Sequences and Series

1. **AJC/2013/I/6**

(a) The first term of an increasing arithmetic progression is 1. S_n is the sum of the first n terms of the arithmetic progression and S_5, S_{10} and S_{20} form a geometric progression. Show that the common difference of the arithmetic progression is 2. [3]

Without the use of a graphing calculator, find the least value of n such that

$$\frac{2S_n}{S_{n+1}-100} > 1.$$ [3]

(b) Mr Wee borrowed \$3400 from an unlicensed money lender on 1st June 2013. The amount he owes the lender at the beginning of each month is twice the outstanding amount at the end of the previous month. From the month of August 2013, Mr Wee decided to repay \$7000 in the middle of each month. Find, in terms of n, the outstanding amount owed at the end of the nth month. Hence find the earliest month when he has fully repaid his loan. [5]

2. **ACJC Prelim/2020/01/Q8**

A hollow circular cylinder of diameter 42 mm is wrapped with n layers of thin film. The first layer of film is in contact with the outer surface of the cylinder and it has length 42π mm. The second layer is in contact with the first layer and has length $(42 + 2x)\pi$ mm, where x mm is the thickness of the thin film. There is no gap between any two layers. The n^{th} layer has length 124π mm.

(i) Show that the thickness of the film satisfies the equation $x(n-1) = 41$. [2]

(ii) Hence find x, given that the total sum of the lengths of the n layers of film is 16766π mm. [3]

Another roll of film which has length 52580 mm is to be cut into n pieces of increasing lengths, where the lengths form a geometric progression with common ratio r. It is given that the sum of the lengths of the first ten pieces of film is k times the sum of the lengths of the first five pieces of film, where k is an unknown constant.

(iii) Show that $k = 1 + r^5$. [2]

(iv) Given that $k = 33$ and that the first piece of film is of length 50 mm, find the largest possible value of n. [3]

3. **TJC/2014/I/7**

A convergent geometric sequence of positive terms, G has first term a and common ratio r.
Write down, in terms of a and r, an expression for the nth odd-numbered term of G. [1]

If the sum of first n odd-numbered terms of G is equal to the sum of all terms of G after the nth odd-numbered term, show that $2r^{2n}+r^{2n-1}-1=0$.

(i) Hence find the value of r when $n=5$. [3]

(ii) In another sequence H, each term is the reciprocal of the corresponding term of G. If the nth term of G and H is denoted by u_n and v_n respectively, show that a new sequence whose nth term is $\ln\left(\dfrac{u_n}{v_n}\right)$, is an arithmetic progression. [4]

4. **CJC/2013/II/1**

(i) A certain machine A is pumping a liquid into an empty container of total volume 850 m^3. The first pumping action fills the container with a volume of 4 m^3. Each subsequent pumping action fills the container with 0.5 m^3 more than the previous pumping action. Pumping continues until the container is completely filled with the liquid. Find the volume of liquid that overflows from the container at the final pumping action. [3]

(ii) Suppose that, after the 50$^{\text{th}}$ pumping action by machine A, a different machine B is used instead to fill the remaining volume of the container. Using machine B, the first pumping action fills the container with a volume of 5 m^3. Each subsequent pumping action fills the container with $\dfrac{5}{6}$ the amount filled in the previous pumping action.

(a) Find the total volume of the container filled after the 10$^{\text{th}}$ pumping action by machine B. [3]

(b) Explain whether the container would be completely filled eventually. [2]

5. **VJC/I/13**

The sum of the first 5 terms of an arithmetic progression, where the common difference is non-zero, is 105. It is further given that the first, third and fourth terms of the arithmetic progression are the first 3 terms of a geometric progression. Show that the common difference of the arithmetic progression is $-\dfrac{21}{2}$. [3]

Deduce that the geometric progression is convergent. [2]

Let S be the sum to infinity of the geometric progression, and A_n be the sum to n terms of the arithmetic progression. Find the least value of n such that the sum of $7S$ and A_n is not positive. [4]

6. **EJC Prelim/2020/02/Q5**

On 1 Jan 2019, Emma takes up a study loan of \$30000 from a bank that offers two repayment plans. Plan A charges a monthly instalment of \$600 at the beginning of every month and increases by \$50 for every subsequent month. There is no interest charged; instead, the bank charges a one-time administrative fee of \$2500 which is to be added to the study loan.

(i) **(a)** Show that, after the n^{th} payment, she would have paid a total amount of \$ $\left(25n^2+575n\right)$. [2]

(b) How many payments will it take for Emma to fully repay her loan? [2]

Plan B charges a fixed monthly instalment of \$$x$ which is to be paid at the beginning of each month, starting from 1 Jan 2019. There is no interest charged for the first month only. Thereafter, interest is charged at the end of each month on the outstanding amount at 1% per month, (i.e. interest will be first charged on 28 February 2019.)

(ii) **(a)** Show that the amount she owes after the fourth payment is $(1.01)^2(30000)-(1.01)^2(2x)-1.01x-x$. [2]

(b) Use the formula for the sum of a geometric progression to find an expression, in terms of n and x, the amount she owes after the n^{th} payment, for $n \geq 2$. [2]

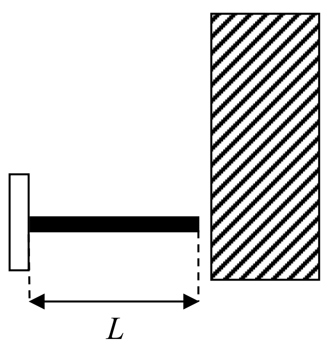
(c) If Emma intends to repay the loan fully after 32 payments, find the least value of x. [2]

(iii) Using the value of x in **part (ii)(c)**, determine which plan will be cheaper for Emma. [1]

7. **NJC/2010/I/8**

(a) The sum, S_n, of the first n terms of an arithmetic progression is given by $S_n = n^2 - 2n$. Write down the expression for $S_n - S_{n-1}$. Hence, find the value of the common difference.

(b) A metal screw of length L (measured in millimetre) is driven into a concrete wall by an electrical screwdriver, such that its distance driven into the wall is proportional to the angle turned by the screwdriver.



Due to some reasons, every subsequent turn by this electrical screwdriver can only achieve an 80% of the angle turned previously.

(i) Given that the initial angle turned by the screwdriver is θ radians, write down the expressions for the first 3 distances driven into the concrete wall, leaving your answers in terms of θ and k, where k is the constant of proportionality.

(ii) Find the total distance driven into the concrete walls after n turns, leaving your answer in the form $ak\theta\left(1-b^n\right)$, where a and b are constants to be determined.

(iii) Given that $k=2$ and assuming that the total distance driven could never exceed the length of the screw, find the minimum length of the metal screw required, giving your answer in terms of θ.

8. **ACJC/2010/I/10**

(a) The first 2 terms of a geometric progression are a and b ($b < a$). If the sum of the first n terms is equal to twice the sum to infinity of the remaining terms, prove that $a^n = 3b^n$. [3]

(b) The terms $u_1, u_2, u_3, \ldots$ form an arithmetic sequence with first term a and having non-zero common difference d.

(i) Given that the sum of the first 10 terms of the sequence is 105 more than $10u_5$, find the common difference. [3]

(ii) If u_{26} is the first term in the sequence which is greater than 542, find the range of values of a. [3]

9. **IJC/2011/I/7**

An arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of a geometric series are equal to the ninth, fifth and second terms respectively of the arithmetic series.

(i) Prove that the geometric series is convergent and find, in terms of a , the sum to infinity. [5]

(ii) Given that $a > 0$, find the largest value of n for which the sum of the first n terms of the geometric series is less than four fifths of the sum to infinity. [3]

10. **ACJC/2011/I/13**

(a) Jerry's brother gave him some game cards for his birthday. He then starts to collect cards for a total period of $3n$ weeks, counting his birthday gift as the first week of collection. Subsequently, the number of cards he buys each week is d more than the number he bought the previous week.

If P is the number of cards collected in the first n weeks and Q the number of cards collected in the last n weeks, show that

$$Q - P = 2n^2 d$$

(b) A fund is established with a single deposit of \$2500 at the beginning of 2011 to provide an annual bursary of \$150. The fund earns interest at 3.5% per annum, paid at the end of each year.

If the first bursary is awarded at the end of 2011 after interest is earned, show that at the end of n years, the amount (in dollars) remaining in the fund is

$$\frac{-12500}{7}(1.035)^n + \frac{30000}{7}.$$

When is the last year that the bursary can be awarded?

11. **NJC/2021/II/2**

An Art teacher teaches her students to create patterns using squares of different sizes. One possible pattern is to begin with the first square with sides of length 2 mm. The first square is inscribed in the second square, where the corners of the first square coincide with the midpoints of the second square. She continues inscribing squares in this manner where the n^{th} square is inscribed in the $(n+1)^{\text{th}}$ square. **Figure 1** shows a piece of artwork after 4 squares are drawn.

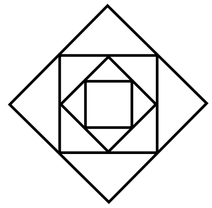


Figure 1

By using this pattern, Student A begins his artwork.

(i) Find, in terms of n, the length of the sides of the n^{th} square. [2]

(ii) A standard A4 paper measures 210 mm by 297 mm. Find the maximum number of complete squares that he can draw on the paper. [2]

Student B uses a giant drawing board and decides to make his artwork more eye-catching. He uses the same pattern and measurements as Student A, but he shades the 1^{st} square and also shades on any protruding areas covered by the 4^{th}, 7^{th} ,…, $(3N+1)^{\text{th}}$ squares, where N is a non-negative integer. A protruding area is defined by the region bounded by the newly drawn square and the square immediately preceding it. **Figure 2** shows a piece of artwork if he draws 4 squares.

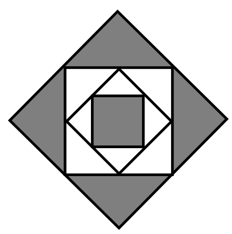


Figure 2

(i) Find, in mm^2, the total shaded area as shown in **Figure 2**. [2]

(ii) Hence or otherwise, find the total shaded area if he draws 30 squares. Give your answer in m^2. [3]

12. **NJC/2012/1/7**

(a) A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of the geometric series are equal to the ninth, fourth and second terms respectively of the arithmetic series.

(i) Show that $d = 3a$. [2]

(ii) Deduce that the geometric series is convergent, and find in terms of a, the sum to infinity. [3]

(b) A mountaineer climbs a mountain of height x metres. The amount of distance he climbs for the first hour is 300 metres. For each subsequent hour, he climbs 10 metres less than the previous hour. The number of whole hours that has passed just before he reaches the summit is n.

(i) Write down an expression for the total distance climbed by the mountaineer and hence show that $pn^2 + qn \le x,$ where p and q are constants to be determined. [2]

(ii) Deduce the value of n if $x = 2500$. [2]

13. **SAJC/2019/II/4(Part of)**

(i) Cauchy's root test states that a series of the form $\sum_{r=0}^{\infty} a_r$ (where $a_r > 0$ for all r) converges when $\lim_{n\to\infty} \sqrt[n]{a_n} < 1,$ and diverges when $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$. When $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, the test is inconclusive. Using the test and given that $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$ for all positive p, explain why the series $\sum_{r=0}^{\infty} \frac{2^r r^x}{3^r}$ converges for all positive values of x. [3]

(ii) By considering $(1-y)^{-2} = 1 + 2y + 3y^2 + 4y^3 + \ldots$, evaluate $\sum_{r=0}^{\infty} \frac{2^r r^x}{3^r}$ for the case when $x = 1$. [2]

14. **DHS/2012/II/4**

A finite sequence $\{a_n\}$ has 50 terms and is such that $a_{n+1} = a_n + 0.15$ for $n = 1, 2, 3, \ldots, 49$.

(i) Given that $a_{50} = 99a_1,$ show that $a_1 = 0.075$. [2]

(ii) Find, without using a calculator, the value of $\sum_{n=1}^{50} a_n$. [2]

Another infinite sequence $\{b_m\}$ is such that $b_1 = a_{50}$ and $\frac{b_m}{b_{m-1}} = 0.98$ for $m \ge 2$.

(iii) Determine the smallest value of k such that $b_k < a_{25}$. [2]

(iv) Find the least value of h such that the sum of the first h terms of $\{b_m\}$ is more than 99% of its sum to infinity. [2]

(v) If $\frac{b_m}{b_{m-1}} = -0.98$ instead, find $\sum_{m=0}^{\infty} b_{1+3m}$. [2]

15. **AJC/2016/II/2**

Anson is closing down his chicken farm and he decides to sell exactly k chickens in the market at the end of every week, where k is a divisor of 1000. Anson has 1000 chickens and the cost of rearing each chicken per week is \$0.50. Show that, when he has sold all his chickens, the total cost of rearing the chickens is \$ $\frac{250}{k}(1000+k)$. [3]

At the end of the first week, Anson sells each chicken at a price of \$12. At the end of every subsequent week, he sells the chickens at a price of 5% less than that of the previous week. Find the smallest value of k in order for Anson to make a profit when he has sold all his chickens. [6]

16. **ACJC/2016/I/3**

(i) Every year Warren Gate's net worth increases by 100% of the previous year. His net worth was estimated to be \$1 000 000 on 31st December 1993. In what year will his fortune first surpass 2.5 billion dollars (1 billion = 10^9)? [3]

(ii) On 1st January 2005, Warren Gates deposits \$100 000 in an investment account and receives an interest of \$1000 on 31st December 2005. After that the amount of interest earned at the end of the year is 1.5 times the amount of interest earned in the previous year. Taking year 2005 as the first year, find the amount of savings that Warren Gates has in his account at the end of the 15th year giving your answer to the nearest integer. [2]

17. **ACJC/2016/II/1**

A triangular region R is drawn on a large sheet of graphing paper marked in 1mm squares. The region R is bounded by the x-axis, the line $y=\frac{1}{20}x$ and the line $y=-\frac{1}{20}x+40$. The scales on both the axes are such that 1 mm represents 1 unit.

By using the table below or otherwise, find the number of complete 1mm squares which lie inside region R.

Range of x	Number of complete 1 mm squares which lies inside region R
$20 \le x \le 40$	
$40 \le x \le 60$	
$60 \le x \le 80$	
⋮	

[3]

18. **CJC/2016/II/4**

Adam and Gregory signed up for a marathon. In preparation for this marathon, Adam and Gregory each planned a 15-week personalised training programme. Adam runs 2.4 km on the first day of Week 1, and on the first day of each subsequent week, the distance covered is increased by 20% of the previous week. Gregory also runs 2.4 km on the first day of Week 1, but on the first day of each subsequent week, the distance covered is increased by d km, where d is a constant. Assume Adam and Gregory only run on the first day of each week.

(i) Find, in terms of d, the total distance covered by Gregory in these 15 weeks. [2]

(ii) Adam targets to cover a total distance of 170 km in these 15 weeks. Can Adam achieve this target? You must show sufficient working to justify your answer. [2]

(iii) It is given that Adam covers a longer distance than Gregory on the first day of the 15th week. Find the maximum value of d, correct to 2 decimal places. Using this value of d, show that the difference in the distance covered by Adam and Gregory for their 15th week training is 0.134 km correct to 3 significant figures. [4]

(iv) Due to unforeseen circumstances, Adam has to end his training programme early. In order for Adam to cover a total distance of 170 km by the end of the 13 weeks, the distance covered has to be increased by x % of the previous week on each subsequent week from Week 1. Find x. [3]

19. **IJC/2016/II/2**

Analysts estimate that when a viral video is posted online, the video attracts comments in such a way that at the end of every hour, the number of comments added for the video is thrice the number of comments at the start of that hour.

In a particular instance, a viral video was posted online and there was one comment immediately after the video was posted. Using the above model proposed by analysts, there will be 3 additional comments by the end of the first hour, 12 additional comments by the end of the second hour, and so on.

(i) Find the number of complete hours for the total number of comments posted online to exceed 200 000. [3]

When the number of these comments posted online reaches 200 000 exactly, Software X is immediately activated to remove the comments. Software X works in such a way that it removes x comments at the start of each day. Once Software X is activated, it is also known that the number of comments at the end of the day is 2% more than the number of comments at the start of the day.

(ii) Show that the number of comments at the end of day n is

$$1.02^n\left(200\,000\right)-51x\left(1.02^n-1\right),$$

where day 1 is the day that the number of comments is exactly 200 000. [3]

(iii) Hence find the range of values of x such that all comments are removed by the end of day 30. Leave your answer to the nearest integer. [2]

Software Y is able to remove comments at the following rate.

- Day 1: 15 000 comments removed
- Subsequent Day: 90% of the number of comments removed on the preceding day

Without using Software X, explain whether Software Y alone is able to remove all 200 000 comments eventually. [2]

20. **TMJC/2021/I/7**

A geometric progression has first term a and common ratio r, and an arithmetic progression has first term b and common difference d, where a, b, d and r are non-zero real numbers. The first, third and eighth term of the geometric progression are equal to the second, third and fifth term of the arithmetic progression respectively.

(i) Show that $r^7 - 3r^2 + 2 = 0$. [2]

(ii) Find the values of r, giving your answer correct to 5 decimal places. Hence explain why the sum to infinity of the geometric progression exists. [2]

It is now given that $a = 12$ and r is the positive value found in part **(ii)**.

(iii) Let E be the sum of the first n even-numbered terms of the arithmetic progression and S_∞ be the sum to infinity of the geometric progression. Find the largest value of n such that the difference between E and S_∞ is less than 1000. [6]

21. **SAJC/2019/I/8**

(a) Meredith owns a set of screwdrivers numbered 1 to 17 in decreasing lengths. The lengths of the screwdrivers form a geometric progression. It is given that the total length of the longest 3 screwdrivers is equal to three times the total length of the 5 shortest screwdrivers. It is also given that the total length of all the odd-numbered screwdrivers is 120 cm. Find the total length of all the screwdrivers, giving your answer correct to 2 decimal places. [4]

(b) Meredith is building a DIY workbench, and she needs to secure several screws by twisting them with a screwdriver drill. Each time Meredith presses the button on the drill, the screw is rotated clockwise by u_n radians, where n is the number of times the button is pressed. Each press rotates the screw more than the previous twist, and on the first press, the screw is rotated by $\frac{2\pi}{3}$ radians.

It is given that $\cos u_{n+1} = \frac{1}{2}\cos u_n - \frac{\sqrt{3}}{2}\sin u_n$ and $\sin u_{n+1} = \frac{1}{2}\sin u_n + \frac{\sqrt{3}}{2}\cos u_n$ for all $n \geq 1$.

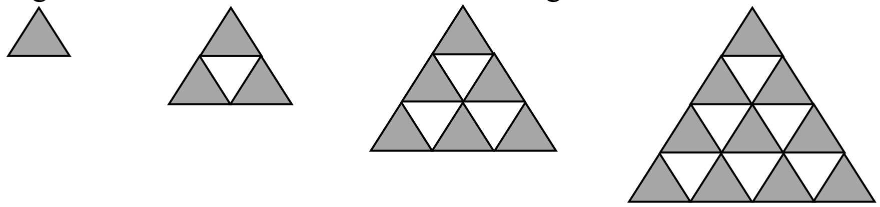
(i) By considering $\cos(u_{n+1} - u_n)$ or otherwise, and assuming that the increase in rotation in successive twists is less than π radians, prove that $\{u_n\}$ is an arithmetic progression with common difference $\frac{\pi}{3}$ radians. [3]

(ii) Each screw requires at least 25 complete revolutions to ensure that it does not fall out. Find the minimum number of times Meredith has to press the drill button to ensure the screw is fixed in place. [3]

(iii) The distance the screw is driven into the workbench on the nth press of the drill, d_n, is proportional to the angle of rotation u_n. If the total distance the screw is driven into the workbench after 21 presses is 144mm, find the distance the screw is driven into the workbench on the first press. [2]

22. **MJC/2016/II/4**

(a) Judith is making a pattern consisting of rows of matchstick triangles as shown. She uses three matchsticks to complete a triangle. She adds two more triangles in the second row, three more triangles in the third row and four more triangles in the fourth row.



Judith has completed $n-1$ rows in the pattern. How many matchsticks does she need in order to form the n^{th} row? [1]

Show that the total number of matchsticks used in making a pattern with n rows is $\dfrac{3n(n+1)}{2}$.

Hence find the maximum number of complete rows she is able to make with two thousand matchsticks. [4]

(b) A geometric progression has first term a and second term b, where a and b are non-zero constants. Given that the sum to infinity of the series is $a+2b$, find the common ratio. [2]
The sum of the first n terms is denoted by G_n. Find G_n in terms of a and n.

Hence show that $\sum_{n=1}^{N} G_n = 2aN - G_N$. [4]

23. **ACJC/2017/I/10**

Abbie and Benny each take a \$50 000 study loan for their 3-year undergraduate program, disbursed on the first day of the program. The terms of the loan are such that during the 3-year period of their studies, interest is charged at 0.1% of the outstanding amount at the end of each month. Upon graduation, interest is charged at 0.375% of the outstanding amount at the end of each month.

(a) Since the interest rate is lower during her studies, Abbie decides that she will make a constant payment at the beginning of each month from the start of the program for its entire duration.

(i) Find the amount, correct to the nearest cent, Abbie needs to pay at the beginning of each month so that the outstanding amount after interest is charged remains at \$50 000 at the end of every month. [2]

(ii) After graduating, Abbie intends to increase her payment to a constant \$$k$ at the beginning of every month. Show that the outstanding amount Abbie owes the bank at the end of n months after graduation, and after interest is charged, is

$$\$\left[1.00375^n(50000) - \frac{803}{3}k\left(1.00375^n - 1\right)\right].$$ [2]

(iii) Abbie plans to repay her loan within 10 years after graduation. Determine if she can do this with a monthly instalment of \$500, justifying your answer. [1]
Find the amount she needs to pay so that she fully repays her loan at the end of exactly 10 years after graduation, leaving your answer to the nearest cent. [2]

(b) Benny wishes to begin his loan repayment only after graduation. Like Abbie, he aims to repay the loan at the end of exactly 10 years after graduation. Leaving your answer to the nearest cent, find

(i) the constant amount Benny needs to pay each month in order to do this, [3]

(ii) the amount of interest Benny pays altogether. [2]

24. **JJC/2018/2/2**

In 2004, 10 000 cases of obesity among Singaporeans aged 18-30 years old were reported. Each year after that, the number of cases reported increased by 7%. If this pattern were to continue, how many obesity cases would be reported in 2018? Leave your answer to the nearest whole number. [3]

John, who was obese, started on a weight-loss programme. The number of calories he burned in the first week of his exercise regime was a. As the intensity of the exercise regime increased, the number of calories John burned each week was increased by d. On the other hand, the number of calories John consumed each week is a geometric sequence such that the numbers of calories he consumed in the first, second and third week equal the numbers of calories he burned through exercising in the seventh, third and first week respectively.

(i) Show that $d = \frac{a}{2}$. [2]

(ii) John burned 3000 calories in the seventh week through exercising. Find the least number of weeks required for the total number of calories John burned to exceed the total number of calories he consumed by at least 200 000. [5]

25. **CJC/2018/2/1(Modified)**

(i) The sum of the first n terms of a sequence is given by $S_n = n(2n+7)$.

Show that $u_n = 4n+5$ and prove that the sequence is an Arithmetic Progression. [2]

It is given that $\sum_{n=1}^{N} \frac{1}{\sqrt{u_{n+1}} + \sqrt{u_n}} = \frac{1}{4}\left(\sqrt{4N+9} - 3\right)$, where u_n is the nth term of the arithmetic series in part **(i)**.

(ii) Hence find the exact value of $\frac{1}{\sqrt{53}+\sqrt{49}} + \frac{1}{\sqrt{57}+\sqrt{53}} + \cdots + \frac{1}{\sqrt{361}+\sqrt{357}}$. [2]

26. **CJC/2022/1/11**

A new coating is applied to TX tennis balls to improve its elasticity. To examine its effect, a TX tennis ball is dropped vertically from a fixed height H cm onto the floor. The height reached after each bounce decreased by 5 cm from the height reached by the previous bounce.

Find, in terms of H and n,

(i) the height reached by the tennis ball after the $(\boldsymbol{n-1})^{\text{th}}$ bounce, [1]

(ii) the total distance travelled by the tennis ball just before it rebounds off the floor for the n^{th} bounce. [5]

(iii) State an assumption used in your calculation for part **(ii).** [1]

The time interval between any successive bounces is also measured. Given that the time interval between the first and second bounce is 1.2 seconds, and the time interval between subsequent consecutive bounces is 0.75 times of the time interval of the previous bounce, find

(iv) the least integer m such that it takes less than 0.02 seconds between the m^{th} and $(m+1)^{\text{th}}$ bounce, [3]

(v) total time taken before the tennis ball comes to a stop given that it takes 2 seconds for the tennis ball to first hit the ground. [2]

27. **EJC/2022/2/3**

The sum S_n of the first n terms of a sequence u_1, u_2, u_3, … is given by

$$S_n = \ln\left(\frac{e^n}{3^{n^2}}\right).$$

(a) Show that $u_n = 1+(1-2n)\ln 3$. [2]

(b) Hence, show that the sequence is an arithmetic progression. [2]

(c) Find the sum of the first ten odd-numbered terms. [2]

(d) A geometric sequence $e^{u_1}, e^{u_2}, e^{u_3}, \ldots$ has a common ratio, r. Find the value of r. [2]

(e) Find the least value of n for which the sum of the first n terms of this geometric sequence is within 10^{-8} of its sum to infinity. [3]

28. **ACJC Promo 9758/2024/Q8**

(a) It is given that $\displaystyle\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{n}{2n+1}$.

(i) Explain why the series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{4r^2-1}$ converges and write down the value to which it converges. [2]

(ii) Find $\displaystyle\sum_{r=6}^{N}\frac{1}{(2r+1)(2r+3)}$ in terms of N, express your answer in a single fraction. [3]

(b) The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$, $u_{n+1} = 1-\dfrac{1}{u_n}$, $n \geq 1$.

Find the value of u_2, u_3 and u_4. Hence find the value of $\displaystyle\sum_{r=1}^{50}u_r$. [4]

29. **EJC Promo 9758/2024/Q11**

A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 2$, $u_{n+1} = ku_n - 3$ for all $n \geq 1$, where k is a constant.

(a) Let $k = \dfrac{1}{5}$.

(i) Write down the values of u_2, u_3 and u_4. [2]

(ii) Given that the sequence converges to l, find the value of l. [1]

(b) Let $k = 1$.

(i) Given that the sequence $u_1, u_2, u_3, \ldots$ is an arithmetic progression, find the value of $\displaystyle\sum_{r=1}^{n}u_{r+4}$. [3]

(ii) Show that the sequence $u_7, u_{10}, u_{13}, u_{16}, \ldots, u_{3n+4}, \ldots$ is also an arithmetic progression. [2]

(iii) Hence, find the value of $\displaystyle\sum_{r=1}^{100}u_{3r+4}$. [2]

30. **NYJC Promo 9758/2024/Q4**

During a clinical trial, the concentration U, of a specific blood agent is measured at one-hour intervals following the initial administration of the trial drug to a patient.
The following readings were obtained

$$U_3 = 88,\ U_4 = 76 \text{ and } U_5 = 70,$$

where U_t denotes the reading t hours after the drug was first administered.
It is believed that U satisfies the relationship

$$U_{t+1} = p + qU_t,\ t \geq 0,$$

where p and q are constants.

(a) Find the values of p and q. [2]

(b) Determine the initial concentration of the blood agent when the drug was initially administered. [3]

(c) Describe how the concentration behaves over a long period of time. [1]

31. **RI Promo 9758/2024/Q5**

(a) Find $\sum_{r=0}^{n}[(n+2)r+n]$, giving your answer in terms of n. [3]

[You may use the result $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for the rest of this question.]

(b) By writing down the first two and the last two terms in the series, find $\sum_{r=1}^{n}(r+2)^3$, giving your answer in terms of n. [3]

(c) Find $1^3 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + \ldots + (2n-1)^3 - (2n)^3$, giving your answer in terms of n. [3]

32. **CJC Prelim 9758/2024/01/Q4**

(a) A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = -1$ and $u_r = u_{r-1} - 0.5^r + r^3$, where $r \geq 2$. By considering $\sum_{r=2}^{n}(u_r - u_{r-1})$, find an expression for u_n in terms of n.

[It is given that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.] [5]

The divergence test states that for a sequence $a_1, a_2, a_3, \ldots$, if $\lim_{n\to\infty} a_n = c$, where c is a non-zero constant or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{r=1}^{\infty} a_r$ diverges.

(b) Determine whether $\sum_{n=1}^{\infty} u_n$ diverges, justifying your answer. [2]

33. **HCI Prelim 9758/2024/02/Q2**

A sequence is defined by the recurrence relation

$$u_{n+1} = \frac{1+u_n}{1-u_n},$$

for $n \geq 1$.

(a) State what happens to the sequence when $u_1 = 0$. [1]

It is now given that $u_1 = 2$.

(b) Find u_2, u_3, u_4, u_5 and u_6 . [2]

(c) By observing the pattern in part **(b)**, find $\sum_{r=1}^{4n} u_r$ in terms of *n*. [2]

34. **DHS Prelim 9758/2025/P1/Q4**

A sequence of real numbers x_1, x_2, x_3, … satisfies the recurrence relation

$$x_{n+1} = \frac{5x_n + 2}{2x_n + 3} \quad \text{for all } n \geq 1.$$

(a) Given that the sequence converges to *l*, find the possible exact values of *l*. [3]

(b) Describe how the sequence behaves when $x_1 = 3$. [1]

(c) Given that $x_5 = \frac{3503}{2158}$, find the value of x_1. [3]

35. **MI Prelim 9758/2025/P2/Q4**

A sequence is defined by $T_{n+2} = T_{n+1} + T_n$ with $T_1 = 3$, $T_2 = 1$ for $n \geq 1$. Another sequence is defined by $r_n = \frac{T_{n+1}}{T_n}$ for $n \geq 1$.

(a) Find T_4. [1]

(b) Show that $r_{n+1} = 1 + \frac{1}{r_n}$ for all for $n \geq 1$. [2]

(c) It is given that r_n converges to a limit *L*. Find the exact value of *L*. [3]

A third sequence u_n for $n \geq 1$, is an arithmetic sequence with first term 5 and common difference 2.

(d) Find the least value of *n* such that $\sum_{k=1}^{n} u_k > 930$. [2]

36. **VJC Prelim 9758/2025/P1/Q6**

It is given that $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)} = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n}$.

(a) Show that $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)}$ is less than $\frac{1}{4}$. [2]

(b) Give a reason why the series $\sum_{r=2}^{\infty} \frac{1}{r\left(r^2-1\right)}$ converges, and write down its value. [2]

(c) Find the smallest value of n for which $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)}$ differs from $\sum_{r=2}^{\infty} \frac{1}{r\left(r^2-1\right)}$ by less than 0007.

[2]

(d) Find $\sum_{r=m+1}^{N} \frac{1}{(r-m)(r-m+1)(r-m+2)}$, where m and N are integers with $N > m > 0$.

(There is no need to express your answer as a single algebraic fraction.) [2]

37. **ACJC Promo 9758/2025/Q3**

Given that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$.

(a) Show that $\sum_{r=2}^{n} \frac{1}{r(r-1)} = 1 - \frac{1}{n}$. [2]

(b) Find $\sum_{r=2}^{\infty} \left(\frac{1}{2^r} + \frac{1}{r(r-1)} \right)$. [3]

38. **RI Promo 9758/2025/Q9**

(a) It is given that $\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$.

(i) State the sum to infinity of the series. [1]

(ii) Find the sum of the first n terms of the series

$$\frac{2}{17 \times 18 \times 19} + \frac{2}{18 \times 19 \times 20} + \frac{2}{19 \times 20 \times 21} + \cdots.$$ [2]

(iii) Find $\sum_{r=2}^{n} \frac{1}{r\left(r^2-1\right)}$. [3]

(b) The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 3$, $u_{n+1} = 2 - \frac{4}{u_n}$, $n \geq 1$.

(i) Find the values of u_2, u_3, u_4 and u_{100}. [2]

(ii) Hence find $\sum_{r=1}^{3n-2} u_r$ in terms of n, simplifying your answer. [2]

Answers

1(a) 10 (b) $n = 8$; Jan 2014

2(ii) $x = 0.204$, (iv) $n = 10$

(b)(i) $S_n = \frac{n\ln 2}{2}\left[(n+1)b+4\right]$ (ii) $-0.194 \le b \le -0.0913$

3 $U_{2n-1} = ar^{2n-2}$ (i) 0.892

4(i) 21 (ii)(a)838

5 Least n =16

6(i)(b)27 (ii)(b) $30000(1.01)^{n-2} - 2x(1.01)^{n-2} - 100x\left(1.01^{n-2} - 1\right)$

(ii)(c) Least value of x is \$1078.84 (iii) Plan B

7(a) $d = 2$ (b)(i) $d = k\theta$; $0.8k\theta$; $0.64k\theta$ (ii) $a = 5$, $b = \frac{4}{5}$ (iii) 10θ

8(b)(i) $d = 21$ (ii) $17 < a \le 38$

9(i) $8a$ (ii) Largest value of n is 5

10(b) 2035

11(i) $2^{\frac{n+1}{2}}$ (ii) 14 (iii) 20 (iv) 307

12(a)(ii) $\frac{125}{3}a$ (b)(i) $p = -5,\ q = 305$ (ii) $n = 9$

13(ii) 6

14(ii) 187.5 (iii) 36 (iv) 228 (v) 3.82

15 $k = 40$

16(i) 2005 (ii) \$ 973 788

17 7600

18(i) $36 + 105d$ (ii) Yes (iii) 2.02 (iv) 25.4%

19(i) 9 (iii) $x \ge 8755$; unable

20(ii) $r = 1$ or $r = 0.93725$ or $r = -0.77986$ (5 d.p.) (iii) largest $n = 28$

21(a) r = 0.882854, total length = 222.38 cm, (b)(i) $\frac{\pi}{3}$(proof), (ii) 16, (iii) $\frac{8}{7}$ mm

22(a) $3n$; 36 (b) $\frac{1}{2}$; $2a\left(1-\left(\frac{1}{2}\right)^n\right)$

23(a)(i) \$49.95 (iii) \$516.26 (b)(i) \$535.17 (ii) \$14220.43

24 25785 obesity cases (ii) 32

25(iii) 3

26(i) $H-(n-1)(5)$ (ii) $H+2(n-1)H-5n(n-1)$ (iv) 16 (v) 6.8s

27(c) $10-190\ln 3$ (d) $r = \frac{1}{9}$ (e) $n = 9$

28(ai) $\frac{1}{2}$ (aii) $\frac{N-5}{13(2N+3)}$ (b) 26.5

29(a)(i) $u_2 = -2.6, u_3 = -3.52, u_4 = -3.704$ (a)(ii) -3.75

(b)(i) $-\frac{n(3n+17)}{2}$ (b)(iii) -46150

30(a) $p = 32, q = 0.5$ (b) 256

31(a) $\frac{n}{2}(n+1)(n+4)$ (b) $\frac{1}{4}(n+2)^2(n+3)^2-9$ (c) $-n^2(4n+3)$

32(a) $u_n=\frac{n^2(n+1)^2}{4}-2.5+0.5^n$ for $n\geq 1$

33(b) $u_2=-3$, $u_3=-\frac{1}{2}$, $u_4=\frac{1}{3}$, $u_5=2$, $u_6=-3$ (c) $-\frac{7}{6}n$

34(a) $l=\frac{1+\sqrt{5}}{2}$ or $l=\frac{1-\sqrt{5}}{2}$ (c) $x_1=2.9$

35(a) $T_4=5$ (c) $L=\frac{1+\sqrt{5}}{2}$ (d) 29

36(b)¼ (c) 27 (d) $\frac{1}{4}+\frac{1}{2(N-m+2)}-\frac{1}{2(N-m+1)}$

37(b) 1.5

38(a)(i) $\frac{1}{2}$ (ii) $\frac{1}{306}-\frac{1}{(n+17)(n+18)}$ (iii) $\frac{1}{4}-\frac{1}{2n(n+1)}$ (b)(i) $\frac{2}{3},-4,-3$ and 3 (ii) $\frac{10-n}{3}$