Q1	Functions and Graphs	
(a) [3m]	$I_n = \int_0^{\frac{1}{2}} (1-2x)^n e^x \ \mathrm{d}x$ $\qquad u = (1-2x)^n \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = e^x$ $= \left[(1-2x)^n e^x \right]_0^{\frac{1}{2}} + 2n\int_0^{\frac{1}{2}} (1-2x)^{n-1} e^x \ \mathrm{d}x$ $\qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -2n(1-2x)^{n-1} \qquad v = e^x$ $= -1 + 2nI_{n-1}$	
(b)(i) [3m]	$I_0 = \int_0^{\frac{1}{2}} (1-2x)^0 e^x \ \mathrm{d}x$ $= \int_0^{\frac{1}{2}} e^x \ \mathrm{d}x$ $= \left[e^x \right]_0^{\frac{1}{2}}$ $= e^{\frac{1}{2}} - 1$ $I_3 = -1 + 6I_2$ $= -1 + 6(-1 + 4I_1)$ $= -7 + 24I_1$ $= -7 + 24(-1 + 2I_0)$ $= -31 + 48I_0$ $= -31 + 48\left(e^{\frac{1}{2}} - 1 \right)$ $= -79 + 48e^{\frac{1}{2}}$	
(b)(ii) [2m]	$\int_0^{\frac{1}{2}} (1-2x)^4 e^x \ \mathrm{d}x$ $= I_4$ $= -1 + 8I_3$ $= -1 + 8\left(-79 + 48e^{\frac{1}{2}} \right)$ $= -633 + 384e^{\frac{1}{2}}$	

(c) **[3m]**	$I_n = \int_0^{\frac{1}{2}} (1-2x)^n e^x \ \mathrm{d}x$ For $x \in \left(0, \frac{1}{2}\right)$, $0 < 1-2x < 1$. As $n \to \infty$, $(1-2x)^n \to 0$. Since e^x is continuous and bounded for $x \in \left[0, \frac{1}{2}\right]$, As $n \to \infty, (1-2x)^n e^x \to 0$, $\therefore I_n = \int_0^{\frac{1}{2}} (1-2x)^n e^x \ \mathrm{d}x \to 0.$	
(d) **[3m]**	Given $I_n = -1 + 2nI_{n-1}$, $I_{n-1} = \frac{1}{2n}(I_n + 1)$ $I_0 = \frac{1}{2}(I_1 + 1)$ $= \frac{1}{2}\left[\frac{1}{4}(I_2 + 1) + 1\right]$ $= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) I_2 + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \frac{1}{2}$ $= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left[\frac{1}{6}(I_3 + 1)\right] + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \frac{1}{2}$ $= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{6}\right) I_3 + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \frac{1}{2}$ $= \lim_{n\to\infty}\left[\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\ldots\left(\frac{1}{2n}\right) I_n\right]$ $+ \lim_{n\to\infty}\left[\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\ldots\left(\frac{1}{2n}\right) + \ldots + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \frac{1}{2}\right]$ As $n \to \infty, I_n \to 0, \frac{1}{2n} \to 0$ $\therefore \frac{1}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{6}\right) + \ldots = e^{\frac{1}{2}} - 1$	

Q2	Numbers and Proofs	
(a)(i) [3m]	$(1+x)^{2m}=\sum_{k=0}^{2m}\binom{2m}{k}x^k$ Let $x=1$ $2^{2m}=\sum_{k=0}^{2m}\binom{2m}{k}$ $2^{2m}=\binom{2m}{0}+\binom{2m}{1}+\ldots+\binom{2m}{m}+\binom{2m}{m+1}+...+\binom{2m}{2m}$ since each term $\binom{2m}{k}$ is positive, $\binom{2m}{m}<2^{2m}$	
(a)(ii) [3m]	$\binom{2m}{m}=\dfrac{(2m)(2m-1)\ldots(m+1)(m)(m-1)\ldots(2)(1)}{\left[(m)(m-1)\ldots(2)(1)\right]\left[(m)(m-1)\ldots(2)(1)\right]}$ $\binom{2m}{m}=\dfrac{(2m)(2m-1)\ldots(m+1)}{(m)(m-1)\ldots(2)(1)}$ This is an integer which implies all factors of $m!$ cancel with the factors in the numerator. We note that $\mathrm{P}_{m+1,2m}$ is part of the numerator. Therefore $\mathrm{P}_{m+1,2m}$ divides $\binom{2m}{m}$.	
(b)(i) [3m]	By the Cauchy-Schwarz Inequality, $\left(\dfrac{a^2}{\left(\sqrt{b+c}\right)^2}+\dfrac{b^2}{\left(\sqrt{c+a}\right)^2}+\dfrac{c^2}{\left(\sqrt{a+b}\right)^2}\right)\left[\left(\sqrt{b+c}\right)^2+\left(\sqrt{c+a}\right)^2+\left(\sqrt{a+b}\right)^2\right]\geq\left(a+b+c\right)^2$ $\left(\dfrac{a^2}{b+c}+\dfrac{b^2}{c+a}+\dfrac{c^2}{a+b}\right)\left(b+c+c+a+a+b\right)\geq\left(a+b+c\right)^2$ $\left(\dfrac{a^2}{b+c}+\dfrac{b^2}{c+a}+\dfrac{c^2}{a+b}\right)\cdot 2\left(a+b+c\right)\geq\left(a+b+c\right)^2$ $\therefore\ 2\left(\dfrac{a^2}{b+c}+\dfrac{b^2}{c+a}+\dfrac{c^2}{a+b}\right)\geq a+b+c$	

[Turn over

(b)(ii) **[5m]**	$\dfrac{yz}{x^2(y+z)}+\dfrac{zx}{y^2(z+x)}+\dfrac{xy}{z^2(x+y)}$ $=\dfrac{\frac{1}{x^2}}{\frac{(y+z)}{yz}}+\dfrac{\frac{1}{y^2}}{\frac{(z+x)}{zx}}+\dfrac{\frac{1}{z^2}}{\frac{(x+y)}{xy}}$ $=\dfrac{\frac{1}{x^2}}{\frac{1}{y}+\frac{1}{z}}+\dfrac{\frac{1}{y^2}}{\frac{1}{z}+\frac{1}{x}}+\dfrac{\frac{1}{z^2}}{\frac{1}{x}+\frac{1}{y}}$ $\ge\dfrac{1}{2}\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$ using result from (b)(i) $\ge\dfrac{1}{2}\cdot 3\sqrt[3]{\dfrac{1}{xyz}}$ using AM-GM Inequality $=\dfrac{3}{4}$ since $xyz=8$ Therefore the minimum value is $\dfrac{3}{4}$. It occurs when $\dfrac{1}{x}=\dfrac{1}{y}=\dfrac{1}{z}=\dfrac{1}{2}$ or $x=y=z=2$	

Q3	**Sequences and Series**	
(a) **[4m]**	Let P_n be the statement $f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}$ for $n \geq 1,\ n \in \mathbb{Z}$. <u>Base case</u>: $n = 1$ $\text{L.H.S.} = f_1 f_3 - f_2^2 = f_1(f_2 + f_1) - f_2^2 = f_1 f_2 + f_1^2 - f_2^2 = (1)(1) + (1)^2 - (1)^2$ $= (-1)^{1+1}$ $= \text{R.H.S.}$ $\therefore P_1$ is true. <u>Inductive Step</u> Assume that P_k is true for some $k \geq 1$, i.e., $f_k f_{k+2} - f_{k+1}^2 = (-1)^{k+1}$. To prove that P_{k+1} is true, i.e., $f_{k+1} f_{k+3} - f_{k+2}^2 = (-1)^{k+2}$. $\text{L.H.S.} = f_{k+1} f_{k+3} - f_{k+2}^2$ $= f_{k+1}(f_{k+2} + f_{k+1}) - f_{k+2}^2$ [by definition of Fibonacci numbers] $= f_{k+1} f_{k+2} + f_{k+1}^2 - f_{k+2}^2$ $= f_{k+1}^2 - f_{k+2}(f_{k+2} - f_{k+1})$ $= f_{k+1}^2 - f_{k+2}(f_k)$ [by definition of Fibonacci numbers] $= -(f_k f_{k+2} - f_{k+1}^2)$ $= (-1)(-1)^{k+1}$ [by inductive hypothesis] $= (-1)^{k+2}$ $= \text{R.H.S.}$ $\therefore P_k$ true $\Rightarrow P_{k+1}$ true By the Principle of Mathematical Induction, P_n is true for $n \geq 1,\ n \in \mathbb{Z}$.	

(b) **[4m]**	Given that $r_n = \dfrac{f_{n+1}}{f_n}$. As the terms in the Fibonacci numbers f_n, are positive, $r_n \geq 1$ As the Fibonacci sequence f_n is increasing, $\dfrac{f_{n+1}}{f_n} = \dfrac{f_{n-1}+f_n}{f_n} = \dfrac{f_{n-1}}{f_n} + 1 \leq 2$ Since $1 \leq r_n \leq 2$, it is a bounded sequence. Hence subsequences $r_2, r_4, r_6, r_8, \ldots$ and $r_1, r_3, r_5, r_7, \ldots$ would naturally be bounded as well Consider $r_{2n} - r_{2n+2} = \dfrac{f_{2n+1}}{f_{2n}} - \dfrac{f_{2n+3}}{f_{2n+2}}$ $= \dfrac{f_{2n+1}f_{2n+2} - f_{2n}f_{2n+3}}{f_{2n}f_{2n+2}}$ $= \dfrac{f_{2n+1}\left(f_{2n+1}+f_{2n}\right) - f_{2n}\left(f_{2n+2}+f_{2n+1}\right)}{f_{2n}f_{2n+2}}$ [by definition of Fibonacci numbers] $= \dfrac{f_{2n+1}^2 - f_{2n}f_{2n+2}}{f_{2n}f_{2n+2}}$ $= \dfrac{-(-1)^{2n+1}}{f_{2n}f_{2n+2}}$ [using the identity from part (a)] $= \dfrac{1}{f_{2n}f_{2n+2}}$ > 0 $\Rightarrow r_{2n} > r_{2n+2}$ The sequence $r_2, r_4, r_6, r_8, \ldots$ is decreasing. Consider $r_{2n+1} - r_{2n-1} = \dfrac{f_{2n+2}}{f_{2n+1}} - \dfrac{f_{2n}}{f_{2n-1}}$ $= \dfrac{f_{2n+2}f_{2n-1} - f_{2n}f_{2n+1}}{f_{2n+1}f_{2n-1}}$ $= \dfrac{\left(f_{2n+1}+f_{2n}\right)f_{2n-1} - f_{2n}\left(f_{2n}+f_{2n-1}\right)}{f_{2n+1}f_{2n-1}}$ [by definition of Fibonacci numbers] $= \dfrac{f_{2n-1}f_{2n+1} - f_{2n}^2}{f_{2n+1}f_{2n-1}}$ $= \dfrac{(-1)^{2n}}{f_{2n+1}f_{2n-1}}$ [using the identity from part (a)] $= \dfrac{1}{f_{2n+1}f_{2n-1}}$ > 0 $\Rightarrow r_{2n+1} > r_{2n-1}$ The sequence $r_1, r_3, r_5, r_7, \ldots$ is increasing.	

(c)
[6m]

Since $r_2, r_4, r_6, r_8, \ldots$ and $r_1, r_3, r_5, r_7, \ldots$ are bounded and monotone,
By the Monotone Convergence Theorem,
$r_2, r_4, r_6, r_8, \ldots$ and $r_1, r_3, r_5, r_7, \ldots$ are convergent.

Consider

$$\begin{aligned} r_{2n} - r_{2n-1} &= \frac{f_{2n+1}}{f_{2n}} - \frac{f_{2n}}{f_{2n-1}} \\ &= \frac{f_{2n+1}f_{2n-1} - f_{2n}^2}{f_{2n}f_{2n-1}} \\ &= \frac{(-1)^{2n}}{f_{2n}f_{2n-1}} \quad \text{[using the identity from part (a)]} \\ &= \frac{1}{f_{2n}f_{2n-1}} \end{aligned}$$

Since f_n is an unbounded and increasing sequence, when the denominator of the preceding fraction is arbitrarily large and consequently $r_{2n} - r_{2n-1}$ is arbitrarily small.

Proof by contradiction:

Let $\lim_{n\to\infty}(r_{2n}) = L_e$ and $\lim_{n\to\infty}(r_{2n-1}) = L_0$, where $L_e \neq L_o$ and $\varepsilon = |L_e - L_o|$.

Since $\lim_{n\to\infty}(r_{2n}) = L_e$, there exists an $N_1 \in \mathbb{N}$ such that $|r_{2n} - L_e| < \frac{\varepsilon}{4}$ $\forall\ n \geq N_1$.

Likewise, $\lim_{n\to\infty}(r_{2n-1}) = L_0$, there exists an $N_2 \in \mathbb{N}$ such that $|r_{2n-1} - L_o| < \frac{\varepsilon}{4}$ $\forall\ n \geq N_2$.

There exists an $N_3 \in \mathbb{N}$ such that $|r_{2n} - r_{2n-1}| < \frac{\varepsilon}{4}$ $\forall\ n \geq N_3$.

Let $N = \max\{N_1, N_2, N_3\}$ and pick any $n \geq N$.

Then by the triangle inequality,

$$\varepsilon = |L_e - L_0| \leq |L_e - r_{2n}| + |r_{2n} - r_{2n-1}| + |r_{2n-1} - L_o| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}.$$

But this implies $\varepsilon < \frac{3\varepsilon}{4}$, which is a clear contradiction. Therefore, $L_e = L_o$.

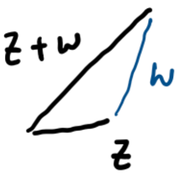
Since $L_e = L_o$, we conclude that sequence r_n converges to the same limit

Now $$r_n = \frac{f_{n+1}}{f_n} = \frac{f_n + f_{n-1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} = 1 + \frac{1}{r_{n-1}}$$

As $n \to \infty$, $r_n \to L$ and $r_{n-1} \to L$,

	$L=1+\frac{1}{L}$ $L^2-L-1=0$ $L=\frac{1+\sqrt{5}}{2}$ or $L=\frac{1-\sqrt{5}}{2}$ (rejected $\because f_n>0\ \forall\ n\in\mathbb{N}$)	

Q4	**Counting / Inequalities**	
(a) **[1m]**	Number of 2-element subsets of $S=\binom{500}{2}=124750$	
(b) **[4m]**	<u>Case 1: Both x and y are multiples of 3</u> Number of multiples of 3 = $\left\lfloor\frac{500}{3}\right\rfloor=166$ Thus, the number of 2-element subsets $\{x,y\}$ of $S=\binom{166}{2}$. <u>Case 2: Exactly one of x and y is a multiple of 3</u> Number of ways of choosing a multiple of 3 = 166 Number of ways of choosing a non-multiple of 3 = 500-166 = 334 Thus, the number of 2-element subsets $\{x,y\}$ of $S=166\times334=55444$ By addition principle, the number of 2-element subsets $\{x,y\}$ of S $=\binom{166}{2}+166\times334=69139$	
(c) **[5m]**	<u>**Case 1**: Both x and y are multiples of 3.</u> From (ii), the number of 2-element subsets $\{x,y\}$ of $S=\binom{166}{2}$ <u>**Case 2**: One of x and y leaves a remainder of 1 when divided by 3 and the other leaves a remainder of 2 when divided by 3.</u> Number of ways of choosing a number which leaves a remainder of 1 when divided by 3 = $\left\lfloor\frac{500}{3}\right\rfloor+1=167$. (the numbers are 1, 4, 7, …, 499) Number of ways of choosing a number which leaves a remainder of 2 when divided by 3 = 500 – 166 – 167 = 167 (the numbers are 2, 5, 8, …, 500) Thus, the number of ways give 167×167 Thus, the total number of ways are $\binom{166}{2}+167^2=41584$	

Q5	**Complex Numbers**	
(a) **[3m]**	$\lvert z+w\rvert^2=(z+w)(z+w)^*$ $=(z+w)(z^*+w^*)$ $=\lvert z\rvert^2+zw^*+wz^*+\lvert w\rvert^2$ $=\lvert z\rvert^2+zw^*+(zw^*)^*+\lvert w\rvert^2$ $=\lvert z\rvert^2+2\operatorname{Re}(zw^*)+\lvert w\rvert^2$ $\quad \lvert a\rvert^2=a\cdot a^*$ $\le\lvert z\rvert^2+2\lvert zw^*\rvert+\lvert w\rvert^2$ $\le\lvert z\rvert^2+2\lvert z\rvert\lvert w^*\rvert+\lvert w\rvert^2$ $\le\lvert z\rvert^2+2\lvert z\rvert\lvert w\rvert+\lvert w\rvert^2$ $\le(\lvert z\rvert+\lvert w\rvert)^2$ $\therefore \lvert z+w\rvert\le\lvert z\rvert+\lvert w\rvert$  Equality holds when $\lvert z+w\rvert=\lvert z\rvert+\lvert w\rvert$, Hence $\arg(z)=\arg(w)$ 	
(b) **[3m]**	$(a-b)(c-d)+(b-c)(a-d)=ac-ad-bc+bd+ab-bd-ac+cd$ $=-ad-bc+ab+cd$ $=b(a-c)-d(a-c)$ $=(a-c)(b-d)$ $\lvert a-c\rvert\lvert b-d\rvert=\lvert(a-c)(b-d)\rvert$ $=\lvert(a-b)(c-d)+(b-c)(a-d)\rvert$ $\le\lvert(a-b)(c-d)\rvert+\lvert(b-c)(a-d)\rvert$ by (i) $\le\lvert a-b\rvert\lvert c-d\rvert+\lvert b-c\rvert\lvert a-d\rvert$ (shown)	

(c) **[5m]**	For equality to happen, by (a), $\arg\left[(a-b)(c-d)\right]=\arg\left[(b-c)(a-d)\right]$ $\arg\left[\dfrac{(a-b)(c-d)}{(b-c)(a-d)}\right]=0$ $\arg\left(\dfrac{b-a}{d-a}\cdot\dfrac{d-c}{b-c}\right)=\pi$ $\arg\left(\dfrac{b-a}{d-a}\right)+\arg\left(\dfrac{d-c}{b-c}\right)=\pi+2k\pi$ for integer k $\arg(b-a)-\arg(d-a)+\arg(d-c)-\arg(b-c)=\pi+2k'\pi$ for integer k' $\arg(b-a)-\arg(d-a)=\angle DAB+2m\pi$ $\arg(d-c)-\arg(b-c)=\angle BCD+2n\pi$ for integer m and n $\angle DAB+\angle BCD=\arg(b-a)-\arg(d-a)+\arg(d-c)-\arg(b-c)-2(m+n)\pi$ $=\pi+2k'\pi-2(m+n)\pi$ $\therefore \angle DAB+\angle BCD=\pi$ since $ABCD$ is a quadrilateral	

Q6	**Mathematical Text and Investigation**	
(a) **[5m]**	$RHS=F_{k+1}F_{n-k}+F_kF_{n-k-1}$ $=\frac{1}{5}\{\left[\varphi^{k+1}-(-\varphi)^{-k-1}\right]\left[\varphi^{n-k}-(-\varphi)^{-n+k}\right]$ $+\left[\varphi^{k}-(-\varphi)^{-k}\right]\left[\varphi^{n-k-1}-(-\varphi)^{-n+k+1}\right]\}$ $=\frac{1}{5}[\ \varphi^{n+1}-\varphi^{k+1}(-\varphi)^{-n+k}-\varphi^{n-k}(-\varphi)^{-k-1}+(-\varphi)^{-n-1}$ $+\varphi^{n-1}-\varphi^{k}(-\varphi)^{-n+k+1}-\varphi^{n-k-1}(-\varphi)^{-k}+(-\varphi)^{-n+1}\ ]$ $=\frac{1}{5}[\ \varphi^{n+1}+\varphi^{n-1}+(-\varphi)^{-n-1}+(-\varphi)^{-n+1}$ $-\varphi^{k}(-\varphi)^{-n+k}\{\varphi-\varphi\}-\varphi^{n-k-1}(-\varphi)^{-k-1}\{\varphi-\varphi\}\ ]$ $=\frac{1}{5}[\ \varphi^{n}(\varphi+\varphi^{-1})-(-\varphi)^{-n}(\varphi+\varphi^{-1})\ ]$ $=\frac{1}{5}\left[\left(\varphi+\varphi^{-1}\right)\left(\varphi^{n}-(-\varphi)^{-n}\ \right)\right]$ Since $\varphi+\varphi^{-1}=\dfrac{1+\sqrt{5}}{2}+\dfrac{2}{1+\sqrt{5}}=\dfrac{1+\sqrt{5}}{2}-\dfrac{1-\sqrt{5}}{2}=\sqrt{5}$,	

$$RHS = \frac{1}{5}\left[\left(\varphi+\varphi^{-1}\right)\left(\varphi^{n}-(-\varphi)^{-n}\ \right)\right]$$

$$= \frac{\sqrt{5}}{5}[\ \varphi^{n}-(-\varphi)^{-n}\ ]$$

$$= \frac{\varphi^{n}-(-\varphi)^{-n}}{\sqrt{5}}$$

$$= F_{n} \qquad \text{(shown)}$$

(b)
[2m]

$$RHS = \binom{n-1}{k}+\binom{n-1}{k-1}$$

$$= \frac{(n-1)!}{k!(n-1-k)!}+\frac{(n-1)!}{(k-1)!(n-k)!}$$

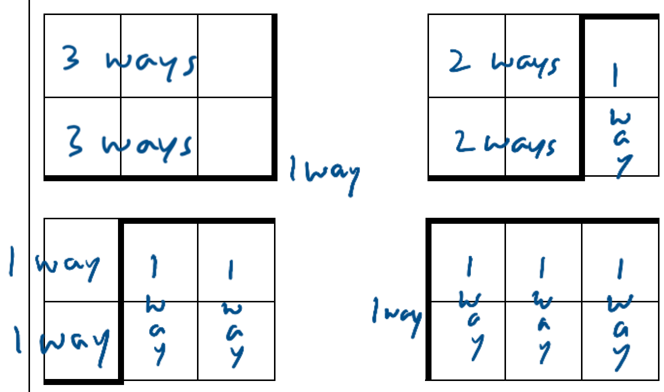
$$= \frac{(n-1)!(n-k)+(n-1)!(k)}{k!(n-k)!}$$

$$= \frac{(n-1)![(n-k)+k]}{k!(n-k)!}$$

$$= \frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k} = LHS$$

(c)
[4m]



$$\binom{5}{3}_{F} = \text{number of ways to tile}$$

$$= (3\times3\times1)+(2\times2\times1)+(1\times1\times1\times1)+(1\times1\times1\times1)$$

$$=15$$

(d)
[3m]

For a row of r squares,

Case 1: Start with a square, then the remaining $r-1$ squares can be tiled in T_{r-1} number of ways.

Case 2: Start with a domino, then the remaining $r-2$ squares can be tiled in T_{r-1} number of ways.

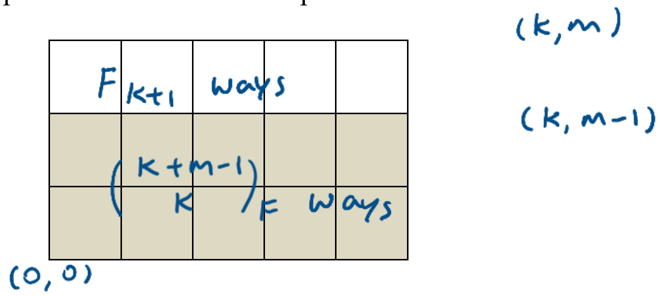
$\therefore$ Total number of ways, $T_r = T_{r-1} + T_{r-2}$.

Hence T_r follows the Fibonacci sequence with $T_1 = 1$ (A row of 1 square has 1 way of tiling), $T_2 = 2$ (A row of 2 squares has 2 ways of tiling) and $T_3 = 3$ (A row of 3 squares has 3 ways of tiling).

$\therefore T_r = F_{r+1}$

(e)
[3m]

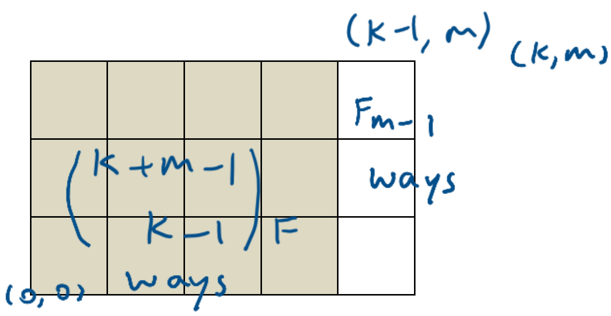
__Case 1:__ The lattice path ends in a vertical step



Number of ways
= no of ways to tile a row with k squares × no of ways to tile a m−1 by k grid

$$= F_{k+1}\binom{k+m-1}{k}_F$$

__Case 2:__ The lattice path ends in a horizontal step



Number of ways
= no of ways to tile a column with m squares × no of ways to tile a m by k−1 grid

$$= F_{m-1}\binom{k+m-1}{k-1}_F$$

$$\therefore \binom{k+m}{k}_F = F_{k+1}\binom{k+m-1}{k}_F + F_{m-1}\binom{k+m-1}{k-1}_F$$