M}{\frac{k_1 k_2}{k_1 + k_2}}\right)^{1/2} = 2\pi\left(\frac{M\left(k_1 + k_2\right)}{k_1 k_2}\right)^{1/2}$$

(d) **(i)** At equilibrium,



$T_1 = T_2$

$k_1 e_1 = k_2 e_2$ ----- (1)

$l + l + e_1 + e_2 = D$

$e_2 = D - 2l - e_1$ ----- (2)

subst. (2) into (1)

$k_1 e_1 = k_2\left(D - 2l - e_1\right)$

$k_1 e_1 + k_2 e_1 = k_2 D - 2k_2 l$

$$e_1 = \frac{k_2 D - 2k_2 l}{k_1 + k_2}$$

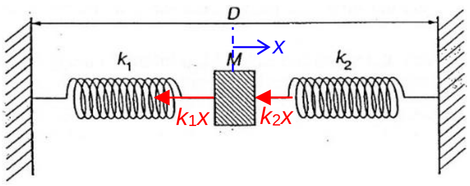
$$d = l + e_1 = l + \frac{k_2 D - 2k_2 l}{k_1 + k_2} = \frac{k_1 l + k_2 l + k_2 D - 2k_2 l}{k_1 + k_2} = \frac{k_1 l + k_2 D - k_2 l}{k_1 + k_2}$$

$$= \frac{k_1 l + k_2\left(D - l\right)}{k_1 + k_2}$$

(ii) $F = -(k_1 + k_2)x$

$Ma = -(k_1 + k_2)x$

$a = -\left(\dfrac{k_1 + k_2}{M}\right)x$



Since $\left(\dfrac{k_1 + k_2}{M}\right)$ is a constant, $a \propto (-x)$. Hence the mass moves with simple harmonic motion.

Compare with SHM equation $a = -\omega^2 x$

$\omega^2 = \dfrac{k_1 + k_2}{M}$

$\dfrac{2\pi}{T} = \sqrt{\dfrac{k_1 + k_2}{M}}$

$T = \dfrac{2\pi}{\sqrt{\dfrac{k_1 + k_2}{M}}} = 2\pi\sqrt{\dfrac{M}{k_1 + k_2}}$

3 (a) (i) $x = A\sin\omega t$

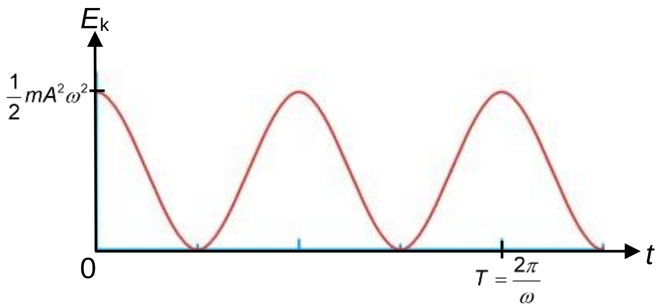
$v = \dfrac{dx}{dt} = A\omega\cos\omega t$

(ii) $E_k = \dfrac{1}{2}mv^2 = \dfrac{1}{2}m(A\omega\cos\omega t)^2 = \dfrac{1}{2}mA^2\omega^2\cos^2\omega t$

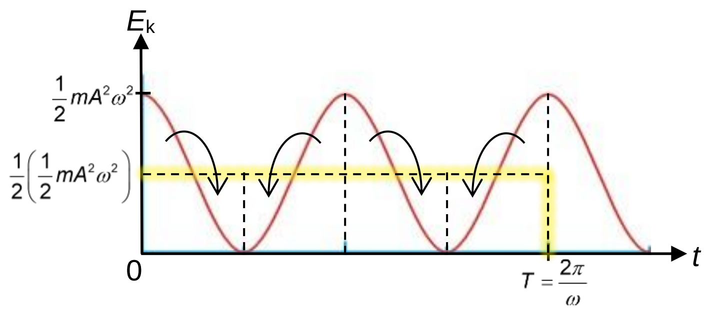
E_k is maximum when $\cos\omega t = 1$

$E_{k,\max} = \dfrac{1}{2}mA^2\omega^2$

period, $T = \dfrac{2\pi}{\omega}$



(iii)



$$\langle E_k \rangle = \frac{\text{area under graph in (ii) over one period or cycle}}{\text{period}}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{2}mA^2\omega^2\right)\times T}{T}$$

$$= \frac{1}{4}mA^2\omega^2 \quad \text{(shown)}$$

$$\langle E_p \rangle = E_{k,\max} - \langle E_k \rangle = \frac{1}{2}mA^2\omega^2 - \frac{1}{4}mA^2\omega^2 = \frac{1}{4}mA^2\omega^2$$

NOTE: The average (mean) value of the kinetic energy is ½ its peak value (since kinetic energy graph is a $\sin^2$ / $\cos^2$ graph and the average of a $\sin^2$ / $\cos^2$ graph is ½).

(b) **(i)**

$$a = -\omega^2 x = -\omega^2\left(A\sin\omega t\right) = -A\omega^2\sin\omega t$$

OR $a = \frac{dv}{dt} = \frac{d\left(A\omega\cos\omega t\right)}{dt} = -A\omega^2\sin\omega t$

$$a^2 = A^2\omega^4\sin^2\omega t$$

$$\langle a^2 \rangle = \frac{1}{2}\left(\text{peak value of } a^2\right) = \frac{1}{2}A^2\omega^4$$

$$\langle P \rangle = \frac{e^2\langle a^2 \rangle}{6\pi\varepsilon_0 c^3} = \frac{e^2\left(\frac{1}{2}A^2\omega^4\right)}{6\pi\varepsilon_0 c^3} = \frac{e^2A^2\omega^4}{12\pi\varepsilon_0 c^3} \quad \text{(shown)}$$

(ii)

$$E_{tot} = E_{k,\max} = \frac{1}{2}mA^2\omega^2 \quad \text{(from (a)(ii))}$$

$$= \frac{1}{2}m_e\left(A_0e^{-\gamma t}\right)^2\omega^2$$

$$= \frac{1}{2}m_eA_0^{\,2}\omega^2e^{-2\gamma t}$$

(iii) Rate of change of E_{tot} with t

$$\frac{dE_{tot}}{dt}=\frac{1}{2}m_eA_0^{\,2}\omega^2\left(-2\gamma e^{-2\gamma t}\right)=-m_eA_0^{\,2}\omega^2\gamma e^{-2\gamma t}$$

$$P=\frac{e^2\left(A_0e^{-\gamma t}\right)^2\omega^4}{12\pi\varepsilon_0c^3}=\frac{e^2A_0^{\,2}\omega^4}{12\pi\varepsilon_0c^3}e^{-2\gamma t}$$

rate at which energy is lost by electron = rate at which energy is radiated

$$\left|\frac{dE_{tot}}{dt}\right|=P$$

$$m_eA_0^{\,2}\omega^2\gamma e^{-2\gamma t}=\frac{e^2A_0^{\,2}\omega^4}{12\pi\varepsilon_0c^3}e^{-2\gamma t}$$

$$\gamma=\frac{e^2\omega^2}{12\pi\varepsilon_0m_ec^3}\qquad\text{(shown)}$$

(iv) According to classical theory, $I\propto A^2$

Since $A=A_0e^{-\gamma t}$,

$$A^2=A_0^{\,2}e^{-2\gamma t}$$

Hence $I=I_0e^{-2\gamma t}$

When $I=\frac{I_0}{2}$,

$$\frac{I_0}{2}=I_0e^{-2\gamma t_{1/2}}$$

$$\ln\frac{1}{2}=-2\gamma t_{1/2}$$

$$\ln 2=2\gamma t_{1/2}$$

$$t_{1/2}=\frac{\ln 2}{2\gamma}$$

$$\gamma=\frac{e^2\omega^2}{12\pi\varepsilon_0m_ec^3}=\frac{e^2\left(2\pi f\right)^2}{12\pi\varepsilon_0m_ec^3}=\frac{e^2\left(2\pi\right)^2\left(c/\lambda\right)^2}{12\pi\varepsilon_0m_ec^3}=\frac{\pi e^2}{3\varepsilon_0m_ec\lambda^2}$$

$$=\frac{\pi\left(1.60\times10^{-19}\right)^2}{3\left(8.85\times10^{-12}\right)\left(9.11\times10^{-31}\right)\left(3.00\times10^{8}\right)\left(500\times10^{-9}\right)^2}$$

$$t_{1/2}=\frac{\ln 2}{2\gamma}$$

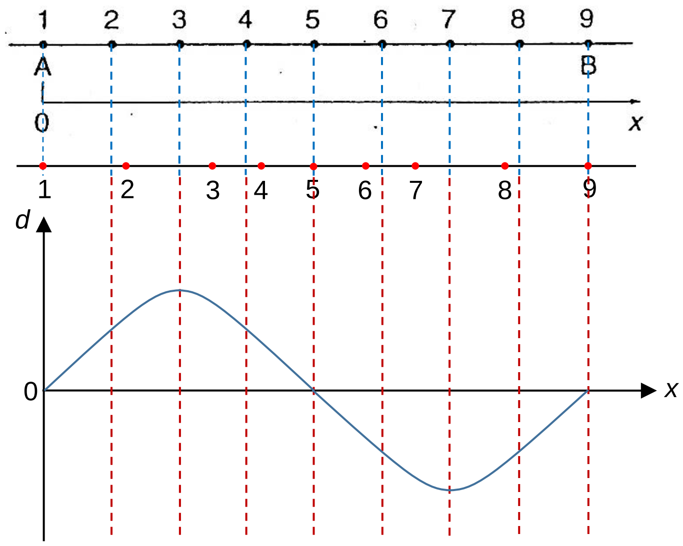
$$=\frac{\ln 2}{2}\left(\frac{3\left(8.85\times10^{-12}\right)\left(9.11\times10^{-31}\right)\left(3.00\times10^{8}\right)\left(500\times10^{-9}\right)^2}{\pi\left(1.60\times10^{-19}\right)^2}\right)$$

$$=7.817\times10^{-9}=7.82\times10^{-9}\text{ s}$$

Wave Motion

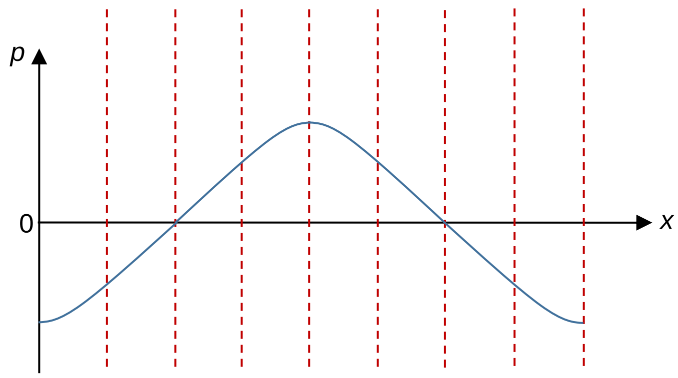
4 **(a)** **(i)** The particles in a longitudinal wave oscillate with simple harmonic motion about their equilibrium positions in a direction parallel to the direction of energy transfer of the wave.

(ii)



(iii) Points of high pressure occur at the regions of compressions. Regions of compressions are where the air particles are closer to one another than when at their equilibrium positions. At the centre of a compression, air particles move towards each other from both sides and cause the air pressure to increase above atmospheric pressure.

Points of low pressure occur at the regions of rarefactions. Regions of rarefactions are where the air particles are further from one another than when at their equilibrium positions. At the centre of a rarefaction, air particles move away from each other on both sides and cause the air pressure to decrease below atmospheric pressure.



p: pressure with respect to atmospheric pressure
The phase difference between the displacement wave and pressure wave is $\frac{\pi}{2}$ radians.

(b) **(i)** The intensity of a wave is defined as the rate of transfer of energy per unit area normal to the direction of energy transfer.

(ii) power, P

$= \text{energy of wave per unit time}$

$= \text{sum of the energies of the paticles within a length } c$

$$= \frac{1}{2}\left(\rho_0 \frac{V}{t}\right) {d_0}^2 \omega^2$$

$$= \frac{1}{2}\rho_0 \left(\text{Area} \times c\right) {d_0}^2 \omega^2$$

Since

$$I = \frac{P}{\text{Area}} = A{d_o}^2$$

$$\frac{\frac{1}{2}\rho_0 \left(\text{Area} \times c\right) {d_0}^2 \omega^2}{\text{Area}} = A{d_o}^2$$

$$A = \frac{1}{2}\rho_0 c \omega^2$$

(c) (i) For constant cross-sectional area, since the rate of loss of energy with respect to x is proportional to the intensity of the wave at the beginning

$$\frac{\delta I}{\delta x} = -kI$$

$$\frac{1}{I}\delta I = -k\delta x$$

Integrating both sides,

$$\int_{I_0}^{I} \frac{1}{I}\, dI = \int_0^x (-k) dx$$

$$\left[\ln I\right]_{I_0}^{I} = -k\left[x\right]_0^x$$

$$\ln I - \ln I_0 = -kx$$

$$\ln\left(\frac{I}{I_0}\right) = -kx$$

$$I = I_0 e^{-kx}$$

(ii) $I = I_0 e^{-kx}$

$\ln\left(\frac{I}{I_0}\right) = -kx$

$k = -\frac{1}{x}\ln\left(\frac{I}{I_0}\right) = -\frac{1}{0.80}\ln\left(\frac{4.2\times10^{-10}}{4.8\times10^{-10}}\right) = 0.1669 = 0.167\ \text{m}^{-1}$

for $I = \frac{1}{2}I_0$,

$\ln\left(\frac{\frac{1}{2}I_0}{I_0}\right) = -kx$

$x = -\frac{1}{k}\ln\left(\frac{1}{2}\right) = -\frac{1}{0.1669}\ln\left(\frac{1}{2}\right) = 4.153 = 4.15\ \text{m}$

5 (a) (i) Let D be the distance between the source of the earthquake and station.

time taken by P-wave, $t_P = \frac{D}{v_P}$

time taken by S-wave, $t_S = \frac{D}{v_S}$

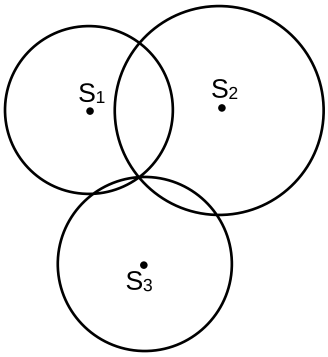
$t_S - t_P = 6.5$

$\frac{D}{v_S} - \frac{D}{v_P} = 6.5$

$D\left(\frac{1}{v_S} - \frac{1}{v_P}\right) = 6.5$

$D = 6.5 \div \left(\frac{1}{5.6} - \frac{1}{8.4}\right) = 109.2 = 109\ \text{km}$

(ii) A minimum of 3 stations is required.
Each station is able to determine the distance of the source from the method in (i). Combining the distance found from each station, the location of the source can be determined accurately as shown in the diagram below.



Knowing the distance of the source of the earthquake from each station S_1, S_2 and S_3, a circle can be drawn around each of the source with the radius of each circle equal to the respective distance from each station. The location of the source is anywhere along the circumference of the circles.

With 2 circles, there will be 2 intersection points which give 2 possible locations of the source. A third circle is required to determine which of these 2 possible locations is correct. The point of intersection of the 3 circles will give the location of the source accurately.

(b) **(i)**

$$M_1 = \lg\left(\frac{A_1}{A_0}\right)$$

$$M_2 = \lg\left(\frac{A_2}{A_0}\right)$$

$$M_2 - M_1 = \lg\left(\frac{A_2}{A_0}\right) - \lg\left(\frac{A_1}{A_0}\right) = \lg\left(\frac{A_2}{A_1}\right)$$

If $A_2 = 10A_1$,

$$M_2 - M_1 = \lg\left(\frac{10A_1}{A_1}\right) = \lg 10 = 1$$

(Shown)

(ii)

$$M = \lg\left(\frac{A}{A_0}\right)$$

$$7.2 = \lg\left(\frac{A}{4.0}\right)$$

$$10^{7.2} = \frac{A}{4.0}$$

$$A = 4.0 \times 10^{7.2} = 6.34 \times 10^7 \text{ units}$$

(iii) From $M = \lg\left(\frac{A}{A_0}\right)$, $M < 0$ (i.e. negative) if $A < A_0$

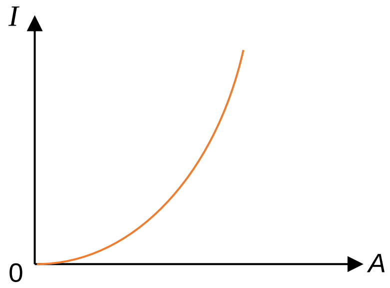
Hence a negative magnitude on the Richter scale means the seismograph output is less than that produced by an earthquake of magnitude just detectable by the instruments in 1935.

6 **(a)** **(i)** The intensity of a progressive wave is defined as the rate of transfer of energy per unit area normal to the direction of energy transfer.

(ii) Intensity I of a wave is proportional to the square of its amplitude A.

$I = kA^2$ where k is the constant of proportionality.

Hence the graph of I against A will be a quadratic curve with a minimum turning point, passing through the origin.



(b) These waves that can be polarised are transverse waves.

Another wave that can be polarised is a simple mechanical wave travelling along a rope.

(c) **(i)** transmitted component of E_0 is $E_0 \cos\theta$

$I \propto E^2 \quad \Rightarrow \quad I = kE^2$ where k is the constant of proportionality

$I_t = kE_t^2$

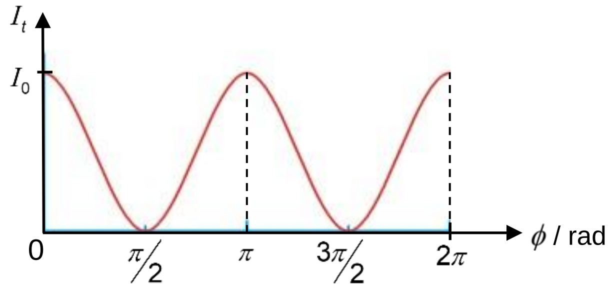
$I_t = k\left(E_0 \cos\theta\right)^2$ ----- (1)

$I_0 = k\left(E_0\right)^2$ ----- (2)

$$\frac{(1)}{(2)} \quad \frac{I_t}{I_0} = \frac{E_0^{\,2}\cos^2\theta}{E_0^{\,2}} = \cos^2\theta$$

$$I_t = I_0 \cos^2\theta$$

(ii) $I_t = I_0 \cos^2 \phi$



(iii) $I_{t_1} = I_0 \cos^2 \alpha$ (intensity after passing through the first two polarisers)

$I_{t_2} = I_{t_1} \cos^2 \alpha = I_0 \cos^2 \alpha \cos^2 \alpha = I_0 \cos^4 \alpha$

$I_{t_3} = I_{t_2} \cos^2 \alpha = I_0 \cos^6 \alpha$

$I_{t_4} = I_{t_3} \cos^2 \alpha = I_0 \cos^8 \alpha$

$\vdots$

$I_{t_N} = I_0 \cos^{2N} \alpha$

$$\begin{aligned}
I_{t_N} &= I_0 \cos^{2N} \alpha \\
&\approx I_0 \left(1 - \frac{1}{2}\alpha^2\right)^{2N} \quad \text{for small } \alpha, \text{ and since } \alpha^2/2 \text{ is small} \\
&\approx I_0 \left(1 - 2N\left(\frac{1}{2}\alpha^2\right)\right) \\
&\approx I_0 \left(1 - 2N\left(\frac{1}{2}\right)\left(\frac{\psi}{N}\right)^2\right)
\end{aligned}$$

$I_t \approx I_0 \left(1 - \frac{\psi^2}{N}\right)$ for (N + 1) polarisers

(Shown)

(iv) For the case in (ii):
when $\phi = \pi/2$ rad, $I_t = 0$

For the case in (iii):

when $\psi = \pi/2$ rad, $I_t \approx I_0 \left(1 - \frac{(\pi/2)^2}{N}\right) \neq 0$ ($I_t \to I_0$ as $N \to \infty$).

When there are only 2 polarisers used and they are crossed perpendicularly with each other, no light is transmitted.

However, if there are more than 2 polarisers whose easy axes do not coincide with that of the first and the last which are perpendicular to each other, light is still transmitted through all the polarisers as light is able to pass through from one polariser to the next, whose axes are not perpendicular.

7 (a)

$$x = x_0 \sin \omega t \quad \Rightarrow \quad \sin \omega t = \frac{x}{x_0}$$

$$v = \frac{dx}{dt} = \omega x_0 \cos \omega t \quad \Rightarrow \quad \cos \omega t = \frac{v}{\omega x_0}$$

$$\sin^2 \omega t + \cos^2 \omega t = 1$$

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{v}{\omega x_0}\right)^2 = 1$$

$$x^2 + \frac{v^2}{\omega^2} = x_0^{\ 2}$$

$$v = \pm\omega\sqrt{x_0^{\ 2} - x^2}$$

Hence, $E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\left(x_0^{\ 2} - x^2\right)$

Resultant force, $F = ma = m\left(-\omega^2 x\right) = -m\omega^2 x$

Hence, $E_p = -\int F dx = \int m\omega^2 x dx = \frac{1}{2}m\omega^2 x^2$

Total energy $= E_k + E_p = \frac{1}{2}m\omega^2\left(x_0^{\ 2} - x^2\right) + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 x_0^{\ 2}$ (shown)

(b) (i) Total energy $= \frac{1}{2}m\omega^2 x_0^{\ 2} = \frac{1}{2}m\left(2\pi f\right)^2 x_0^{\ 2}$

$$2.0\times10^{-20} = \frac{1}{2}\left(1.0\times10^{-25}\right)\left(2\pi\times8.0\times10^{12}\right)^2 x_0^{\ 2}$$

$$x_0 = 1.258\times10^{-11} = 1.3\times10^{-11} \text{ m}$$

(ii) Typical separation of atoms in a solid is about 3×10^{-10} m, which is about 23 times the amplitude of vibration calculated above.
Hence the answer to **(i)** is about 1/23 of the interatomic separation.

(iii) Total vibrational energy $= \left(2.0\times10^{-20}\right)\left(6.02\times10^{23}\right) = 12040 = 1.2\times10^4$ J

(iv) Assume that the total vibrational energy is proportional to the thermodynamic temperature.
Increasing the temperature from 500 K to 501 K, will increase the total vibrational energy by $1/500\times1.2\times10^4 = 24$ J.

(v) Since total vibrational energy $E \propto x_0^{\ 2}$,

$$\frac{\Delta E}{E} = 2\frac{\Delta x_0}{x_0} \quad \Rightarrow \quad \frac{\Delta x_0}{x_0} = \frac{1}{2}\frac{\Delta E}{E} = \frac{1}{2}\left(\frac{1}{500}\right) = \frac{1}{1000}$$

(c) Since total vibrational energy is a constant regardless of mass, let

$\frac{1}{2}m\omega^2 {x_0}^2 = k$ where k is a constant.

Maximum restoring force occurs when acceleration is maximum at the amplitude.

$$F_{max} = ma_{max} = -m\omega^2 x_0$$

$$m\omega^2 x_0 = -F_{max}$$

Since $\frac{1}{2}m\omega^2 {x_0}^2 = k$,

$$\frac{1}{2}(-F_{max})x_0 = k$$

$$x_0 F_{max} = A$$

where $A = -2k$ and is another constant. (shown)

8 **(a)** **(i)** 'a' is the acceleration of the inertial body undergoing simple harmonic motion.
'ω' is the angular frequency of the oscillating body.
'x' is the instantaneous displacement of the body from its equilibrium position.

(ii) on the left hand side:

$$\omega = \frac{2\pi}{T}$$

units of $\omega = \text{s}^{-1}$

on the right hand side:

$k = \frac{F}{e} = \frac{ma}{e}$ where e is the extension of the spring

units of $\sqrt{\frac{k}{m}} = \sqrt{\frac{\text{kg m s}^{-2}\ \text{m}^{-1}}{\text{kg}}} = \text{s}^{-1}$

Since the S.I. base units on the left hand side of the equation are the same as that on the right hand side, the equation is homogeneous.

(b) **(i)** **1** The box alone takes half-a-period to move from the lowest to the highest point.

$$\omega = \sqrt{\frac{k}{M}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{k}{M}}$$

$$\frac{1}{2}T = \pi\sqrt{\frac{M}{k}}$$

2 The distance moved from the lowest to the highest point is twice the amplitude of the oscillation.
The amplitude of the oscillation is equal to the difference in the extension of the spring due to a load of $(M+m)g$ and Mg.

Using Hooke's law:
Extension of spring due to $(M+m)g$ is $(M+m)g/k$
Extension of spring due to Mg alone is Mg/k
Amplitude $=(M+m)g/k-Mg/k=mg/k$
Distance moved $=2mg/k$

3 The plasticine falls from rest under gravity for a time of $\pi\sqrt{\dfrac{M}{k}}$ as obtained in (i)1.

distance fallen, $s=ut+\dfrac{1}{2}at^2=0+\dfrac{1}{2}g\left(\pi\sqrt{\dfrac{M}{k}}\right)^2=\dfrac{1}{2}g\pi^2\dfrac{M}{k}$

4 angular frequency of the combined mass, $\omega=\sqrt{\dfrac{k}{M+m}}$

(ii) Since the plasticine reaches the bottom of the box just in time when the box reaches the top of its motion, the height of the box is the sum of the distance fallen by the plasticine and the distance risen by the box.

$$h=\frac{2mg}{k}+\frac{1}{2}g\pi^2\frac{M}{k}=\frac{4mg+\pi^2Mg}{2k} \text{ (shown)}$$

(iii)
$$v^2=u^2+2as=0+2g\left(\frac{1}{2}g\pi^2\frac{M}{k}\right)$$
$$v=\sqrt{g^2\pi^2\frac{M}{k}}=\pi g\sqrt{\frac{M}{k}} \text{ (shown)}$$

(iv) By the law of conservation of linear momentum,

downward momentum of the box and plasticine immediately after impact = downward momentum of the plasticine just before impact

(momentum of the box just before impact is zero since it is momentarily stationary at the highest point)

$$(M+m)v=m\left(\pi g\sqrt{\frac{M}{k}}\right)$$
$$v=\left(\frac{m}{M+m}\right)\left(\pi g\sqrt{\frac{M}{k}}\right)$$

(v) For the combined mass undergoing S.H.M.: $v = \pm\omega\sqrt{{x_0}^2 - x^2}$ where

$\omega = \sqrt{\dfrac{k}{M+m}}$ from (i)4.

$|v| = \left(\dfrac{m}{M+m}\right)\left(\pi g\sqrt{\dfrac{M}{k}}\right)$ from (iv)

at $x = \dfrac{2mg}{k}$ from (i)2.

Considering the magnitude of $|v| = \omega\sqrt{{x_0}^2 - x^2}$,

$$\left(\frac{m}{M+m}\right)\left(\pi g\sqrt{\frac{M}{k}}\right) = \sqrt{\frac{k}{M+m}}\sqrt{{x_0}^2 - \left(\frac{2mg}{k}\right)^2}$$

$$\left(\frac{m^2}{(M+m)^2}\right)\left(\pi^2 g^2\left(\frac{M}{k}\right)\right) = \frac{k}{M+m}\left({x_0}^2 - \frac{4m^2g^2}{k^2}\right)$$

$$x_0 = \sqrt{\left(\frac{\pi^2 g^2 M m^2}{(M+m)k^2}\right) + \left(\frac{4m^2g^2}{k^2}\right)}$$