RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 4: Functions

SYLLABUS INCLUDES

- concepts of function, domain and range;
- inverse functions and composite functions;
- conditions for the existence of inverse functions and composite functions;
- domain restriction to obtain an inverse function;
- relationship between graphs of a one-to-one function and its inverse.

PRE-REQUISITES

- Basic exponential, logarithmic and trigonometric expressions
- Algebraic manipulations involving exponential, logarithmic and trigonometric expressions.
- Solving inequalities

CONTENT

INTRODUCTION

One of the most important themes in calculus is the analysis of relationships between physical and mathematical quantities. These relationships can be represented in terms of formulae, graphs or numerical data. In this chapter, we will develop the concept of a function, which is a platform for almost all mathematical and physical relationships.

1 Relations and Functions

1.1 Relations

A **relation** from a given set A to a given set B is a rule for connecting elements in A (inputs) to elements in B (outputs). Set A is known as the **domain** and set B the **codomain** of the relation.

Consider $A=\{0,1,2\}$ and $B=\{0,1,2,3\}$. Let's say that an element x in A is related (or mapped) to an element y in B if and only if x is less than y. This relation connects the elements in set A to the elements in set B (e.g. 1 is related to 2 and 3 since 1 is less than 2 and 3). We can illustrate this relation from set A to set B by using an arrow diagram as shown in Fig 1.1.

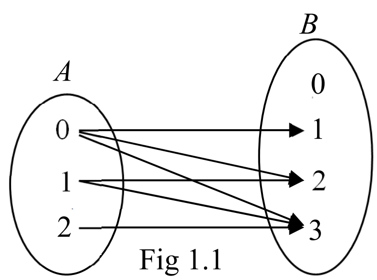


Fig 1.1

There are countless examples of relations in our daily life. Can you formulate some of them?

Note that there are many different types of relations. For example, we can have a **one to one relation** as shown in Fig. 1.2, a **many to one relation** as shown in Fig. 1.3 and a **one to many relation** as shown in Fig. 1.4.

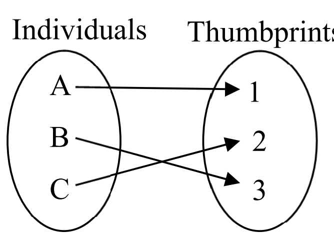


Fig 1.2

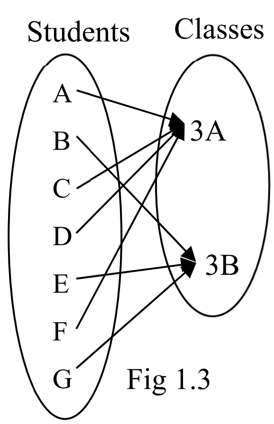


Fig 1.3

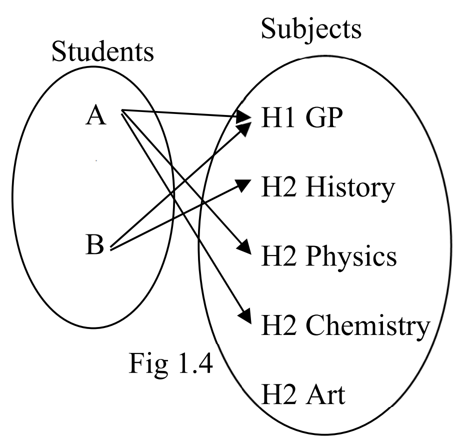


Fig 1.4

The domain of the relation shown in Fig. 1.1 is $\{0,1,2\}$. Can you identify the domains for the examples in Fig 1.2, 1.3 and 1.4?

1.2 Functions

A **function** f: $X \to Y$ is a relation which maps **each** element in the set X, to **one and only one** element in the codomain Y.

The set X is called the domain of f, denoted by D_f, and the set Y is called the codomain of f.

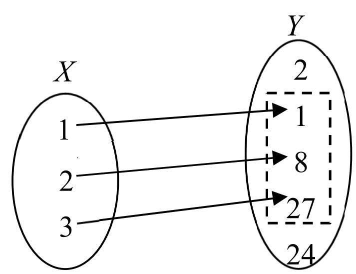
Based on the definition of a function, which of the relations shown in Fig. 1.1 – 1.4 are functions?

Notes:

(1) We usually use lower case letters such as f or g to name a function.

(2) If the element $a \in X$ is mapped to the element $b \in Y$, then we write $f(a) = b$.
b is called the image of a under f.

The range of f is the set of images of X under f. i.e. Range of f $= \{f(x) : x \in X\}$.



In this case, $X = \{1,2,3\}$ is the domain and $Y = \{1,2,8,24,27\}$ is the codomain.

Range of f $= \{1,8,27\}$.

* Range of f $\subseteq$ Codomain of f.

(3) Two functions are **equal** if and only if they have the **same rule and domain**.

1.3 Representation of a Function

At 'A' Levels, most functions considered are relations between real numbers where the rule can be stated algebraically. It is not necessary to state the codomain in this context, and it suffices to represent a function by defining its domain and rule. In this case, we take range of f as the codomain of f. Eg. we write

(1) $f : x \mapsto x^2 + x$, where $x \in \mathbb{R}^+$,

(2) $f(x) = x^2 + x$, where $x \in \mathbb{R}^+$.

1.4 Existence of a Function

A relation f is more often shown as a graph on which the points $(x,\ f(x))$ are plotted. The equation of the resulting curve can be written in the form $y = f(x)$.

Graphically, to decide whether a relation is a function, any *vertical line* $x = k,\ k \in D_f$ drawn should cut the graph of f at **one and only one** point.

1.5 Graphical Method to determine Range of a Function

Consider a function f with domain denoted by D_f.

The range of f, denoted by R_f, is the set of images $=\{\mathrm{f}(x): x \in X\}$.

One of the more commonly used methods for finding the range of a function is by sketching its graph with equation $y = \mathrm{f}(x)$, where $x \in \mathrm{D}_\mathrm{f}$. When sketching graphs, one should take note of characteristics such as symmetry, intersections with the axes, turning points and asymptotes.

A graphing calculator is a useful tool in sketching graphs. However, it has certain limitations. For example, it cannot draw vertical asymptotes, and it also does not know what critical features of a graph to display. It is thus useful to be acquainted with the equations of basic functions and understand the properties of their graphs.

Example 1

(a) Sketch the graphs of the following functions and state its range:

(i) $\mathrm{f}: x \mapsto x^2 - 8x,\ x \in \mathbb{R},\ 1 \le x \le 10,$

(ii) $\mathrm{g}: x \mapsto |\ln x|,\ x \in \mathbb{R}^+,$

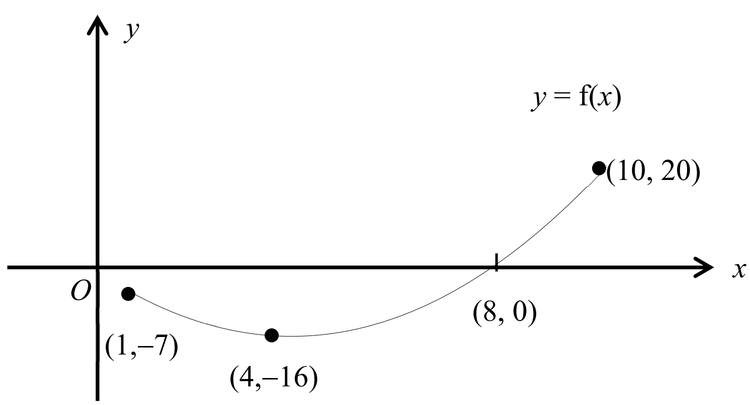
(iii) $\mathrm{h}: x \mapsto 1 + \mathrm{e}^x,\ x \in \mathbb{R},\ x < 2,$

(iv) $\mathrm{k}(x) = \begin{cases} x^2, & x \in \mathbb{R},\ x \le 0, \\ x - 1, & x \in \mathbb{R},\ x > 0. \end{cases}$

(b) A curve C has equation $y^2 = 2x$, $x \in \mathbb{R}_0^+$. By considering the graph of C, show that C does not represent a function.

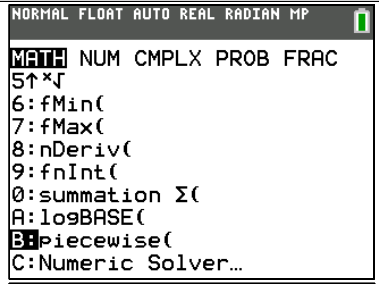
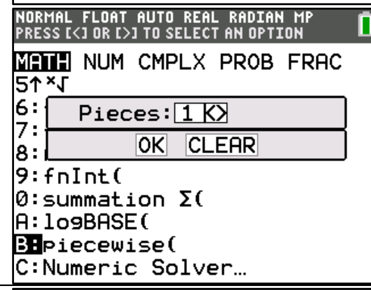
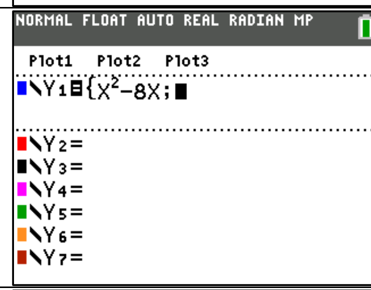
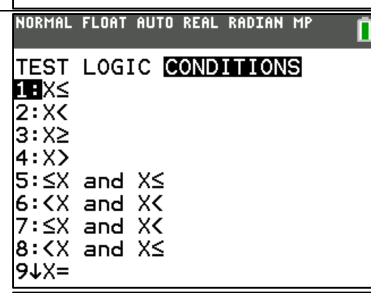
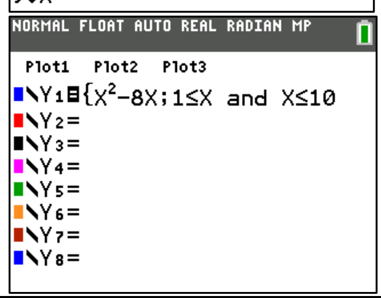
Solution :

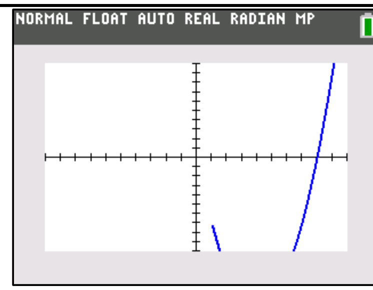
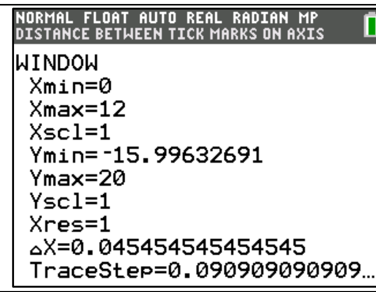
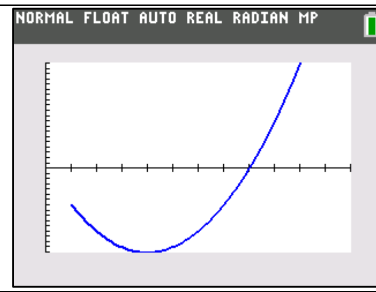
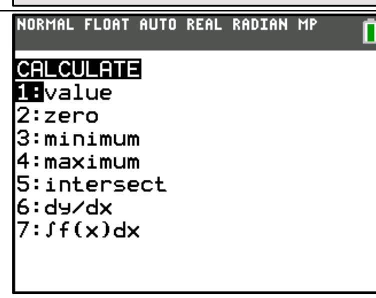
(a)(i) $\mathrm{f}: x \mapsto x^2 - 8x,\ x \in \mathbb{R},\ 1 \le x \le 10,$



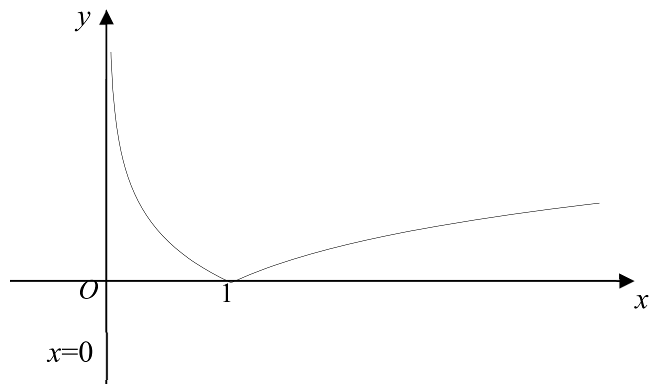
$\mathrm{f}(1) = -7$, $\mathrm{f}(4) = -16$, $\mathrm{f}(10) = 20$

$\mathrm{R}_\mathrm{f} = [-16, 20]$

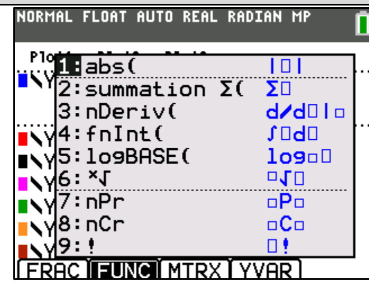
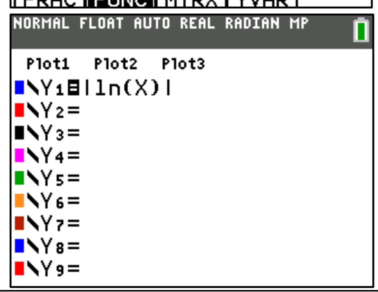
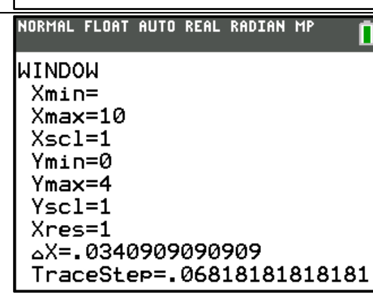
GC Keystrokes	***Screenshots***
1. Press [y=] 2. In order to draw the graph based on the given domain, use the piecewise function in the MATH Menu by pressing [math] [▲] [▲]. 3. Press [enter] to select **B:piecewise(** and [◄] until Pieces show only "1". Press [▼] [enter] to select **OK**. **<u>Note:</u>** The "piecewise" function is only available for GC with updated OS of 5.3.0.0037.	 
4. Press [X,T,θ,n] [x²] [−] [8] [X,T,θ,n] for $y = x^2 - 8x$. Press [►] to go to the box for entering the domain.	
5. Press [1] for the lower limit of the domain 6. Press [2nd] [math] [test] [►] [►] to access the CONDITIONS Menu. 7. Press [5] to select **5: ≤ X and X ≤ .** Press [1] [0] for the upper limit of the domain. Press [enter] to confirm the entry.	 

8. Press $\boxed{\text{graph}}$ to draw the graph. The graph is drawn based on **ZStandard** but it does not give the complete graph (see figure on right). The “bottom” part seems to be missing. **ZStandard** graphs the function in the standard viewing window, where x and y values are between -10 and 10 inclusive.	
9. To adjust the windows settings to a suitable settings (like in a graph paper), press $\boxed{\text{window}}$. Set Xmin = 0 and Xmax = 12 as shown (based on the domain given).	
10. Press $\boxed{\text{zoom}}$ $\boxed{0}$ to select **0:ZoomFit** to fit all part of the graph given the range of values of x (based on Xmin and Xmax above)	
11. To calculate intercepts and stationary points, press $\boxed{\text{2nd}}$ $\boxed{\text{trace}}$[calc]. Select **1: value** (to calculate y-intercept) **2: zero** (to calculate x-intercept) **3: minimum** (to calculate minimum value of an interval) **4: maximum** (to calculate maximum value of an interval)	

(ii) $\mathrm{g}: x \mapsto |\ln x|,\ x \in \mathbb{R}^+$,

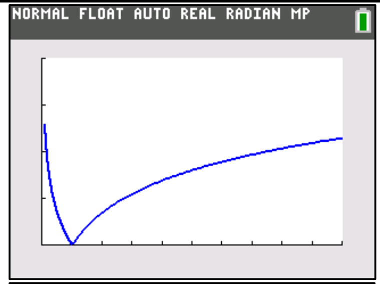


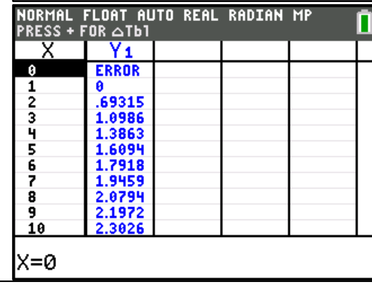
$\mathrm{R_g} = [0, \infty)$

GC Keystrokes	*Screenshots*
1. Press $\boxed{\text{y=}}$ 2. Press $\boxed{\text{alpha}}\boxed{\text{window}}$[f2] to access the shortcut Menu with commonly used functions. 3. Press $\boxed{\text{enter}}$ to select **1:abs(** and $\boxed{\text{ln}}\boxed{X,T,\theta,n}\boxed{)}\boxed{\text{enter}}$ for entry of the function.	
4. To view the graph, as $\|\ln(x)\| \geq 0$ for $x \in \mathbb{R}^+$, you may change the windows settings to the one on the right:	

To check whether the graph of g has a y-intercept, we can refer to the table of values by pressing [2nd] [graph] [table].

As g(0) is not defined, it suggests that $x = 0$ is a vertical asymptote. This is further confirmed by our knowledge of the graph of $y = \ln x$, which has a vertical asymptote $x = 0$.

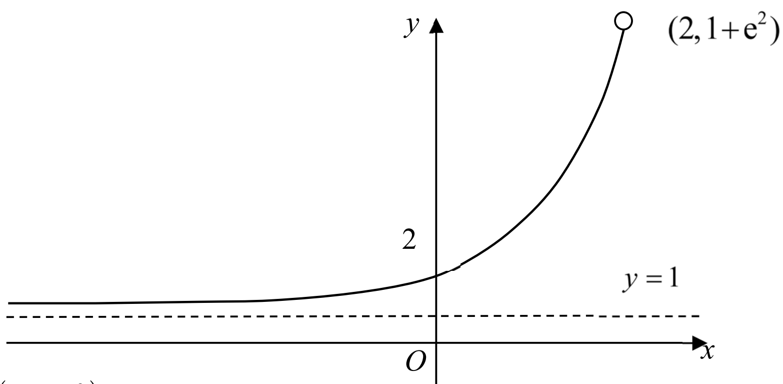



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

X	Y1
0	ERROR
1	0
2	.69315
3	1.0986
4	1.3863
5	1.6094
6	1.7918
7	1.9459
8	2.0794
9	2.1972
10	2.3026

X=0

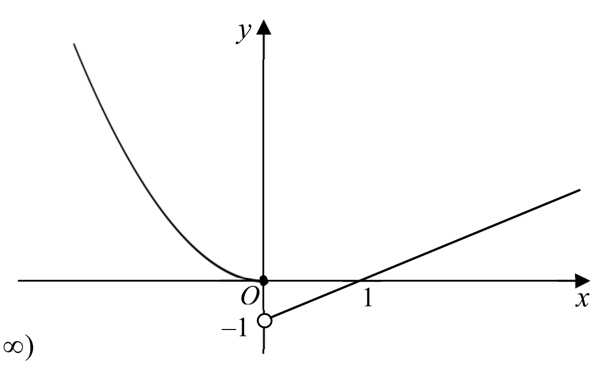
(iii) $\mathrm{h}: x \mapsto 1+\mathrm{e}^{x}$, $x \in \mathbb{R}$, $x < 2$,



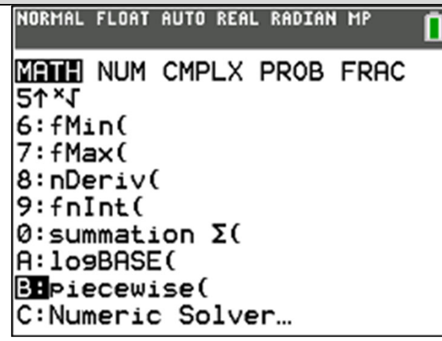
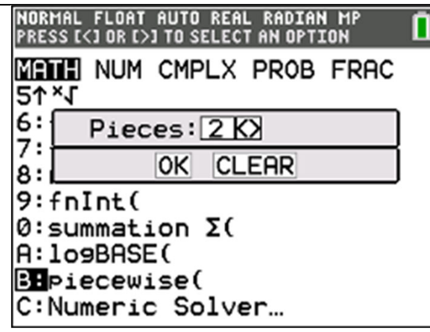
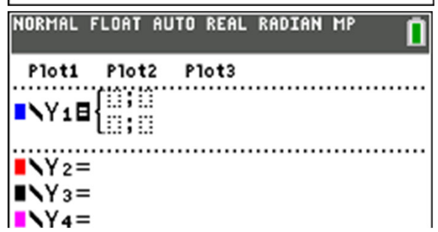
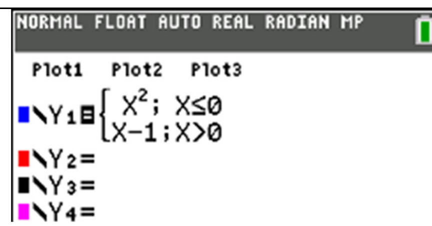
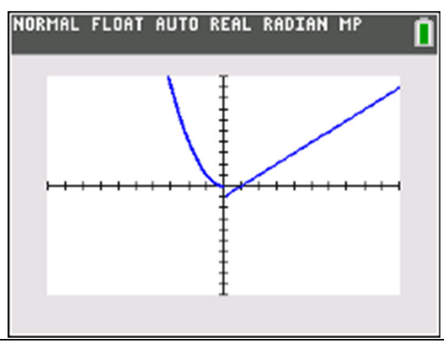
$\mathrm{R_h} = \left(1, 1+\mathrm{e}^2\right)$

(iv) $\mathrm{k}(x) = \begin{cases} x^2, & x \in \mathbb{R},\ x \le 0, \\ x-1, & x \in \mathbb{R},\ x > 0. \end{cases}$

For GC without the updated OS, we can sketch the graph by keying in $Y_1 = x^2(x \le 0)$ and $Y_2 = (x-1)(x > 0)$.

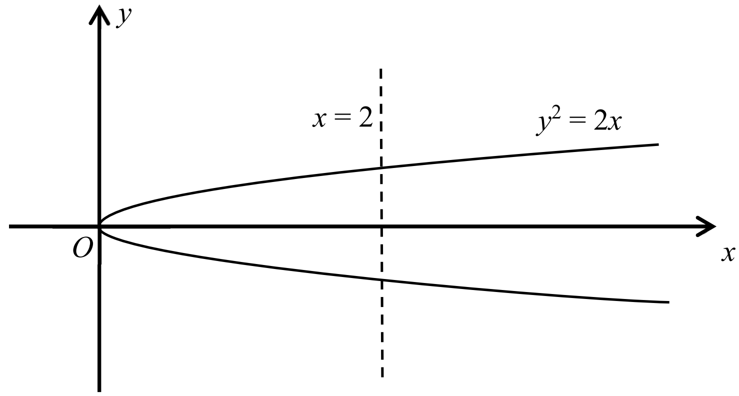


$\mathrm{R_k} = (-1, \infty)$

GC Keystrokes	*Screenshots*
1. Press $\boxed{y=}$. 2. At "Y_1=", press $\boxed{\text{math}}$ and scroll up to select **B: piecewise (** . **<u>Note:</u>** The "piecewise" function is only available for GC with updated OS of 5.3.0.0037.	
3. Use the left or right arrow keys to select the required number of pieces.	 
4. Key in the required equations and the corresponding domains in the appropriate spaces. Once done, press $\boxed{\text{graph}}$ to graph the function.	 

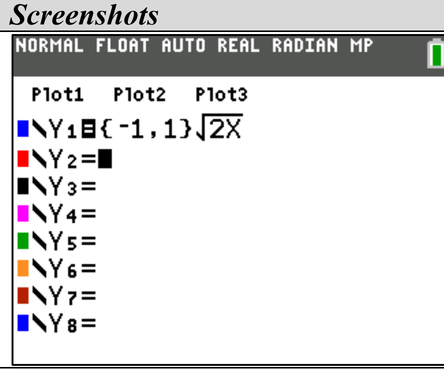
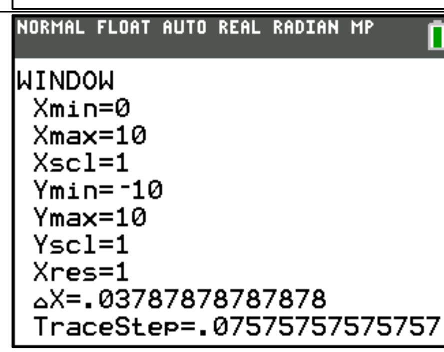
(b) $y^2 = 2x \Rightarrow y = \pm\sqrt{2x}$

Solution:



Since the vertical line $x = 2$, where $2 \in \mathbb{R}_0^+$, does not cut C at one and only one point, so C does not represent a function.

OR: Since there exists $2 \in \mathbb{R}_0^+$ such that $y^2 = 2(2) \Rightarrow y = \pm 2$, C does not represent a function.

GC Keystrokes	***Screenshots***
1. Press [y=] 2. At "Y_1=", press [2nd] [(] [{] [(-)] [1] [,] [1] [2nd] [)] [}] [2nd] [x²] [√] [2] [X,T,θ,n] [enter] for $y = \pm\sqrt{2x}$.	
3. Press [window] to adjust window settings. Since $x \geq 0$, set Xmin = 0. If the domain of f is $\mathbb{R}$, we usually press [zoom] . Select **6: ZStandard** which gives a standard window of $-10 \leq x \leq 10,\ -10 \leq y \leq 10$.	

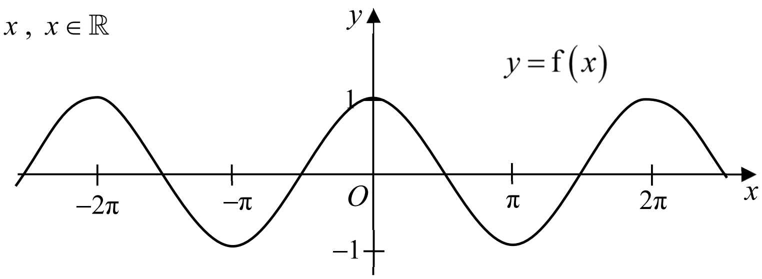
We will now look at a few types of functions.

1.6 Even and Odd Functions

An **even** function f is one such that $\mathrm{f}(x) = \mathrm{f}(-x)$ for all values of x in its domain.

The graph of such a function is therefore symmetrical about the y-axis, i.e. the graph remains the same upon reflection in the y-axis.

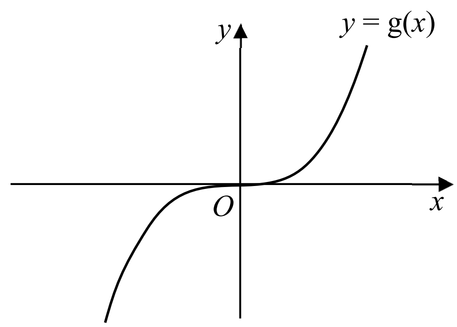
Eg. $\mathrm{f} : x \mapsto \cos x$, $x \in \mathbb{R}$



An **odd** function f is one such that $\mathrm{f}(-x) = -\mathrm{f}(x)$ for all values of x in its domain.

The graph of such a function is therefore symmetrical about the origin,
i.e. the graph remains the same upon a $180°$ rotation about the origin.

Eg. $\mathrm{g} : x \mapsto x^3$, $x \in \mathbb{R}$



1.7 One-One Functions

A function f is said to be **one-one** if no two elements in the given domain have the same image (output).

i.e. for $x_1, x_2 \in \mathrm{D}_\mathrm{f}$, if $x_1 \neq x_2$, then $\mathrm{f}(x_1) \neq \mathrm{f}(x_2)$.

Equivalently, if $\mathrm{f}(x_1) = \mathrm{f}(x_2)$, then $x_1 = x_2$.

Graphically, f is one-one if and only if every horizontal line $y = k$, $k \in \mathrm{R}_\mathrm{f}$, cuts the graph off at **one and only one** point.

Example 2

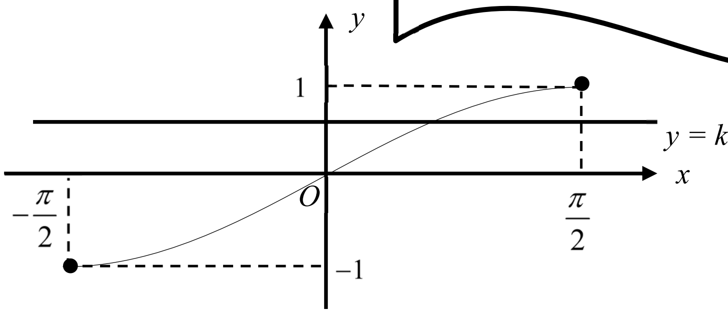
Sketch the graphs of the following functions and determine whether they are one-one.

(a) $\mathrm{f}: x \mapsto \sin x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, **(b)** $\mathrm{g}: x \mapsto 2\mathrm{e}^{-x^2}, x \in \mathbb{R}$

Solution

(a) $\mathrm{f}: x \mapsto \sin x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

GC Tips:
1) Check that you are in radian mode.
2) It's good to use "ZTrig" for sketching trigo curves.

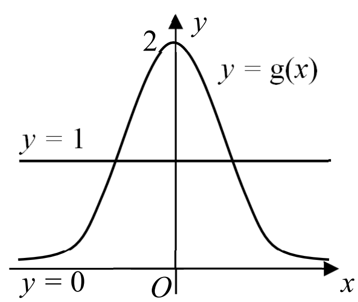


$\mathrm{R_f} = [-1, 1]$

Since every horizontal line $y = k$, $k \in [-1, 1]$, cuts the graph of f at one and only one point, f is a one-one function.

(b) $\mathrm{g}: x \mapsto 2\mathrm{e}^{-x^2}, x \in \mathbb{R}$

It can be verified that g is an even function since $\mathrm{g}(-x) = 2\mathrm{e}^{-(-x)^2} = 2\mathrm{e}^{-x^2} = \mathrm{g}(x)$ for all $x \in \mathbb{R}$, so the graph of g is symmetrical about the y-axis.



$\mathrm{R_g} = (0, 2]$.

Since the horizontal line $y = 1$ cuts the graph of g at more than one point, g is not a one-one function.

Alternatively, to show that a function g is *not* one-one, we can provide a **counterexample**, i.e. find two *different* elements in the domain of g that gives the *same* image.

In the above example, since $\mathrm{g}(1) = \mathrm{g}(-1) = 2\mathrm{e}^{-1}$, g is not a one-one function.

2 Inverse Functions

Let $\mathrm{f}: X \rightarrow Y$ be a function whose arrow diagram is as shown in Fig 2.1. Reversing the arrows, a new function g having domain Y and range X is obtained as seen in Fig. 2.2.

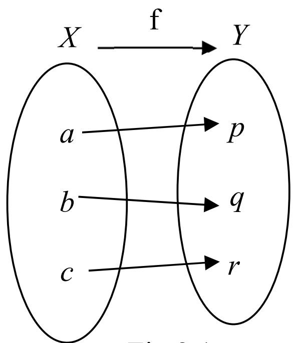


Fig 2.1

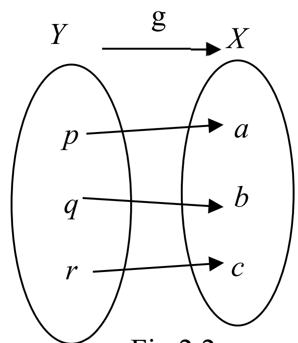


Fig 2.2

This new function g is called the inverse function of f and is denoted by f^{-1}. Note that if the function f maps a to p, then its inverse f^{-1} maps p to a. i.e. if $\mathrm{f}(a) = p$, then $\mathrm{f}^{-1}(p) = a$. In general, if $y = \mathrm{f}(x)$, then $x = \mathrm{f}^{-1}(y)$.

However, not every function has an inverse which is a function. Consider the function $\mathrm{h}: A \rightarrow B$ whose arrow diagram is shown in Fig 2.3. When the arrows in Fig 2.3 are reversed, we see from Fig. 2.4 that $\mathrm{k}: B \rightarrow A$ does not exist as a function as there are elements in set B which have two images. Hence the inverse function for h does not exist.

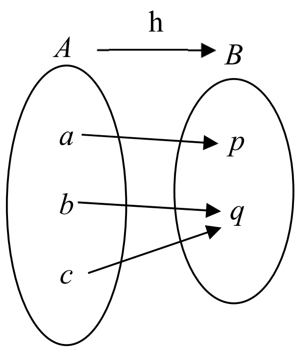


Fig 2.3

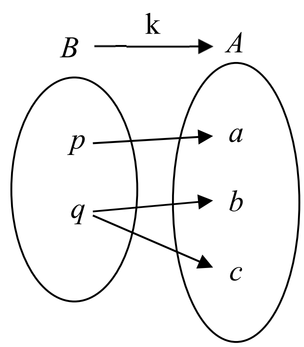


Fig 2.4

2.1 Condition for the Existence of an Inverse Function

Recall that a function f is a relation which maps each element in the domain to a unique image. However, we see from Fig. 2.3 that for the inverse function to exist, the function cannot have elements which are mapped to the same unique image. i.e. the function f has to be one-one.

Thus the **inverse function** of f exists if and only if f is **one-one**.

2.2 Definition and Range of an Inverse Function

Let f be a one-one function. Then f has an inverse function f^{-1} defined by

$$\boxed{\mathrm{f}^{-1}(y) = x \iff y = \mathrm{f}(x) \text{ for } x \in \mathrm{D_f}.}$$

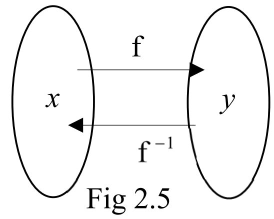


Fig 2.5

E

What about the domain and range of f^{-1}?

From Fig 2.5, it is easy to see that

> **domain of** f^{-1} = range of f, i.e. $\mathrm{D_{f^{-1}}} = \mathrm{R_f}$
>
> and
>
> **range of** f^{-1} = domain of f, i.e. $\mathrm{R_{f^{-1}}} = \mathrm{D_f}$

Note that $\mathrm{f}^{-1}(x) \neq \dfrac{1}{\mathrm{f}(x)}$

We can obtain the rule of f^{-1} as follows:

(i)	Let $y = \mathrm{f}(x)$
(ii)	Make x the subject of the formula in terms of y so that $x = \mathrm{g}(y)$.
(iii)	From **(i)** and **(ii)**, we have $x = \mathrm{f}^{-1}(y) = \mathrm{g}(y)$ Replacing y by x in $\mathrm{f}^{-1}(y) = \mathrm{g}(y)$, we obtain $\mathrm{f}^{-1}(x) = \mathrm{g}(x)$.

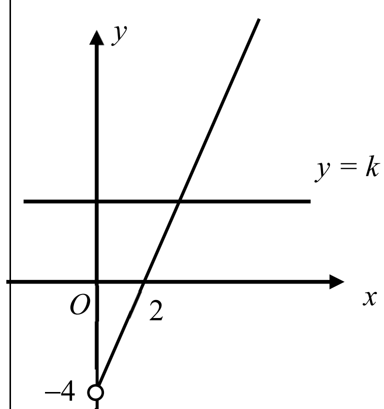
Example 3

(a) Show that the inverse function of f, where $\mathrm{f}(x) = 2x - 4, \quad x \in \mathbb{R}^+$, exists. Find $\mathrm{f}^{-1}(x)$ and state the domain.

(b) Given that $\mathrm{g}: x \mapsto 1 + \mathrm{e}^{-x}, \quad x \in \mathbb{R}$, define its inverse function in similar form and state its range.

Solution

(a) $\mathrm{f}(x) = 2x - 4, \quad x \in \mathbb{R}^+$



$\mathrm{R_f} = (-4, \infty)$

Since every horizontal line $y = k$, $k \in (-4, \infty)$, cuts the graph of f at one and only one point, f is a one-one function.

Hence, the inverse function of f exists.

Let $y = 2x - 4$

$\Rightarrow x = \dfrac{1}{2}(y + 4)$

$\therefore \mathrm{f}^{-1}(x) = \dfrac{1}{2}(x + 4)$ with $\mathrm{D_{f^{-1}}} = (-4, \infty)$

(b) Let $y=1+\mathrm{e}^{-x}$

$$\mathrm{e}^{-x}=y-1$$
$$-x=\ln(y-1)$$
$$x=-\ln(y-1)$$
$$\mathrm{g}^{-1}(x)=-\ln(x-1)$$
$$\mathrm{D}_{\mathrm{g}^{-1}}=\mathrm{R}_{\mathrm{g}}=(1,\infty)$$

Thus, $\mathrm{g}^{-1}:x\mapsto-\ln(x-1), \quad x>1,$

$\mathrm{R}_{\mathrm{g}^{-1}}=\mathrm{D}_{\mathrm{g}}=\mathbb{R}$

Note: You need not verify that the inverse function exists if it is not required by the question.

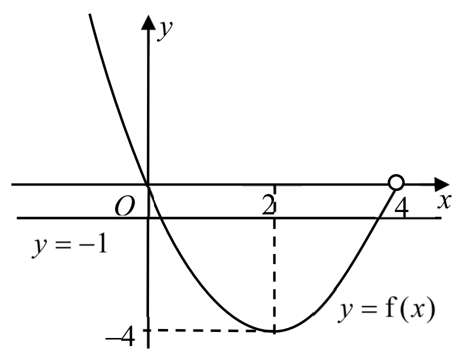
Example 4

Show that the inverse of the function $\mathrm{f}:x\mapsto x^2-4x, \quad x<4$ does not exist. State the greatest value of k for which the function $\mathrm{g}:x\mapsto x^2-4x, \quad x\le k$ is one-one. Using this value of k, obtain the inverse function of g in a similar form.

Solution

$\mathrm{f}(x)=x^2-4x=(x-2)^2-4$

Since the horizontal line $y=-1$ cuts the graph of f at more than one point, f is not one-one and hence f^{-1} does not exist.



For g to be one-one, greatest value of k is 2.

Let $y=(x-2)^2-4, x\le 2$

$$x=2\pm\sqrt{y+4}$$
$$x=2-\sqrt{y+4} \quad (\text{since } x\le 2)$$

$\mathrm{D}_{\mathrm{g}^{-1}}=\mathrm{R}_{\mathrm{g}}=[-4,\infty)$

Thus, $\mathrm{g}^{-1}:x\mapsto 2-\sqrt{x+4}, \ x\ge -4$.

2.3 Graphical relationship between a Function and its Inverse

Fig. 2.6 shows the graph of $y = \mathrm{g}(x)$ where $\mathrm{g}(x) = 1 + \mathrm{e}^{-x}$, $x \in \mathbb{R}$. From Example 3(b), we found that the inverse function of g is given by $\mathrm{g}^{-1}(x) = -\ln(x-1)$, $x > 1$. By sketching the graph of $y = \mathrm{g}^{-1}(x)$ on the same diagram (using the **same scale** for both the x- and y -axis), what is the graphical relationship that you observe between the two graphs?

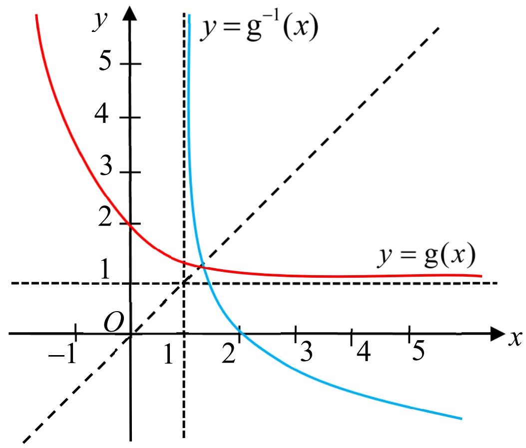


Fig. 2.6

From Fig. 2.6, we observe the following:

- $\mathrm{g}(0) = 2$ so $(0, 2)$ is found on the curve $y = \mathrm{g}(x)$.
- $\mathrm{g}^{-1}(2) = 0$ so $(2, 0)$ is found on the curve $y = \mathrm{g}^{-1}(x)$.
- $(0, 2)$ and $(2, 0)$ are points that are reflections of each other in the line $y = x$.

In general, if (a, b) is a point on the curve $y = \mathrm{g}(x)$, then (b, a) is on the curve $y = \mathrm{g}^{-1}(x)$ since $\mathrm{g}(a) = b \Leftrightarrow \mathrm{g}^{-1}(b) = a$.

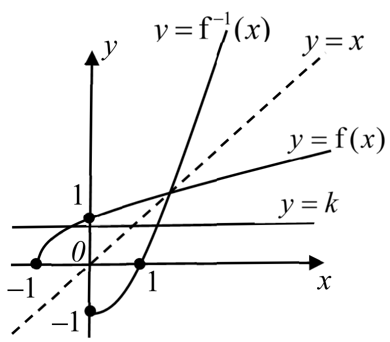
Graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$ are **reflections** of each other **in the line** $y = x$**.**

Example 5

The function f is defined as $\mathrm{f}(x)=\sqrt{x+1}, \quad x \geq -1$. Show that the inverse function of f exists. Sketch the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ on the same diagram, showing clearly the graphical relationship between the two graphs.

Hence find the exact value of x which satisfies the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ and deduce the solution set to $\mathrm{f}(x) \leq \mathrm{f}^{-1}(x)$.

Solution



Since every horizontal line $y=k,\ k \geq 0$ cuts the graph of f at one and only one point, f is a one-one function. Hence, the inverse function of f exists.

Note: The graph of f^{-1} can be obtained by reflecting the graph of f in the line $y = x$.

To solve $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, we can consider solving $\mathrm{f}(x)=x$ as the point of intersection of the graphs of f and f^{-1} also lies on the line $y = x$.

Thus $\quad \mathrm{f}(x)=\mathrm{f}^{-1}(x) \Rightarrow \sqrt{x+1} = x \Rightarrow x+1 = x^2 \Rightarrow x^2 - x - 1 = 0$

$$\Rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}.$$

But $x \geq 0 \Rightarrow x = \dfrac{1+\sqrt{5}}{2}$.

The solution set of to $\mathrm{f}(x) \leq \mathrm{f}^{-1}(x)$ is $\left[\dfrac{1+\sqrt{5}}{2}, \infty\right)$.

3 Composite Functions

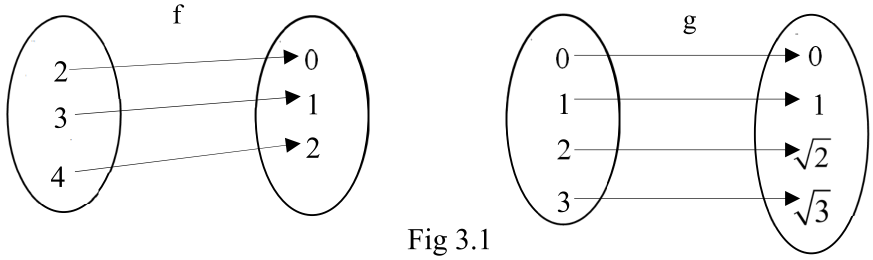


Fig 3.1

Consider the functions f and g given by $\mathrm{f}(x)=x-2,\ x\in\{2, 3, 4\}$ and $\mathrm{g}(x)=\sqrt{x}$, $x\in\{0, 1, 2, 3\}$ and whose arrow diagrams are as shown in Fig 3.1.

In this case, it is quite clear that the outputs of the function f can be used as inputs of the function g. For example, if we input $x=4$ into f, the output under the function f is 2 and this value can be used as an input for g to give a final output value of $\sqrt{2}$. This idea is illustrated in the arrow diagram shown in Fig 3.2.

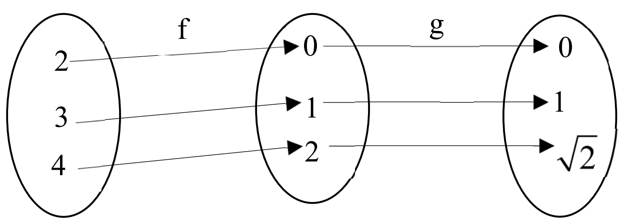


Fig 3.2

In general, we observe that for any $x\in \mathrm{D}_{\mathrm{f}}$, $x\xrightarrow{\mathrm{f}}x-2\xrightarrow{\mathrm{g}}\sqrt{x-2}$.

In fact, there is a direct mapping $x\mapsto\sqrt{x-2}$ which will map an input of f directly to an output value of g. The new function formed in the way described above is known as "g composed with f", and is denoted by gf (or $\mathrm{g}\circ\mathrm{f}$). It maps the elements in the domain of f directly to the elements in the range of g.

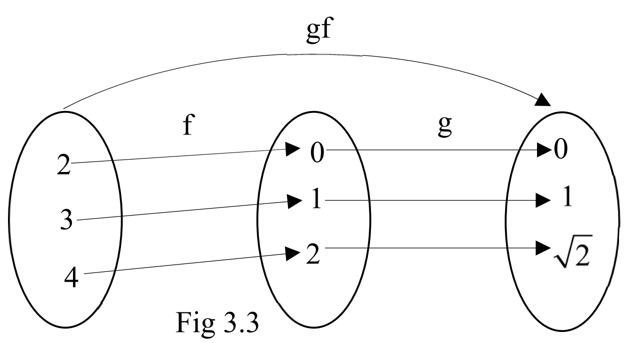


Fig 3.3

We write the rule for this composite function as $\mathrm{gf}(x)=\sqrt{x-2}$.

3.1 Condition for the Existence of a Composite Function

For arbitrary functions f and g, not all ‘outputs of f’ can be used as ‘inputs for g’ as not all images of f may be elements in the domain of g.

For example, consider the functions f and g given by

$$\mathrm{f}(x)=x-2,\quad x\in\mathbb{R} \quad \text{and} \quad \mathrm{g}(x)=\sqrt{x},\quad x\in\mathbb{R},\ x\geq 0.$$

Clearly, $\mathrm{f}(0)=-2$ but $-2\notin \mathrm{D_g}$ and hence there is no 'final' output.
i.e. the element 0 has no image under gf and the composite function gf does not exist.
Thus, for the composite function gf to exist as a function; each $x\in \mathrm{D_f}$ must have a unique image $\mathrm{f}(x)$ and for each image $\mathrm{f}(x)$, it has to have one and only one image under the rule of g. It is thus necessary for the range of f to be a subset of the domain of g.

Thus, for any given functions f and g, • the composite function gf exists when **range of f ⊆ domain of g**, i.e. $\mathrm{R_f}\subseteq \mathrm{D_g}$ • $\mathrm{D_{gf}}=\mathrm{D_f}$

3.2 Definition of a Composite Function

Let f and g be two functions such that range of f ⊆ domain of g.
Then the composite function gf is defined by

$$\boxed{\mathrm{gf}(x)=\mathrm{g}(\mathrm{f}(x)) \text{ for all } x\in\mathrm{D_f}.}$$

Note:

(1) In general, $\mathrm{fg}\neq\mathrm{gf}$. i.e. composition of functions is not commutative.
Also, even if gf and fg both exist, their domains may be different since $\mathrm{D_{gf}}=\mathrm{D_f}$ while $\mathrm{D_{fg}}=\mathrm{D_g}$.

(2) The composite function ff, if it exists, is written as f^2,
i.e. $\mathrm{f}^2(x)=\mathrm{ff}(x)=\mathrm{f}\left(\mathrm{f}(x)\right)$.
In general, we have $\mathrm{f}^n(x)=\mathrm{ff}^{n-1}(x)=\mathrm{f}\left(\mathrm{f}^{n-1}(x)\right)$.

Example 6

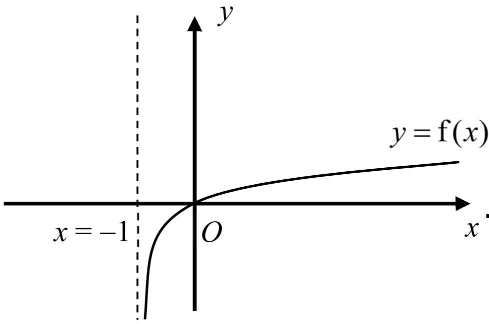
The functions f and g are defined by

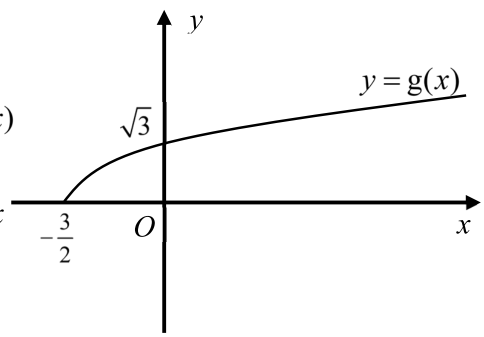
$$\mathrm{f}: x \mapsto \ln(x+1), \quad x \in \mathbb{R},\ x > -1,$$

$$\mathrm{g}: x \mapsto \sqrt{2x+3}, \quad x \in \mathbb{R},\ x \geq -\frac{3}{2}.$$

(i) Show that fg exist and define the function in a similar form.
(ii) Show that gf does not exist.

Solution





(i)

$\mathrm{R_g} = [0, \infty)$, $\mathrm{D_f} = (-1, \infty)$. Since $\mathrm{R_g} \subseteq \mathrm{D_f}$, fg exists.

$\mathrm{fg}(x) = \mathrm{f}\left(\mathrm{g}(x)\right) = \mathrm{f}(\sqrt{2x+3}) = \ln\left(\sqrt{2x+3}+1\right)$ and $\mathrm{D_{fg}} = \mathrm{D_g} = \left[-\frac{3}{2}, \infty\right)$

Hence $\mathrm{fg}: x \mapsto \ln\left(\sqrt{2x+3}+1\right), \quad x \in \mathbb{R},\ x \geq -\frac{3}{2}$.

(ii)

$\mathrm{R_f} = \mathbb{R}$, $\mathrm{D_g} = \left[-\frac{3}{2}, \infty\right)$. Since $\mathrm{R_f} \not\subseteq \mathrm{D_g}$ gf does not exist.

Example 7

The function g is defined by $\mathrm{g}(x) = \frac{4}{2-x}$, $x \in \mathbb{R}, x \neq 0, 2$. Find an expression for $\mathrm{g}^2(x)$ and $\mathrm{g}^3(x)$. Hence, determine the value for $\mathrm{g}\left(\frac{1}{2}\right)$ and $\mathrm{g}^{100}\left(\frac{1}{2}\right)$.

Solution

$$\mathrm{g}^2(x) = \mathrm{g}\left(\mathrm{g}(x)\right) = \mathrm{g}\left(\frac{4}{2-x}\right) = \frac{4}{2-\frac{4}{2-x}} = \frac{4(2-x)}{2(2-x)-4} = \frac{4-2x}{-x} = 2-\frac{4}{x}$$

$$\mathrm{g}^3(x) = \mathrm{g}\left(\mathrm{g}^2(x)\right) = \mathrm{g}\left(2-\frac{4}{x}\right) = \frac{4}{2-\left(2-\frac{4}{x}\right)} = \frac{4}{\frac{4}{x}} = x$$

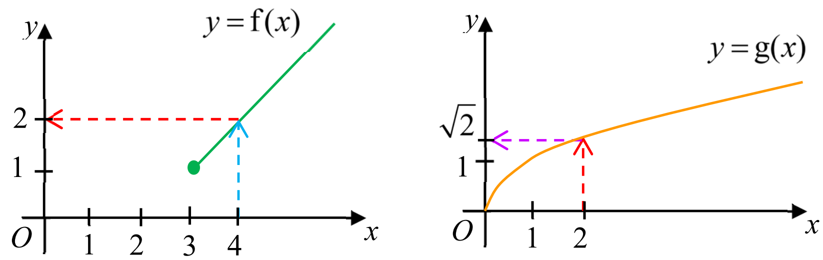
$$\mathrm{g}\left(\frac{1}{2}\right) = \frac{8}{3}; \qquad \mathrm{g}^{100}\left(\frac{1}{2}\right) = \mathrm{g}\left(\mathrm{g}^{99}\left(\frac{1}{2}\right)\right) = \mathrm{g}\left(\frac{1}{2}\right) = \frac{8}{3}$$

3.3 Range of a Composite Function

We can find the range of a composite function gf by sketching the graph of gf.

Alternatively, we can obtain the range of gf by sketching the graphs of f and g.
Let us explore how this alternative method works.

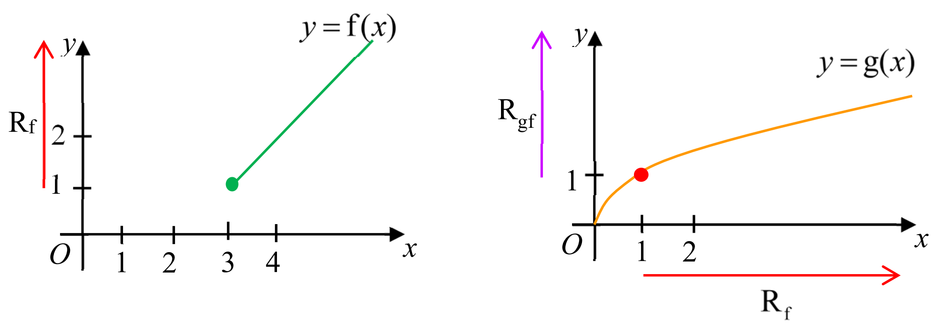
Consider functions $\mathrm{f}(x)=x-2,\ x\geq 3$ and $\mathrm{g}(x)=\sqrt{x},\ x\geq 0$.
For the composite function gf, the rule for f is first applied to each $x\in \mathrm{D_f}$ to find the image $\mathrm{f}(x)$, and then the rule for g is applied to $\mathrm{f}(x)$ to obtain the image $\mathrm{g}(\mathrm{f}(x))$. i.e. $\mathrm{gf}(x)=\mathrm{g}(\mathrm{f}(x))$.
Let us consider $x=4$. Then $\mathrm{gf}(4)=\mathrm{g}(\mathrm{f}(4))=\mathrm{g}(2)=\sqrt{2}$. Graphically, we have



Extending this to the domain of f (= domain of gf), we can obtain the

range of gf as the range of g whose domain is restricted to range of f.

i.e. $\mathrm{R_{gf}}=\mathrm{R_{g_1}}$ where $\mathrm{g_1}(x)=\sqrt{x},\quad x\geq 1$.



In this case, we see that $[3,\infty)\xrightarrow{\mathrm{f}}[1,\infty)\xrightarrow{\mathrm{g}}[1,\infty)$. Thus $\mathrm{R_{gf}}=[1,\infty)$.

Note: From the above illustration, we can conclude that $\mathrm{R_{gf}}\subseteq \mathrm{R_g}$.

Example 8

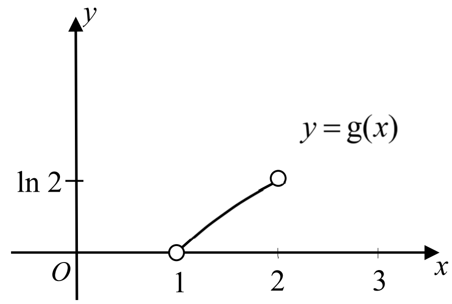
Let f and g be functions defined by $\mathrm{f}(x)=\mathrm{e}^{-x}+1,\ x\in\mathbb{R}^{+}$ and $\mathrm{g}(x)=\ln x,\ 1<x<2$. Determine whether the composite function fg exists. If it does, define the composite function in a similar form and find its range.

Solution

$\mathrm{R_g}=(0,\ln 2)$, $\mathrm{D_f}=(0,\infty)$.

Since $\mathrm{R_g}\subseteq\mathrm{D_f}$, fg exists.

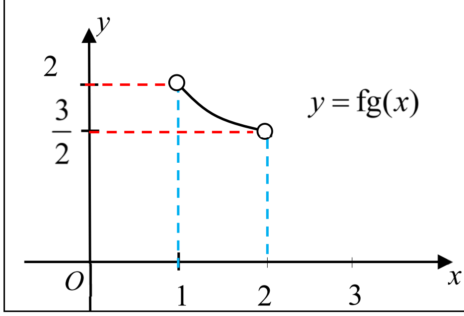
$$\mathrm{fg}(x)=\mathrm{e}^{-\ln x}+1$$
$$=\frac{1}{x}+1,\quad 1<x<2.$$

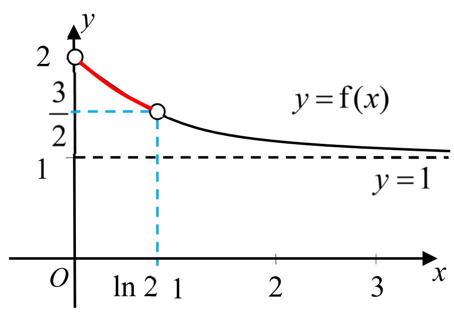


Using the graphs of g and f,

$(1,\ 2)\xrightarrow{\mathrm{g}}(0,\ \ln 2)\xrightarrow{\mathrm{f}}\left(\frac{3}{2},\ 2\right)$. So $\mathrm{R_{fg}}=\left(\frac{3}{2},\ 2\right)$

Alternatively, we can sketch the graph of fg to obtain $\mathrm{R_{fg}}$.





3.4 Composition of a Function and its Inverse

Consider a function $\mathrm{f}:x\mapsto y$ and its inverse function $\mathrm{f}^{-1}:y\mapsto x$. What can we say about the composite function $\mathrm{f}^{-1}\mathrm{f}$?

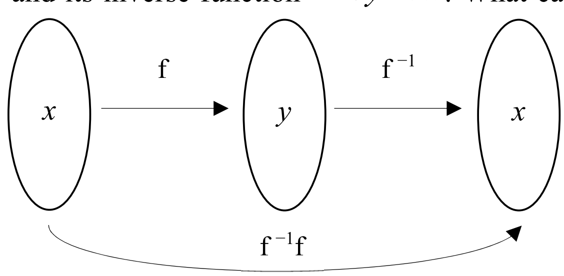


Fig 3.4

It is clear from Fig 3.4 that $\mathrm{f}^{-1}\mathrm{f}(x)=\mathrm{f}^{-1}\left(\mathrm{f}\left(x\right)\right)=x$. Similarly, we have $\mathrm{ff}^{-1}(x)=\mathrm{f}\left(\mathrm{f}^{-1}\left(x\right)\right)=x$. i.e. the composite function $\mathrm{f}^{-1}\mathrm{f}$ or ff^{-1} will map any element x in the domain to itself.

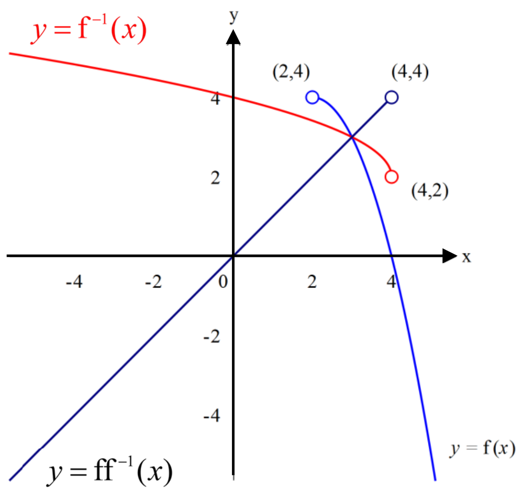
However, though the composite functions $\mathrm{f}^{-1}\mathrm{f}$ and ff^{-1} have the same rule, they may have different domains as $\mathrm{D}_{\mathrm{f}^{-1}\mathrm{f}}=\mathrm{D_f}$ while $\mathrm{D}_{\mathrm{ff}^{-1}}=\mathrm{D}_{\mathrm{f}^{-1}}$.

Example 9 [AJCPrelim9233/2004/02/Q4 modified]

The function f is defined by $\mathrm{f}: x \mapsto x(4-x), \quad x \in \mathbb{R}, \quad x > 2$.

On a single clearly labelled diagram, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$. State, with a reason, whether the equation $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$ has a solution.

Solution



To solve $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$, observe that $\mathrm{ff}^{-1}(x) = x$ for $x \in \mathrm{D}_{\mathrm{ff}^{-1}}$

and $\mathrm{f}^{-1}\mathrm{f}(x) = x$ for $x \in \mathrm{D}_{\mathrm{f}^{-1}\mathrm{f}}$

$\therefore x \in \mathrm{D}_{\mathrm{ff}^{-1}} \cap \mathrm{D}_{\mathrm{f}^{-1}\mathrm{f}}$, i.e. $x \in \mathrm{D}_{\mathrm{f}^{-1}} \cap \mathrm{D}_{\mathrm{f}}$

Since $\mathrm{D}_{\mathrm{f}} = (2, \infty)$ and $\mathrm{D}_{\mathrm{f}^{-1}} = (-\infty, 4)$, $\mathrm{D}_{\mathrm{f}^{-1}} \cap \mathrm{D}_{\mathrm{f}} \neq \varnothing$, the equation has a solution.

Example 10 [DHSJ1Mid-Year9740/2013/Q10]

The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \begin{cases} -\dfrac{x^2}{4}, & -2 \le x < 0, \\ x^3, & 0 \le x \le 2. \end{cases}$$

$$\mathrm{g}: x \mapsto x, \quad -1 \le x < 0.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$ and show that f^{-1} exists. **[3]**

(ii) Find f^{-1} in a similar form. **[4]**

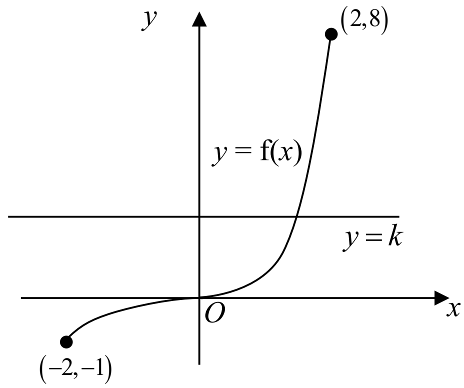
The function h is defined by $\mathrm{h}: x \mapsto \mathrm{f}(x), \quad -2 \le x < 0$.

(iii) Show that the composite function hg exists. **[1]**

(iv) Solve $\mathrm{gh}(x) = \mathrm{hg}(x)$, showing your working clearly. **[2]**

Solution

(i)



$R_f = [-1,8]$

Since every horizontal line $y = k, k \in [-1,8]$, cuts the graph of f at one and only one point, f is a one-one function and thus f^{-1} exists.

(ii) For $-2 \le x < 0$,

$$y = -\frac{x^2}{4} \Rightarrow x = \pm\sqrt{-4y} \Rightarrow x = -\sqrt{-4y} \text{ (since } -2 \le x < 0)$$

For $0 \le x \le 2$, $y = x^3 \Rightarrow x = \sqrt[3]{y}$

$$f^{-1} : x \mapsto \begin{cases} -\sqrt{-4x}, & -1 \le x < 0, \\ \sqrt[3]{x}, & 0 \le x \le 8. \end{cases}$$

(iii) $R_g = [-1,0)$ and $D_h = [-2,0)$. Since $R_g \subseteq D_h$, hg exists.

(iv) $gh(x) = h(x)$ and $hg(x)=h(x)$

For $gh(x) = hg(x), x \in D_{gh} \cap D_{hg}$.

$D_{gh} = D_h = [-2,0)$ and $D_{hg} = D_g = [-1,0) \Rightarrow D_{gh} \cap D_{hg} = [-1,0)$

So, $x \in [-1,0)$

Example 11 [SRJCJC1Mid-Year9740/2013/Q2]

(a) The function g is defined by

$$g : x \mapsto ax + b, \quad x \in \mathbb{R}^+,$$

where a and b are positive real numbers.

Show that g^2 exist and hence determine the range of g^2, leaving your answer in terms of a and b. **[2]**

(b) The function h is defined by

$$h : x \mapsto \frac{x+7}{x-1}, \quad x \in \mathbb{R}, x \ne 1.$$

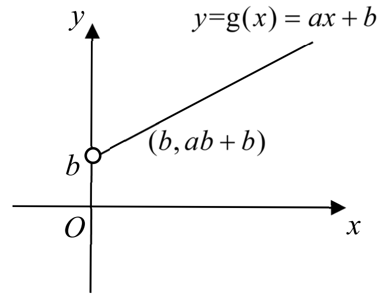
Given that $h^{-1} = h$, evaluate $h^{55}\left(\frac{1}{3}\right)$. **[2]**

Solution

(a) For g^2 to exist, $R_g \subseteq D_g$.

$D_g = \mathbb{R}^+$ and from the graph of g, $R_g = (b, \infty)$

Since $b > 0$, $R_g \subseteq D_g$. Hence g^2 exist.



Using the graph of g, $\mathbb{R}^+ \xrightarrow{g} (b, \infty) \xrightarrow{g} (ab + b, \infty)$.

$R_{g^2} = (ab + b, \infty)$

(b) $h : x \mapsto \dfrac{x+7}{x-1}, x \in \mathbb{R}, x \neq 1$

$$h^{55}(x) = \underbrace{hh^{-1}hh^{-1}...hh^{-1}}_{54 \text{ times}}h(x) = h(x) \text{ since } h = h^{-1} \text{and } hh^{-1}(x) = x$$

$$\therefore h^{55}\left(\frac{1}{3}\right) = h\left(\frac{1}{3}\right) = \frac{\frac{1}{3}+7}{\frac{1}{3}-1} = -11$$

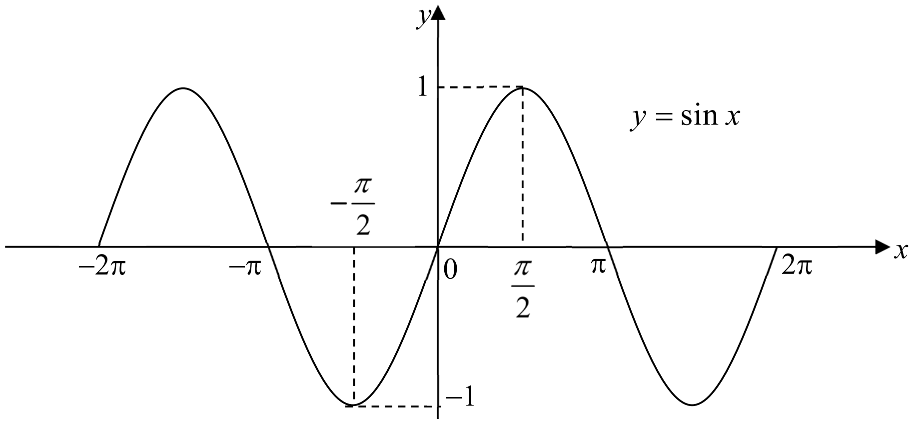
4 Trigonometric and Inverse Trigonometric Functions

Trigonometric functions are periodic functions. This means the function values are repeated in regular periods and therefore trigonometric functions are not one-one functions in general. However, the inverse trigonometric functions can be defined by restricting the domains of trigonometric functions such that it is a one-one function in the **restricted domain**.

Since trigonometric functions are periodic, more than one value of x give the same function values. For example, consider the function $y = \sin x$, $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}$ give the same function value of $y = \dfrac{1}{2}$. However when we want to find the value of x given value of $y = \dfrac{1}{2}$, we give the **principal value**, which is the value contained within the restricted domain.

Similar to other functions in general, the graphs of trigonometric functions and its inverse functions are the reflection of each other in the line $y = x$.

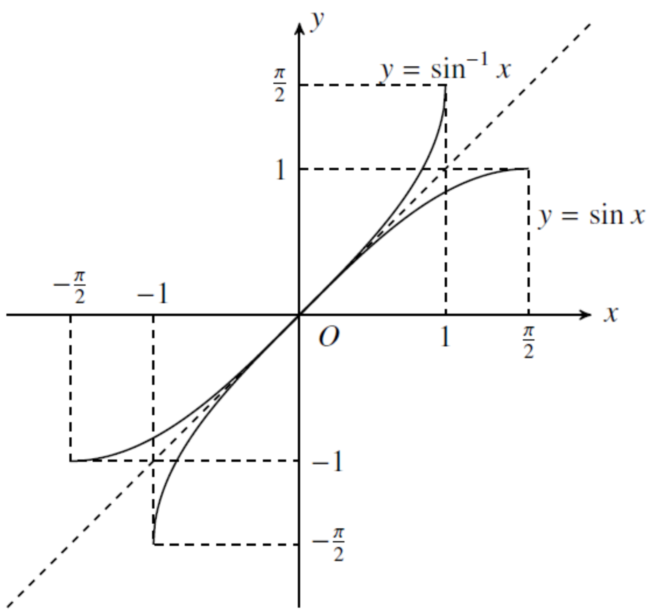
4.1 Sine and Inverse Sine function



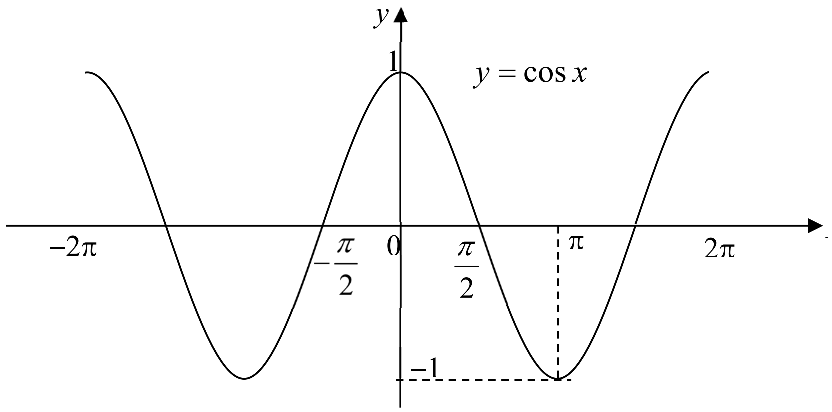
Consider the function $\mathrm{f} : x \mapsto \sin x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\mathrm{R_f} = [-1,1]$.

Since f is one-one for the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, f has an inverse.

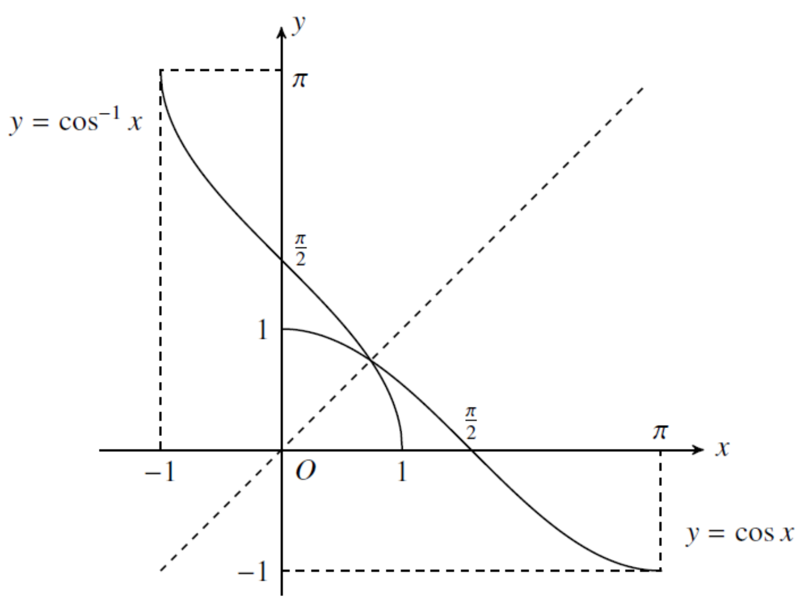
We denote the inverse of a sine function by **arcsin** or
$\mathrm{f}^{-1} : x \mapsto \sin^{-1} x, \quad x \in [-1,1], \quad \mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D_f} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



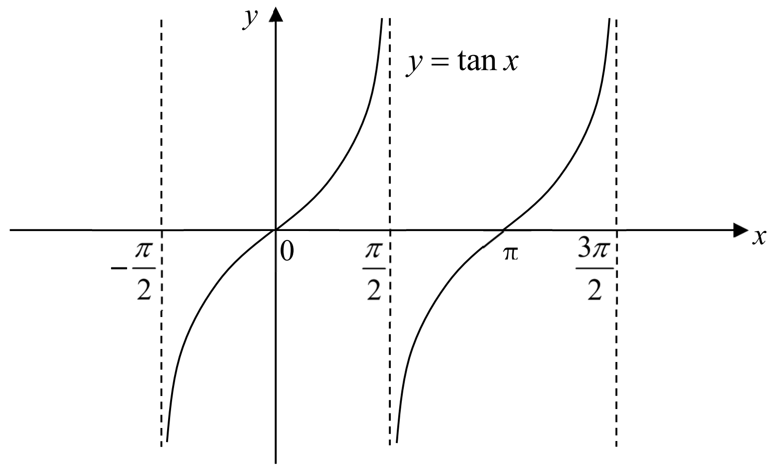
4.2 Cosine and Inverse Cosine function



Consider the function $\mathrm{g}: x \mapsto \cos x$, $x \in [0,\pi]$, then $\mathrm{R_g} = [-1,1]$.
Since g is one-one for the domain $[0,\pi]$, g has an inverse.
We denote the inverse of a cosine function by **arccos** or
$\mathrm{g}^{-1}: x \mapsto \cos^{-1} x, \quad x \in [-1,1], \quad \mathrm{R}_{\mathrm{g}^{-1}} = \mathrm{D_g} = [0,\pi]$.



4.3 Tangent and Inverse Tangent function



Consider the function $\text{h}: x \mapsto \tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $\text{R}_\text{h} = \mathbb{R}$.

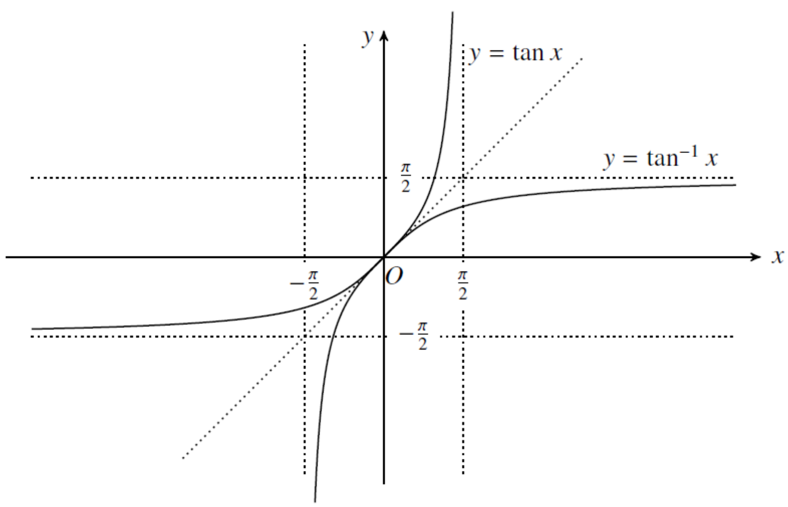
Since h is one-one for the domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, h has an inverse.

We denote the inverse of a tangent function by **arctan** or

$\text{h}^{-1}: x \mapsto \tan^{-1} x, \quad x \in \mathbb{R}, \quad \text{R}_{\text{h}^{-1}} = \text{D}_\text{h} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$

Notice that the graph $y = \tan x$ has vertical asymptotes at $x = -\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Therefore the graph of its inverse, $y = \tan^{-1} x$ has horizontal asymptotes at $y = -\frac{\pi}{2}$ and $\frac{\pi}{2}$.



In summary, the trigonometric functions are only one-one within a restricted domain. Thus, define

$$\mathrm{f}: x \mapsto \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\mathrm{g}: x \mapsto \cos x, \quad 0 \le x \le \pi$$
$$\mathrm{h}: x \mapsto \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \quad \text{for principal values of } x.$$

The table below summarizes the restricted domain and the principal values of each inverse trigonometric function.

Inverse trigonometry function	**Notation**	**Equivalent form**	**Domain**	**Range of principal value**
arcsin	$y = \sin^{-1} x$	$\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
arccos	$y = \cos^{-1} x$	$\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
arctan	$y = \tan^{-1} x$	$\tan y = x$	$\mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Note that $\sin^{-1} x \ne \dfrac{1}{\sin x}$.

Now let us consider the two functions $\mathrm{ff}^{-1}(x)$ and $\mathrm{f}^{-1}\mathrm{f}(x)$.

Recall from section 3.4 that even though the composite functions $\mathrm{f}^{-1}\mathrm{f}$ and ff^{-1} have the same rule, i.e. $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{ff}^{-1}(x) = x$, they may have different domains as $\mathrm{D}_{\mathrm{f}^{-1}\mathrm{f}} = \mathrm{D}_\mathrm{f}$ while $\mathrm{D}_{\mathrm{ff}^{-1}} = \mathrm{D}_{\mathrm{f}^{-1}}$.

$\mathrm{ff}^{-1}(x) = x, \quad x \in \mathrm{D}_{\mathrm{f}^{-1}}$	$\mathrm{f}^{-1}\mathrm{f}(x) = x, \quad x \in \mathrm{D}_\mathrm{f}$
$\sin\left(\sin^{-1} x\right) = x, \quad -1 \le x \le 1$	$\sin^{-1}\left(\sin x\right) = x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
$\cos\left(\cos^{-1} x\right) = x, \quad -1 \le x \le 1$	$\cos^{-1}\left(\cos x\right) = x, \quad 0 \le x \le \pi$
$\tan\left(\tan^{-1} x\right) = x, \quad x \in \mathbb{R}$	$\tan^{-1}\left(\tan x\right) = x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Example 12

If $\alpha = \sin^{-1}\left(\frac{1}{2}\right)$, find the principal value of α and hence obtain the values of $\cos\alpha$ and $\tan\alpha$.

Solution

$\alpha = \sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \sin\alpha = \frac{1}{2}$ and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \alpha = \frac{\pi}{6}$

$\cos\alpha = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\tan\alpha = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

5 Solving Inequalities Graphically

So far, you have learnt how to solve inequalities using analytical methods or sketching of simple graphs without the use of a GC. With the use of a graphing calculator, we are able to sketch more complicated graphs and thus be able to solve any inequalities using graphical methods.

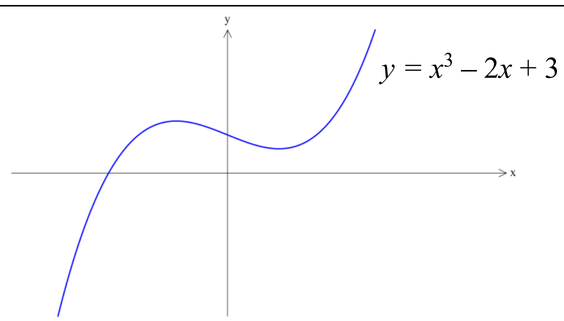
Example 13

Find the solution set of each of the following inequalities:

(a) $x^3 - 2x + 3 \geq 0$, **(b)** $|x-1| + |5-2x| < 7-4x$.

Solution:

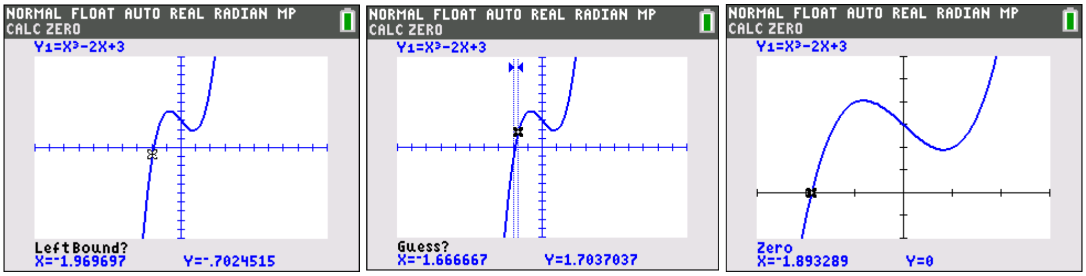
(a)



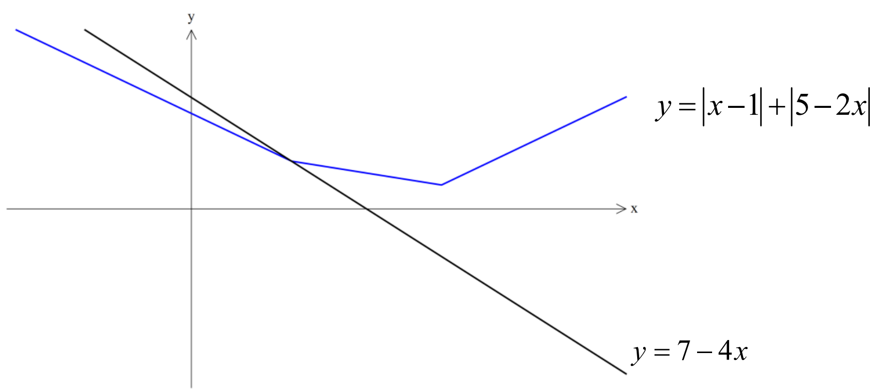
Clearly, from the graph, $x^3 - 2x + 3 \geq 0 \Rightarrow x \geq -1.89$ (3sf)

Hence, solution set $= [-1.89, \infty)$

GC Key Strokes:
From the graphing calculator, key in the function $y = x^3 - 2x + 3$ as in Example 1, and use $\boxed{\text{2nd}}\boxed{\text{trace}}$ **2: zero** (to calculate x-intercept).
You will be prompted to enter a left bound and a right bound of an interval which contains the x-intercept.

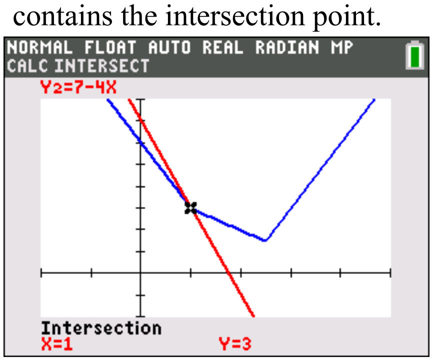


(b)



Clearly, from the graph, $|x-1| + |5-2x| < 7-4x \Rightarrow x < 1$
Hence, solution set $= (-\infty, 1)$

GC Key Strokes:
From the graphing calculator, key in the functions $y = |x-1| + |5-2x|$ and $y = 7-4x$.
Use $\boxed{\text{2nd}}\boxed{\text{trace}}$ **5: intersect** (to calculate intersection points).
Similarly, you will be asked to select a left and right bound of an interval which contains the intersection point.



Alternatively, we could graph $y = |x-1| + |5-2x| - (7-4x)$, and find the values of x such that $|x-1| + |5-2x| - (7-4x) < 0$.

Example 14

The functions f and g are defined as follow.

$$\mathrm{f}: x \mapsto \mathrm{e}^x \sin x, \quad x \in \mathbb{R}, \ -3 \le x \le 3$$

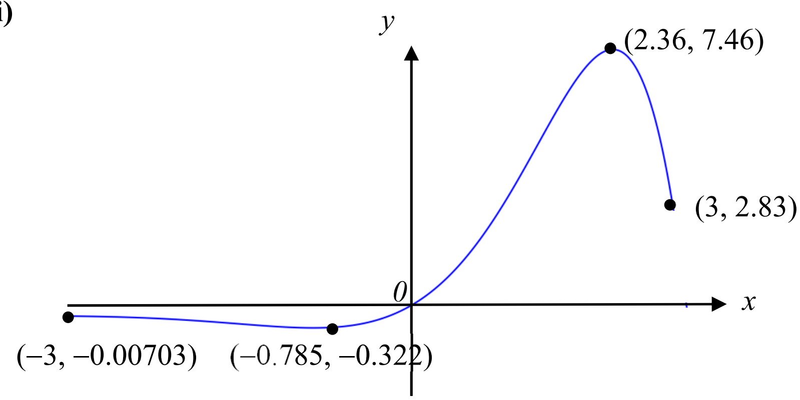
$$\mathrm{g}: x \mapsto x + x^2 + \frac{x^3}{3}, \quad x \in \mathbb{R}, \ -3 \le x \le 3$$

(i) Sketch the graph of $y = \mathrm{f}(x)$ and find the range of f.

(ii) Find the set of values of x for which the value of g(x) is within ± 0.5 of the value of f(x).

Solution:

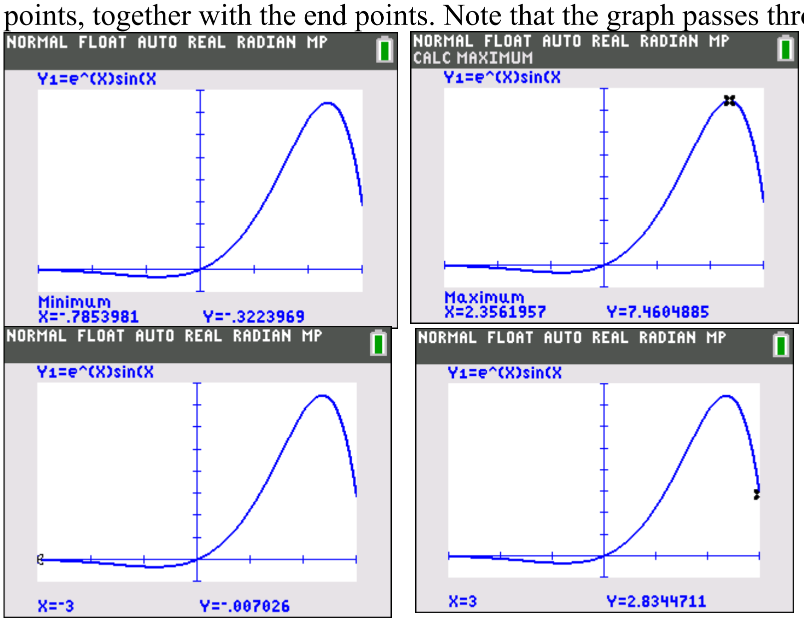
(i)



$\therefore$ The range of f is $[-0.322, 7.46]$.

GC Key Strokes (similar to Example 1):

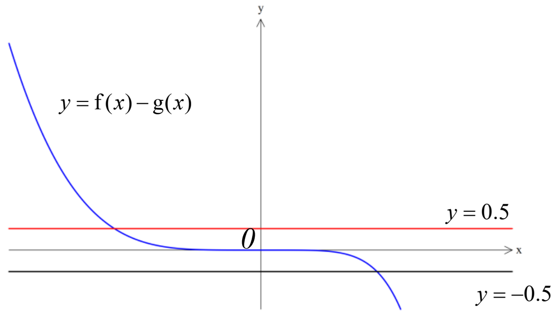
The graph of $y = \mathrm{f}(x)$ is to be sketched with coordinates of maximum and minimum points, together with the end points. Note that the graph passes through the origin.



(ii) $\left|f(x)-g(x)\right| \le 0.5$

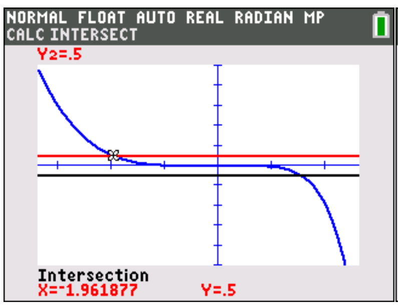
$-0.5 \le f(x)-g(x) \le 0.5$

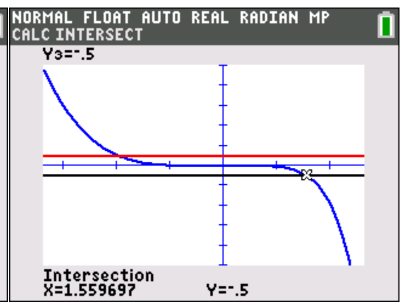
Sketch $y = f(x)-g(x)$, $y=-0.5$ and $y = 0.5$.



The intersection points are at $x=-1.96$ and 1.56 (refer to GC screenshots below).

Set of values is $\{x \in \mathbb{R} : -1.96 \le x \le 1.56\}$.



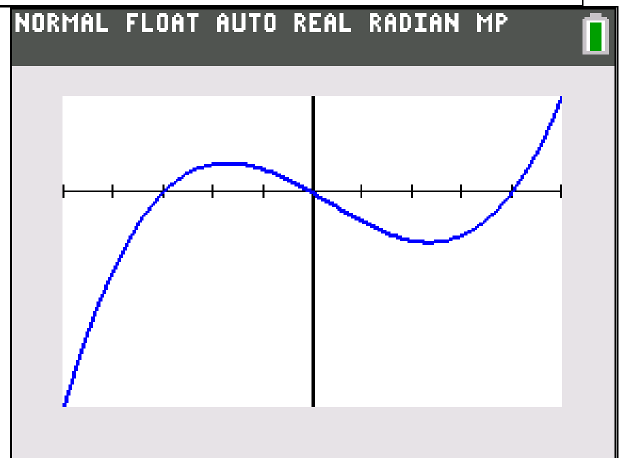
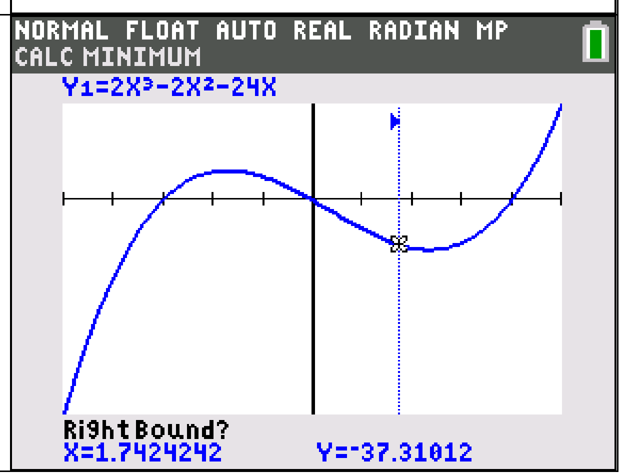


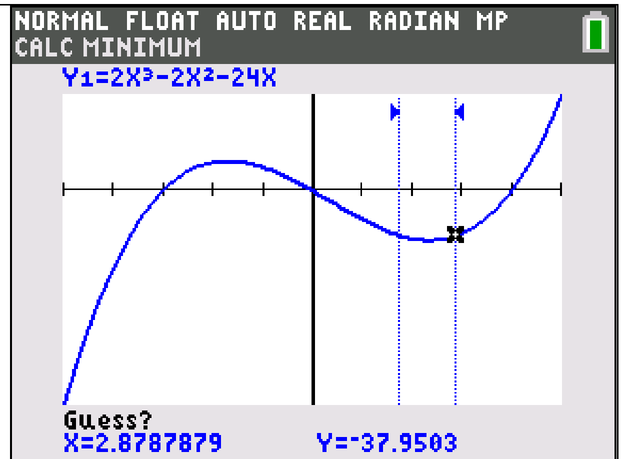
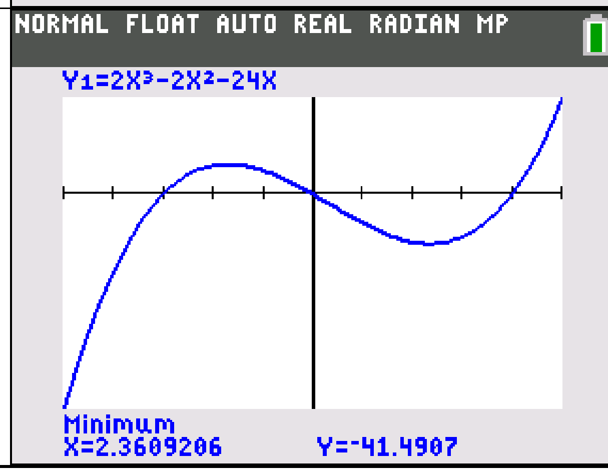
Alternatively, you may sketch $y = \left|f(x)-g(x)\right|$ and $y = 0.5$.

Appendix: Graphing Calculator Task

Using a graphing calculator to find the maximum and minimum turning points of $y = 2x(x-4)(x+3)$

Solution:

In the $\boxed{\text{y=}}$ screen, enter **Y$_1$ = 2X^3 - 2X^2 – 24X**. Press $\boxed{\text{window}}$ to set $X_{\min} = -5$ and $X_{\max} = 5$. Choose 0:**ZoomFit** from the $\boxed{\text{zoom}}$ menu to obtain graph.	
Press $\boxed{\text{2nd}}$ $\boxed{\text{trace}}$ Select **3:minimum** (to calculate min turning point)	
Move the cursor to the left of the minimum point to define a left bound for the minimum and press $\boxed{\text{enter}}$.	

Move the cursor to the right of the minimum point to define a right bound for the minimum and press $\boxed{\text{enter}}$. Press $\boxed{\text{enter}}$ again when the screen prompts for a “guess” value.	
The GC will return you the coordinates of the minimum point.	

Similarly you can find the maximum turning point by selecting **4:maximum** and by following the same procedure.

SUMMARY