Hwa Chong Institution 华侨中学
Mathematics (SMTP)

Name: Wan Run Hao (27) Class: 3I1 Date: 9th July

MA302.5.X1 – Square Matrices

A *square* matrix is a matrix whereby it is of the order $n \times n$, i.e. it has equal number of rows and columns.

An *identity* matrix (usually denoted by **I**) is a square matrix whereby the entries along the diagonal of the matrix, from the top left to the bottom right, are made up of '1's, while the rest are filled up with '0's.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are examples of identity matrices.

X1. Show that for any $n \times n$ identity matrix **I**, and a $n \times n$ matrix **A**, where n is 2 or 3,

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}.$$

Let $n=2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ (shown)}$$

Let $n=3$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ (shown)}$$

This is in fact true for any positive integer n.

S1. Show that $\mathbf{I}^2 = \mathbf{I}$ where $\mathbf{I}$ is a 2×2 square matrix. Show also that $\mathbf{I}^3 = \mathbf{I}$. Can you make a conjecture?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ (shown)}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ (shown).}$$

$I^n = I$, where $n \in \mathbb{Z}^+$ ✓

Determinant and inverse matrix

S2. Given 2×2 square matrices $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, show that $\mathbf{AB} = \mathbf{BA} = k\mathbf{I}$, where the scalar multiple k can be expressed in terms of a, b, c and d and $\mathbf{I}$ is the 2×2 identity matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= (ad-bc)(I) \text{ (shown)}$$ ✓

k is known as the determinant of matrix $\mathbf{A}$ (or $\mathbf{B}$). It is commonly denoted by det($\mathbf{A}$) or $|\mathbf{A}|$ or simply $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Note that the sign $|\cdot|$ ***does not*** represent the absolute value.

If $k = 1$, the matrix $\mathbf{B}$ is known as the inverse of matrix $\mathbf{A}$. Similarly, matrix $\mathbf{A}$ is the inverse of matrix $\mathbf{B}$. The inverse matrix is usually denoted as $\mathbf{A}^{-1}$ (for matrix $\mathbf{A}$).

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = kI$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\frac{1}{k}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right] = I$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{k}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

S3. If $k = -1$, what would be the inverse matrix for $\mathbf{A}$?

$$A^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

$A^{-1} = -B$, when $k = -1$. ✓

S4. Show that the determinant of $\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$ is 1. Hence, write down the inverse matrix $\mathbf{A}^{-1}$ of **A**. Check that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

$\det(A) = 4\times2 - 7\times1$

$= 1$ (shown)

$A^{-1} = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}$

$AA^{-1} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^{-1}A = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$AA^{-1} = A^{-1}A = I$

S5. Suppose $\mathbf{AB} = k\mathbf{I}$ where $k \neq 1$. By dividing the matrix equation by *k*, show that the inverse of **A** is given by $\frac{1}{k}\mathbf{B}$. Similarly, write down the inverse of **B**.

$AB = kI$

$A\frac{B}{k} = I$

$\frac{1}{k}B = A^{-1}$ (shown)

$AB = kI$

$\frac{A}{k}B = I$

$\frac{1}{k}A = B^{-1}$ (shown)

$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

From above, it is clear that a matrix **A** has an inverse if and only if its determinant is non-zero. In this case **A** is said to be **non-singular**. Conversely, **A** is singular if its determinant is zero.

We thus have the following

Given a non-singular 2×2 matrix A such that $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse matrix $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

S6. Find the inverse matrices of the following wherever possible.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.4 & -0.6 \\ 0.2 & 0.2 \end{pmatrix}$$

(b) $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & 9 \end{pmatrix}$

$$= \frac{1}{\det(B)} \begin{pmatrix} 9 & -1 \\ -2 & 0 \end{pmatrix}$$

$$= -\frac{1}{2} \begin{pmatrix} 9 & -1 \\ -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4.5 & 0.5 \\ 1 & 0 \end{pmatrix}$$

(c) $\mathbf{C} = \begin{pmatrix} -2 & 4 \\ -3 & 6 \end{pmatrix}$

$$= \frac{1}{\det(C)} \begin{pmatrix} 6 & -4 \\ 3 & -2 \end{pmatrix}$$

$$= \frac{1}{0} \begin{pmatrix} 6 & -4 \\ 3 & -2 \end{pmatrix}$$

= No solution as C is not non-singular.

What do you notice about the rows of matrix **C**? How does that help you determine whether a 2×2 matrix has an inverse?

They are multiples of each other.

$R_1 = \frac{2}{3} R_2$

$\begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$ where $k = \frac{3}{2}$

Determinant $= kab - kab$

$= 0$

The rows, when multiples of each other, results in no inverse matrix.

S7. Show that, for a 2×2 matrix, if the rows/columns are multiples of each other, the matrix is singular, i.e. it has no inverse.

Case 1: rows are multiples

$$A = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$$

$$\det(A) = kab - kab$$
$$= 0 \text{ (shown)}$$

Case 2: columns are multiplies

$$B = \begin{pmatrix} a & ka \\ b & kb \end{pmatrix}$$

$$\det(B) = kab - kab$$
$$= 0 \text{ (shown)}$$

S8. If **A** is non-singular, make **B** the subject in the following matrix equation $\mathbf{AB} = \mathbf{C}$. Justify your working.

$$AB = C$$
$$A^{-1}AB = A^{-1}C$$
$$IB = A^{-1}C$$
$$B = A^{-1}C$$

$$AB = CD$$
$$ABB^{-1} = CDB^{-1}$$
$$A = CDB^{-1}$$
$$AE = CDB^{-1}E$$
$$AEB = CDB^{-1}EB$$
$$\neq CED$$

S9. Let $\mathbf{X} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 5 & 0 \\ -2 & 4 \end{pmatrix}$.

(a) Find $\mathbf{X}^{-1}$, the inverse of $\mathbf{X}$.

$$X^{-1} = \frac{1}{\det(X)} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} 0 & 0.5 \\ -1 & 1.5 \end{pmatrix}$$

(b) Given that $\mathbf{XP} = \mathbf{Y}$, find the matrix $\mathbf{P}$.

$$XP = Y$$

$$X^{-1}XP = X^{-1}Y$$

$$P = X^{-1}Y$$

$$= \begin{pmatrix} 0 & 0.5 \\ -1 & 1.5 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ -8 & 6 \end{pmatrix}$$