CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

CANDIDATE NAME			
CLASS		INDEX NUMBER	

MATHEMATICS 9758/01

Paper 1

02 Sep 2025
3 hours

Additional Materials: Printed Answer Booklet
List of Formulae (MF27)

READ THESE INSTRUCTIONS FIRST

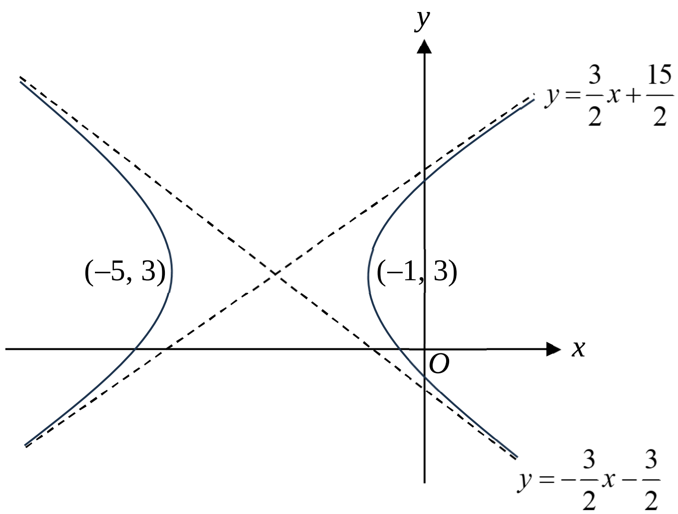
Answer **all** the questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **43** printed pages, including this cover page.

1



The diagram above shows the curve C with equation $\frac{(x+p)^2}{r}-\frac{(y+q)^2}{9}=1$, where p, q and r are constants. The vertices are $(-5,3)$ and $(-1,3)$ and the asymptotes are $y=\frac{3}{2}x+\frac{15}{2}$ and $y=-\frac{3}{2}x-\frac{3}{2}$.

(a) Find the values of p, q and r. [3]

(b) The curve D has equation $(x+3)^2+m(y-3)^2=m$, where m is a positive constant. Find the range of values of m for which curves C and D do not intersect. [2]

Solution:

<table>
<tr><td>Q1</td><td colspan="2"></td></tr>
<tr><td>(a)</td><td>Method ①:
Centre of hyperbola is the mid-point of $(-5,3)$ and $(-1,3)$, i.e.
$\left(\frac{-5-1}{2},\frac{3+3}{2}\right)=(-3,3)$
$$\frac{(x+p)^2}{r}-\frac{(y+q)^2}{9}=1$$
$$\frac{[x-(-p)]^2}{(\sqrt{r})^2}-\frac{[y-(-q)]^2}{9}=1$$
Centre is $(-p,-q)$
Comparing $p=3$ and $q=-3$</td><td>Method ②:
Centre of hyperbola is the point of intersection of the asymptotes.
$y=\frac{3}{2}x+\frac{15}{2}---(1)$
$y=-\frac{3}{2}x-\frac{3}{2}---(2)$
Solving (1) & (2),
$x=-3$
$y=3$
$$\frac{(x+p)^2}{h}-\frac{(y+q)^2}{9}=1$$
$$\frac{[x-(-p)]^2}{(\sqrt{r})^2}-\frac{[y-(-q)]^2}{9}=1$$
Centre is $(-p,-q)$
Comparing $p=3$ and $q=-3$</td></tr>
<tr><td></td><td colspan="2">x-distance from vertex to centre of hyperbola is 2, therefore $\sqrt{r}=2 \Rightarrow r=4$</td></tr>
<tr><td>(b)</td><td colspan="2">$(x+3)^2+m(y-3)^2=m$
$$\frac{(x+3)^2}{(\sqrt{m})^2}+(y-3)^2=1$$
Ellipse, centre (–3, 3)
[Notice that the centre of the ellipse is also the centre of the hyperbola]

$\sqrt{m}<2$
$m<4$

Since m is a positive constant, $0<m<4$.</td></tr>
</table>

2 The nth term of a sequence is given by $u_n = an^2 + bn + c$. The first three terms of this sequence of numbers are 2, 6 and 12.

(a) Find the values of a, b and c. [3]

(b) Given that $\sum_{n=1}^{N} \frac{1}{u_n} = 1 - \frac{1}{N+1}$, find $\sum_{n=8}^{\infty} \frac{1}{u_n}$. [3]

Solution:

Q2	
(a)	When $n=1$, $u_1 = a(1)^2 + b(1) + c = 2$ $a+b+c=2$ ---- (1) When $n=2$, $u_2 = a(2)^2 + b(2) + c = 6$ $4a+2b+c=6$ ---- (2) When $n=3$, $u_3 = a(3)^2 + b(3) + c = 12$ $9a+3b+c=12$ ---- (3) Using G.C., $a=1,\ b=1$ and $c=0$
(b)	**Method ①:** As $N \to \infty$, $\frac{1}{N+1} \to 0$. $\therefore \sum_{n=1}^{\infty} \frac{1}{u_n} = 1$ $\sum_{n=8}^{\infty} \frac{1}{u_n} = \sum_{n=1}^{\infty} \frac{1}{u_n} - \sum_{n=1}^{7} \frac{1}{u_n}$ $= 1 - \left(1 - \frac{1}{7+1}\right)$ $= \frac{1}{8}$ **Method ②:** $\sum_{n=8}^{N} \frac{1}{u_n} = \sum_{n=1}^{N} \frac{1}{u_n} - \sum_{n=1}^{7} \frac{1}{u_n}$ $= \left(1 - \frac{1}{N+1}\right) - \left(1 - \frac{1}{7+1}\right)$ $= \frac{1}{8} - \frac{1}{N+1}$ As $N \to \infty$, $\frac{1}{N+1} \to 0$. $\therefore \sum_{n=8}^{\infty} \frac{1}{u_n} = \frac{1}{8}$

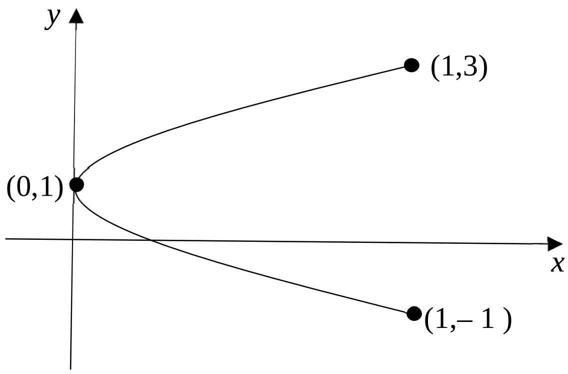
3 A curve C has parametric equations

$$x = \sin^2 t, \quad y = 1 + 2\sin t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

(a) Sketch C, stating clearly the coordinates of the endpoints and the coordinates of the y-intercept(s). [2]

(b) Find the exact cartesian equation of l, the normal to C at the point $\left(\frac{1}{4}, 2\right)$. [4]

Solution:

Q3	
(a)	
(b)	At $\left(\frac{1}{4}, 2\right)$, Since $y = 1 + 2\sin t$, $1 + 2\sin t = 2$ $\sin t = \frac{1}{2}$ $t = \frac{\pi}{6}$ **OR** Since $x = \sin^2 t$ $\sin^2 t = \frac{1}{4}$ $\sin t = \pm\frac{1}{2}$ $t = \frac{\pi}{6}$ or $t = -\frac{\pi}{6}$ (rejected since $y > 0$) $x = \sin^2 t$ $y = 1 + 2\sin t$ $\frac{dx}{dt} = 2\sin t\cos t$ $\frac{dy}{dt} = 2\cos t$ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{2\sin t\cos t} = \frac{1}{\sin t}$ When $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{1}{\frac{1}{2}} = 2$

	Gradient of normal $=-\frac{1}{2}$ Equation of normal: $y-2=-\frac{1}{2}\left(x-\frac{1}{4}\right)$ $y=-\frac{1}{2}x+\frac{17}{8}$ or $8y=-4x+17$ (or any other equivalent form) **OR** Substitute $\left(\frac{1}{4},2\right)$ and gradient of normal $=-\frac{1}{2}$ into $y=mx+c$, $2=-\frac{1}{2}\left(\frac{1}{4}\right)+c$ $c=\frac{17}{8}$ Equation of normal: $y=-\frac{1}{2}x+\frac{17}{8}$

4 It is given that

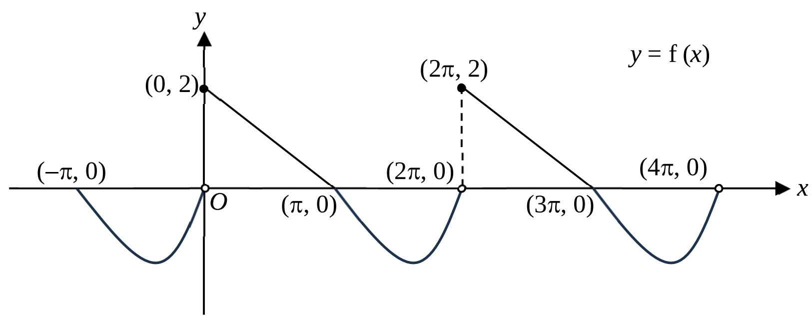
$$f(x) = \begin{cases} 2 - \dfrac{2}{\pi}x & \text{for} \quad 0 \le x < \pi \\ x\sin x & \text{for} \quad \pi \le x < 2\pi \end{cases}$$

and that $f(x) = f(x + 2\pi)$ for all real values of x.

(a) Sketch the graph of $y = f(x)$ for $-\pi \le x < 4\pi$. You do not need to label the coordinates of any stationary point(s). [3]

(b) Find the exact area bounded by the curve $y = f(x)$, the x-axis and the lines $x = 0$ and $x = \dfrac{3\pi}{2}$. [4]

Solution:

Q4	
(a)	
(b)	$\int_{\pi}^{3\pi/2} x\sin x \, dx$ $= \left[-x\cos x\right]_{\pi}^{3\pi/2} - \int_{\pi}^{3\pi/2}(-\cos x)\, dx$ $= \left[\left(-\frac{3\pi}{2}\cos\frac{3\pi}{2}\right) - (-\pi\cos\pi)\right] + \int_{\pi}^{3\pi/2}\cos x \, dx$ $= -\pi + \left[\sin x\right]_{\pi}^{3\pi/2}$ $= -\pi + \sin\frac{3\pi}{2} - \sin\pi$ $= -\pi - 1$ $u = x \quad \frac{dv}{dx} = \sin x$ $\frac{du}{dx} = 1 \quad v = -\cos x$ $\int_{0}^{\frac{3\pi}{2}} f(x)\, dx = \frac{1}{2}(\pi)(2) - \int_{\pi}^{3\pi/2} x\sin x\, dx$ $= \pi - (-\pi - 1)$ $= (2\pi + 1)$ units^2

5 A sequence of negative real numbers $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = -1 \quad \text{and} \quad u_{n+1} = \frac{-2u_n + 6}{u_n - 1}, \text{ for } n \geq 1.$$

(a) Given that $u_n \to l$ as $n \to \infty$, find the value of l. [3]

(b) Sketch the graph of $y = \dfrac{-2x+6}{x-1} - x$, stating the equations of the asymptotes and the coordinates of the points where it crosses the axes. [3]

(c) Hence show that $u_{n+1} > u_n$ if $u_n < l$. [2]

<u>Solution:</u>

<table>
<tr><td>Q5</td><td></td></tr>
<tr><td>(a)</td><td>

As $n \to \infty$, $u_n \to l$, $u_{n+1} \to l$.

$l = \dfrac{-2l+6}{l-1}$

<u>Method ①:</u> Algebraic

$l(l-1) = -2l+6$

$l^2 - l = -2l+6$

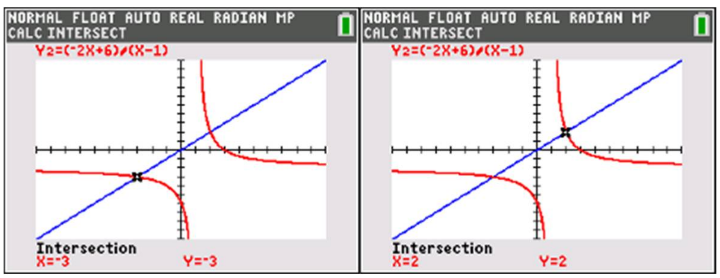
$l^2 + l - 6 = 0$

$(l+3)(l-2) = 0$

$l = -3$ or $l = 2$ $(\text{reject } \because l < 0)$

$\therefore l = -3$

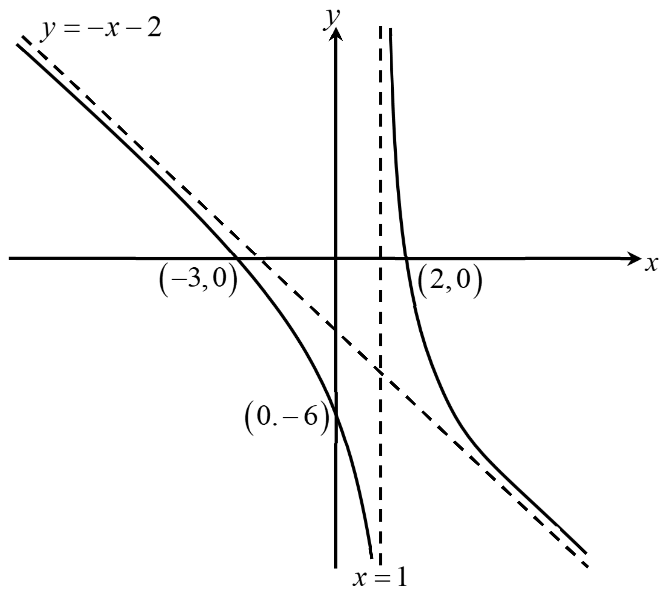
<u>Method ②:</u> Graphical



$l = -3$ or $l = 2$ $(\text{reject } \because l < 0)$

$\therefore l = -3$

</td></tr>
</table>

(b)	
(c)	$u_{n+1}-u_n=\dfrac{-2u_n+6}{u_n-1}-u_n$ From the graph of $y=\dfrac{-2x+6}{x-1}-x$, when $x<-3$, $y>0$. If $u_n<-3$, $u_{n+1}-u_n>0$ $u_{n+1}>u_n$ (shown)

6 Relative to the origin O, the position vectors of points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $\mathbf{a}$ is a unit vector and $\mathbf{b}$ is perpendicular to $\mathbf{b}-\mathbf{a}$.

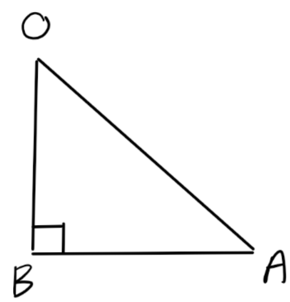
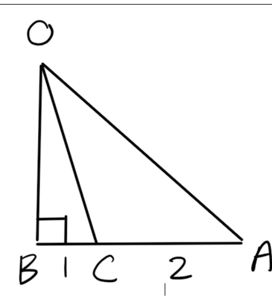
(a) Show that $0<|\mathbf{b}|<1$. [3]

It is given further that $|\mathbf{b}|=\dfrac{1}{2}$ and C lies on AB such that $AC:CB=2:1$.

(b) Find the value of $|\mathbf{a}\cdot\mathbf{c}|$ and state the geometrical interpretation of $|\mathbf{a}\cdot\mathbf{c}|$. [4]

(c) Find the value of $\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}\times\mathbf{c}|}$. [2]

Solution:

Q6	
(a)	Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Since $\underset{\sim}{b}$ and $\underset{\sim}{b}-\underset{\sim}{a}$ are perpendicular, $\underset{\sim}{b}\cdot\left(\underset{\sim}{b}-\underset{\sim}{a}\right)=0$ $\underset{\sim}{b}\cdot\underset{\sim}{b}-\underset{\sim}{b}\cdot\underset{\sim}{a}=0$ $\left\|\underset{\sim}{b}\right\|^2-\left\|\underset{\sim}{a}\right\|\left\|\underset{\sim}{b}\right\|\cos\theta=0$ Since $\underset{\sim}{a}$ is a unit vector, $\left\|\underset{\sim}{b}\right\|^2-\left\|\underset{\sim}{b}\right\|\cos\theta=0$ $\left(\left\|\underset{\sim}{b}\right\|\right)\left(\left\|\underset{\sim}{b}\right\|-\cos\theta\right)=0$ $\left\|\underset{\sim}{b}\right\|-\cos\theta=0$, since $\left\|\underset{\sim}{b}\right\|>0$ $\left\|\underset{\sim}{b}\right\|=\cos\theta$ If $\cos\theta=1$, $\theta=0^\circ$. This implies that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel which contradicts the fact that $\underset{\sim}{b}$ and $\underset{\sim}{b}-\underset{\sim}{a}$ are perpendicular. Furthermore, $\cos\theta=\left\|\underset{\sim}{b}\right\|>0$. Since $0<\cos\theta<1$, $0<\left\|\underset{\sim}{b}\right\|<1$ (shown) OR Since OAB is a right-angled triangle, $\|a\|^2=\|b\|^2+\|b-a\|^2$ Since $\|a\|$ is the hypotenuse and $\|a\|=1$, $\|b\|<1$ Since $\|b\|>0$ $0<\left\|\underset{\sim}{b}\right\|<1$ (shown) 
(b)	By Ratio Theorem, $\underset{\sim}{c}=\frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}$ 

$$|\underset{\sim}{a}\cdot\underset{\sim}{c}| = \left|\underset{\sim}{a}\cdot\left(\frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}\right)\right|$$

$$= \left|\frac{1}{3}\underset{\sim}{a}\cdot\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{a}\cdot\underset{\sim}{b}\right|$$

$$= \left|\frac{1}{3}\underset{\sim}{a}\cdot\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}\cdot\underset{\sim}{b}\right|, \text{ since } \underset{\sim}{a}\cdot\underset{\sim}{b}=\underset{\sim}{b}\cdot\underset{\sim}{b} \text{ from } (a)$$

$$= \left|\frac{1}{3}|\underset{\sim}{a}|^2+\frac{2}{3}|\underset{\sim}{b}|^2\right|$$

$$= \left|\frac{1}{3}(1)^2+\frac{2}{3}\left(\frac{1}{2}\right)^2\right|$$

$$= \frac{1}{2}$$

OR

$$BA = \sqrt{1^2-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$$

$$BC = \frac{BA}{3} = \frac{\sqrt{3}}{6}$$

$$\angle BOC = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{6}}{\frac{1}{2}}\right) = \frac{\pi}{6}$$

$$\angle BOA = \cos^{-1}\left(\frac{\frac{1}{2}}{1}\right) = \frac{\pi}{3}$$

$$\angle AOC = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$$|\underset{\sim}{c}| = \sqrt{\left(\frac{\sqrt{3}}{6}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{3}}$$

$$|\underset{\sim}{a}\cdot\underset{\sim}{c}| = \left||\underset{\sim}{a}||\underset{\sim}{b}|\cos\frac{\pi}{6}\right|$$

$$= \left|(1)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)\right|$$

$$= \frac{1}{2}$$

$|\underset{\sim}{a}\cdot\underset{\sim}{c}| = |\underset{\sim}{c}\cdot\underset{\sim}{a}|$ is the length of projection of $\underset{\sim}{c}$ on $\underset{\sim}{a}$.

(c)	**<u>Method ①:</u> Cross Product**

$$\frac{\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|}{\left|\underset{\sim}{a}\times\underset{\sim}{c}\right|}=\frac{\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|}{\left|\underset{\sim}{a}\times\left(\frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}\right)\right|}$$

$$=\frac{\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|}{\left|\frac{2}{3}\underset{\sim}{a}\times\underset{\sim}{b}\right|}, \text{ where } \underset{\sim}{a}\times\underset{\sim}{a}=0$$

$$=\frac{\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|}{\frac{2}{3}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|}$$

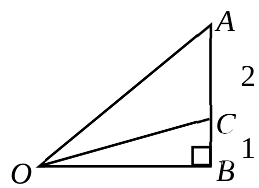
$$=\frac{3}{2}$$

<u>Method ②:</u> Area of triangle

Since $\underset{\sim}{b}$ and $\underset{\sim}{b}-\underset{\sim}{a}$ are perpendicular, and C lies on AB,

$$\frac{\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|}{\left|\underset{\sim}{a}\times\underset{\sim}{c}\right|}=\frac{\frac{1}{2}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|}{\frac{1}{2}\left|\underset{\sim}{a}\times\underset{\sim}{c}\right|}$$

$$=\frac{\text{Area of }\Delta OAB}{\text{Area of }\Delta OAC}$$



$$=\frac{\frac{1}{2}(3)(OB)}{\frac{1}{2}(2)(OB)}$$

$$=\frac{3}{2}$$

<u>Method 3:</u>

$$\frac{\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|}{\left|\underset{\sim}{a}\times\underset{\sim}{c}\right|}=\frac{\left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\left|\sin\frac{\pi}{3}\right|}{\left|\underset{\sim}{a}\right|\left|\underset{\sim}{c}\right|\left|\sin\frac{\pi}{6}\right|}$$

$$=\frac{(1)\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{1\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\right)}$$

$$=\frac{3}{2}$$

7 It is given that the curve $y = \mathrm{f}(x)$ satisfies the equation $\ln y = \frac{\pi}{4} - \tan^{-1}\left(\mathrm{e}^{x}\right)$.

(a) Show that $\left(1+\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + y\mathrm{e}^{x} = 0$. [2]

(b) Find the Maclaurin series for y, up to and including the term in x^2. [5]

(c) Deduce the series expansion for $y = \dfrac{\mathrm{e}^{\frac{\pi}{4} - \tan^{-1}\left(\mathrm{e}^{x}\right)}}{\sqrt{1-x}}$, up to and including the term in x^2. [4]

Solution:

Q7	
(a)	$\ln y = \frac{\pi}{4} - \tan^{-1}\left(e^x\right)$ $\frac{1}{y}\frac{dy}{dx} = -\frac{e^x}{1+\left(e^x\right)^2}$ $\left(1+e^{2x}\right)\frac{dy}{dx} = -ye^x$ $\left(1+e^{2x}\right)\frac{dy}{dx} + ye^x = 0$ (shown)
(b)	$\left(1+e^{2x}\right)\frac{dy}{dx} + ye^x = 0$ $\left(1+e^{2x}\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(2e^{2x}\right) + ye^x + e^x\frac{dy}{dx} = 0$ $\left(1+e^{2x}\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(2e^{2x}+e^x\right) + ye^x = 0$ When $x=0$, $\ln y = \frac{\pi}{4} - \tan^{-1}\left(e^0\right) = \frac{\pi}{4} - \frac{\pi}{4} = 0$ $y = 1$ $\left(1+e^{2x}\right)\frac{dy}{dx} + ye^x = 0$ $\left(1+e^0\right)\frac{dy}{dx} + 1 = 0$ $\frac{dy}{dx} = -\frac{1}{2}$ $\left(1+e^0\right)\frac{d^2y}{dx^2} + \left(-\frac{1}{2}\right)\left(2e^0+e^0\right) + 1 = 0$ $2\frac{d^2y}{dx^2} - \frac{3}{2} + 1 = 0$ $\frac{d^2y}{dx^2} = \frac{1}{4}$ $y = 1 - \frac{1}{2}x + \frac{1}{4}\cdot\left(\frac{x^2}{2}\right) + \ldots$ $y = 1 - \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$

(c)	$\ln y = \frac{\pi}{4} - \tan^{-1}\left(e^x\right)$ $y = e^{\frac{\pi}{4} - \tan^{-1}\left(e^x\right)}$ $y \approx 1 - \frac{1}{2}x + \frac{1}{8}x^2$ from part (b) $y = \frac{e^{\frac{\pi}{4} - \tan^{-1}\left(e^x\right)}}{\sqrt{1-x}}$ $= e^{\frac{\pi}{4} - \tan^{-1}\left(e^x\right)}(1-x)^{-\frac{1}{2}}$ $= \left(1 - \frac{1}{2}x + \frac{1}{8}x^2 + \ldots\right)\left[1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2 + \ldots\right]$ $= \left(1 - \frac{1}{2}x + \frac{1}{8}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right)$ $= 1 - \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{3}{8}x^2 + \ldots$ $= 1 + \frac{1}{4}x^2 + \ldots$

8 The plane p passes through the points A, B and C with coordinates $(1,0,2)$, $(2,-1,3)$ and $(-4,-1,0)$ respectively.

(a) Show that a cartesian equation of p is $x-y-2z=-3$. [3]

The line l has equation $\mathbf{r}=\begin{pmatrix}2\\0\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\-1\end{pmatrix}$, $\lambda\in\mathbb{R}$.

(b) Find the acute angle between l and p. [2]

It is given that a variable point R lies on p and is at a distance of $\sqrt{22}$ from the point Q with coordinates $(3,4,-2)$.

(c) Find the foot of perpendicular from Q to p. [4]

(d) Hence describe geometrically the path traced by R. [2]

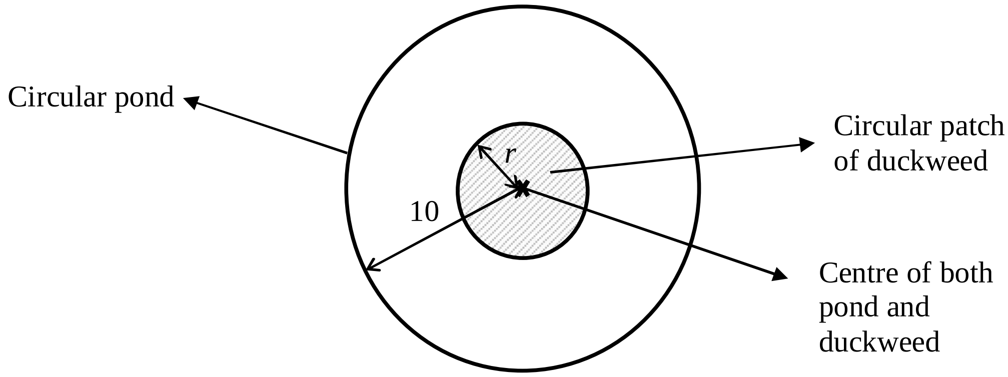
Solution:

Q8	
(a)	$\overrightarrow{OA}=\begin{pmatrix}1\\0\\2\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}2\\-1\\3\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}-4\\-1\\0\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}2\\-1\\3\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $\overrightarrow{AC}=\begin{pmatrix}-4\\-1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}-5\\-1\\-2\end{pmatrix}$ $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}-5\\-1\\-2\end{pmatrix}=\begin{pmatrix}3\\-3\\-6\end{pmatrix}=3\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ A normal of p is $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$. $p:\ \mathbf{r}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=-3$ A cartesian equation of p is $x-y-2z=-3$. (shown)
(b)	**Method ①:** Let the acute angle between l and p be θ. Let the acute angle between the normal of p and the direction vector of l be α. $\alpha=\cos^{-1}\dfrac{\left\|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right\|\left\|\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right\|}=\cos^{-1}\dfrac{5}{\sqrt{6}\sqrt{6}}=33.55730976^\circ$ $\theta=90^\circ-33.55730976$ $=56.44269024^\circ$ $\approx 56.4^\circ$ (to 1 d.p.) or 0.985 (to 3 s.f.)

	Method ②: Let the acute angle between l and p be θ. $\theta = \sin^{-1} \frac{\left\lvert \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\rvert}$ $= \sin^{-1} \frac{5}{\sqrt{6}\sqrt{6}}$ $= 56.44269024°$ $\approx 56.4° \ (\text{to 1 d.p.})$
(c)	Let F be the foot of perpendicular from Q to p. **Method ①:** $l_{QF} : \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \ \mu \in \mathbb{R}$ Since F lies on l_{QF}, $\overrightarrow{OF} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$, for some $\mu \in \mathbb{R}$. Since F also lies on p, $\overrightarrow{OF} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -3$. $\left[\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -3$ $1(3+\mu) - (4-\mu) - 2(-2-2\mu) = -3$ $3 + 6\mu = -3$ $\Rightarrow \mu = -1$ $\therefore \overrightarrow{OF} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$ The foot of perpendicular from Q to p is $(2,5,0)$.

	Method ②: $\overrightarrow{QF}=\left(\overrightarrow{QA}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a normal vector of p $=\left(\begin{pmatrix}1-3\\0-4\\2+2\end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right)\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ $=\frac{1}{6}\left(\begin{pmatrix}-2\\-4\\4\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right)\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ $=-\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ $\therefore\overrightarrow{OF}=\overrightarrow{OQ}+\overrightarrow{QF}=\begin{pmatrix}3\\4\\-2\end{pmatrix}-\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}2\\5\\0\end{pmatrix}$ The foot of perpendicular from Q to p is $(2,5,0)$.
(d)	$\overrightarrow{QF}=\begin{pmatrix}2\\5\\0\end{pmatrix}-\begin{pmatrix}3\\4\\-2\end{pmatrix}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$ $\left\|\overrightarrow{QF}\right\|=\sqrt{(-1)^2+1^2+2^2}=\sqrt{6}$ $RF=\sqrt{22-6}=4$ The path traced by R is a circle that lies on p with **centre** $(2,5,0)$ and **radius** 4.

9 A Koi pond farmer introduces a circular patch of duckweed at the center of his circular pond to enhance water purification and provide supplemental nutrition for his Koi, as shown in the figure below.



The duckweed grows over time t days, forming a continuous circular patch on the pond's surface. It is assumed that the duckweed forms a thin, floating layer with negligible thickness. The rate of increase of the area covered by duckweed is directly proportional to the area of the pond that remains uncovered by the duckweed.

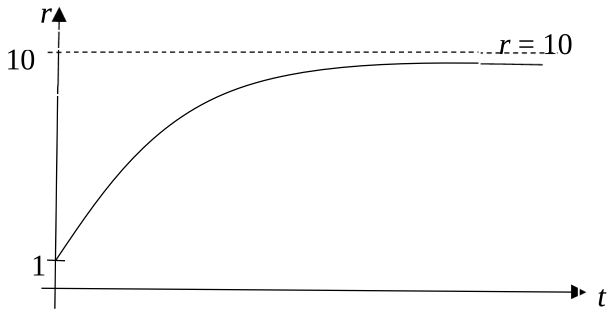
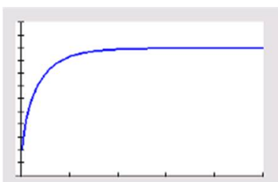
(a) Given that the radius of the pond is 10 m, show that the radius, r m of the circular patch of duckweed satisfies the differential equation $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k(100 - r^2)}{2r}$, $k > 0$. [2]

(b) The initial radius of the circular patch of duckweed is 1 m. Solve the differential equation, expressing r in the form $r = \sqrt{P - Qe^{-kt}}$, where P and Q are constants to be determined. [6]

(c) Sketch the graph of r against t. [3]

Solution:

Q9	
(a)	Area of pond $=\pi\left(10^2\right)=100\pi$ Area of duckweed $=\pi r^2$ Since the duckweed grows at a rate that is directly proportional to the uncovered area of the pond, $\dfrac{\mathrm{d}A}{\mathrm{d}t}=k\left[100\pi-\pi r^2\right]$, $k>0$ But, $A=\pi r^2$ $\therefore \dfrac{\mathrm{d}A}{\mathrm{d}t}=2\pi r\dfrac{\mathrm{d}r}{\mathrm{d}t}$ So, $2\pi r\dfrac{\mathrm{d}r}{\mathrm{d}t}=k\pi(100-r^2)$ $\therefore \dfrac{\mathrm{d}r}{\mathrm{d}t}=\dfrac{k(100-r^2)}{2r}$ (shown)
(b)	$\dfrac{\mathrm{d}r}{\mathrm{d}t}=\dfrac{k(100-r^2)}{2r}$ Separating the variables and integrating wrt t, $\int\dfrac{r}{100-r^2}\ \mathrm{d}r=\dfrac{k}{2}\int 1\ \mathrm{d}t$ $-\dfrac{1}{2}\int\dfrac{-2r}{100-r^2}\ \mathrm{d}r=\dfrac{k}{2}t+C$ $-\dfrac{1}{2}\ln\left\|100-r^2\right\|=\dfrac{k}{2}t+C$ $\ln\left\|100-r^2\right\|=-kt-2C$ $\left\|100-r^2\right\|=e^{-kt-2C}$ $100-r^2=\pm e^{-2C}e^{-kt}$ $100-r^2=Qe^{-kt}$, where $Q=\pm e^{-2C}$ $r^2=100-Qe^{-kt}$ $r=\pm\sqrt{100-Qe^{-kt}}$ Since $r>0$, $\therefore$ $r=\sqrt{100-Qe^{-kt}}$ When $\mathrm{t}=0$, $r=1$, $1=\sqrt{100-Qe^0}$ $1=100-Q$ $Q=100-1=99$

	Hence, $r=\sqrt{100-99e^{-kt}}$ Where $P=100$ and $Q=99$
(c)	Since $k>0$,  

10 Mr Huat is saving for a car that he plans to buy in the future. He needs to save a minimum of \$150,000. A savings plan allows him to deposit \$6,000 into a savings account on the first day of every month. At the end of each month, the total amount in the savings account (including interest) is increased by 3%. Mr Huat makes an initial deposit on 1 January 2026.

(a) Show that the total amount, correct to 2 decimal places, in the savings account at the end of December 2026 is \$87,706.74. [2]

(b) Find the month and year in which the total amount in the savings account will first exceed \$150,000. Explain whether this occurs on the first or last day of the month. [6]

Mr Huat can also choose to save regularly in a different savings account which offers no interest. He makes an initial deposit on 1 January 2026 of \$6,000. On the first day of each subsequent month, he deposits \$$d$ more than he deposited in the previous month.

(c) Find, in terms of d, the total amount in the savings account at the end of December 2026. [2]

(d) Hence, find the minimum value of d such that Mr Huat can buy the car by end of December 2026. Give your answer correct to the nearest integer. [2]

Solution:

Q10	
(a)	See table and working below

Month n	Start of month	End of month
1	6000	$6000(1.03)$
2	$6000(1.03)+6000$	$\left[6000(1.03)+6000\right](1.03)$ $=6000(1.03)^2+6000(1.03)$
3	$6000(1.03)^2+6000(1.03)+6000$	$\left[6000(1.03)^2+6000(1.03)+6000\right](1.03)$ $=6000(1.03)^3+6000(1.03)^2+6000(1.03)$
$\vdots$	$\vdots$	$\vdots$
12	$6000(1.03)^{11}+6000(1.03)^{10}+\ldots+6000(1.03)+6000$	$6000(1.03)^{12}+6000(1.03)^{11}+\ldots+6000(1.03)$
$\vdots$	$\vdots$	$\vdots$
n	$6000(1.03)^{n-1}+6000(1.03)^{n-2}+\ldots+6000(1.03)+6000$	$6000(1.03)^{n}+6000(1.03)^{n-1}+\ldots+6000(1.03)$

Total amount in the savings account at the end of December 2026

$$=\underbrace{6000(1.03)^{12}+6000(1.03)^{11}+\ldots+6000(1.03)}_{\text{Sum of GP: First term=6000(1.03), Common Ratio=1.03, No. of terms=12}}$$

$$=\frac{6000(1.03)\left(1.03^{12}-1\right)}{1.03-1}$$

$$=87706.74269$$

$$\approx 87706.74 \ (\text{to 2 d.p.})$$

(b)

Total amount in the savings account at the end of the *n*th month

$$=\underbrace{6000(1.03)^{n}+6000(1.03)^{n-1}+\ldots+6000(1.03)}_{\text{Sum of GP: First term=6000(1.03), Common Ratio=1.03, No. of terms=}n}$$

$$=\frac{6000(1.03)\left(1.03^{n}-1\right)}{1.03-1}$$

$$=206000\left(1.03^{n}-1\right)$$

$$206000\left(1.03^{n}-1\right)>150000$$

$$1.03^{n}-1>\frac{150000}{206000}$$

$$1.03^{n}>1.7282$$

<u>Method ① to find least *n*:</u> Algebraic

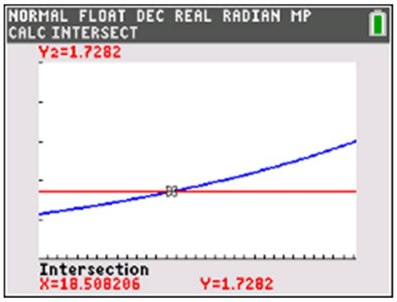
$n\ln 1.03 > \ln 1.7282$

$n > 18.507$

Least $n = 19$

Total amount in the savings account will first exceed \$150,000 on July 2027.

<u>Method ② to find least *n*:</u> Graphical



From the graph,

$n > 18.507$

Least $n = 19$

Total amount in the savings account will first exceed \$150,000 on July 2027.

When $n = 18$,

Total amount in the savings account at the end of the 18th month (i.e. June 2027)

$= 206000\left(1.03^{18} - 1\right)$

$= 144701.2106$

Total amount in the savings account at start of the 19th month (i.e. July 2027)

$= 144701.2106 + 6000$

$= 150701.2106 > 150000$

Hence, this occurs on the first day of the month.

(c) Total amount in the savings account at the end of December 2026

$= 6000 + (6000 + d) + \ldots + (6000 + 11d)$

$= \dfrac{12}{2}\left[6000 + (6000 + 11d)\right]$

$= 72000 + 66d$

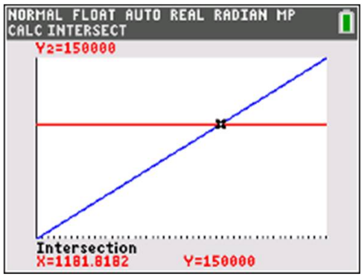
(d)

<u>Method ① to find minimum *d*:</u> Algebraic

$$72000 + 66d \geq 150000$$
$$66d \geq 78000$$
$$d \geq 1181.818182$$

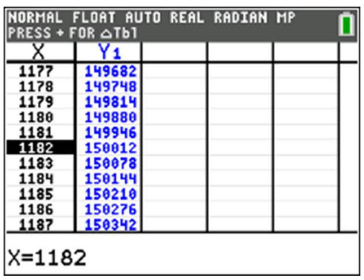
Hence, minimum value of $d = 1182$ $\left(\text{to nearest integer}\right)$.

<u>Method ② to find minimum *d*:</u> Graphical



Hence, minimum value of $d = 1182$ $\left(\text{to nearest integer}\right)$.

<u>Method ③ to find minimum *d*:</u> GC Table


NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

X	Y1
1177	149682
1178	149748
1179	149814
1180	149880
1181	149946
1182	150012
1183	150078
1184	150144
1185	150210
1186	150276
1187	150342

X=1182

Hence, minimum value of $d = 1182$ $\left(\text{to nearest integer}\right)$.

11 **(a)** **(i)** Show that $\dfrac{3x^2}{(x+1)(3x^2+x+1)}$ can be expressed as

$$\frac{A}{x+1}+\frac{Bx+C}{3x^2+x+1},$$

where A and B are constants to be determined. [2]

(ii) Hence, find $\displaystyle\int \frac{3x^2}{(x+1)(3x^2+x+1)}\,\mathrm{d}x$. [4]

(b) The region R is bounded by the curve $y=\dfrac{3x^2}{(x+1)(3x^2+x+1)}$, the line $x=4$ and the x-axis.

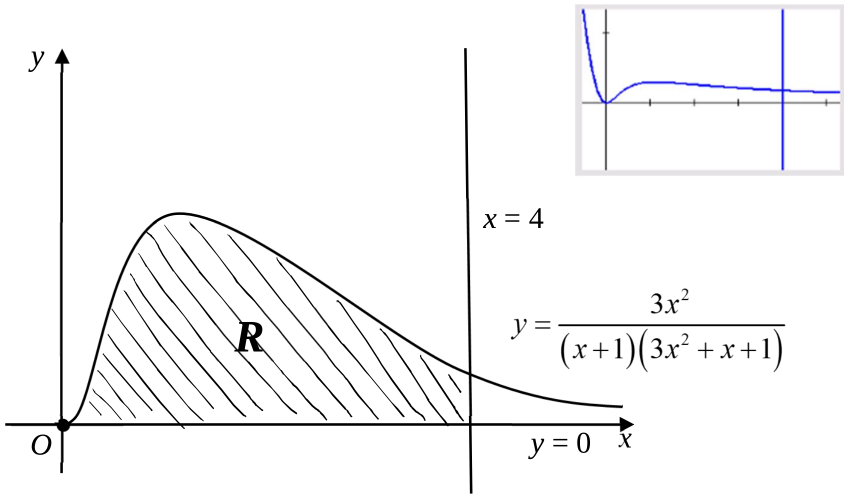
(i) Sketch the graph of $y=\dfrac{3x^2}{(x+1)(3x^2+x+1)}$ for $x\geq 0$, giving the coordinates of any intercepts with the axes and the equations of any asymptotes. Shade the region R on your sketch. [3]

(ii) Find the exact area of R. [3]

(iii) Find the volume of the solid generated when R is rotated through 2π radians about the x-axis. [2]

Solution:

Q11	
(a)(i)	Let $\dfrac{3x^2}{(1+x)(3x^2+x+1)}=\dfrac{A}{1+x}+\dfrac{Bx+C}{3x^2+x+1}$ By comparing numerators, $3x^2=A(3x^2+x+1)+(Bx+C)(1+x)$ Substitute $x=-1$, $3(-1)^2=A[3(-1)^2+(-1)+1]$ $A=1$ Substitute $x=0$, $3(0)^2=(1)[3(0)^2+(0)+1]+[B(0)+C](1+0)$ $C=-1$ Substitute $x=1$, $3(1)^2=(1)[3(1)^2+(1)+1]+[B(1)-1](1+1)$ $B=0$ Hence, $\dfrac{3x^2}{(1+x)(3x^2+x+1)}=\dfrac{1}{1+x}-\dfrac{1}{3x^2+x+1}$
(a)(ii)	$\displaystyle\int\frac{3x^2}{(x+1)(3x^2+x+1)}\,\mathrm{d}x$ $\displaystyle=\int\left(\frac{1}{x+1}-\frac{1}{3x^2+x+1}\right)\mathrm{d}x$ $\displaystyle=\ln\lvert x+1\rvert-\frac{1}{3}\int\frac{1}{x^2+\frac{x}{3}+\frac{1}{3}}\,\mathrm{d}x$ $\displaystyle=\ln\lvert x+1\rvert-\frac{1}{3}\int\frac{1}{x^2+\frac{x}{3}+\left(\frac{1}{6}\right)^2-\left(\frac{1}{6}\right)^2+\frac{1}{3}}\,\mathrm{d}x$ $\displaystyle=\ln\lvert x+1\rvert-\frac{1}{3}\int\frac{1}{\left(x+\frac{1}{6}\right)^2+\frac{11}{36}}\,\mathrm{d}x$ $\displaystyle=\ln\lvert x+1\rvert-\frac{1}{3}\left(\frac{1}{\sqrt{11/36}}\right)\tan^{-1}\left(\frac{x+1/6}{\sqrt{11/36}}\right)+C$ $\displaystyle=\ln\lvert x+1\rvert-\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right)+C$

(b)(i)	
(b)(ii)	Exact area of R $\int_0^4 \dfrac{3x^2}{(x+1)(3x^2+x+1)}\,\mathrm{d}x$ $=\left[\ln\lvert x+1\rvert-\dfrac{2}{\sqrt{11}}\tan^{-1}\left(\dfrac{6x+1}{\sqrt{11}}\right)\right]_0^4$ $=\left[\ln\lvert 4+1\rvert-\dfrac{2}{\sqrt{11}}\tan^{-1}\left(\dfrac{6(4)+1}{\sqrt{11}}\right)\right]-\left[\ln\lvert 0+1\rvert-\dfrac{2}{\sqrt{11}}\tan^{-1}\left(\dfrac{6(0)+1}{\sqrt{11}}\right)\right]$ $=\ln 5-\dfrac{2}{\sqrt{11}}\tan^{-1}\left(\dfrac{25}{\sqrt{11}}\right)+\dfrac{2}{\sqrt{11}}\tan^{-1}\left(\dfrac{1}{\sqrt{11}}\right)$ $=\ln 5+\dfrac{2}{\sqrt{11}}\left[\tan^{-1}\left(\dfrac{1}{\sqrt{11}}\right)-\tan^{-1}\left(\dfrac{25}{\sqrt{11}}\right)\right]$
(b)(iii)	Volume required $=\pi\int_0^4\left[\dfrac{3x^2}{(x+1)(3x^2+x+1)}\right]^2\,\mathrm{d}x$ $=0.715$ (3s.f.)