Name: ________________________________ (   )   **Class:** 25 / ___

Additional Notes

Topic 7 Gravitational Field

Content

- Gravitational Field
- Gravitational force between point masses
- Gravitational field of a point mass
- Gravitational field near to the surface of the Earth
- Gravitational potential
- Circular orbits

Learning Outcomes:
Candidates should be able to:

(a) show an understanding of the concept of a gravitational field as an example of field of force and define gravitational field strength at a point as the gravitational force exerted per unit mass placed at that point.

(b) recognise the analogy between certain qualitative and quantitative aspects of gravitational and electric fields. (will be discussed in JC2)

(c) recall and use Newton's law of gravitation in the form $F = \frac{Gm_1m_2}{r^2}$.

(d) derive, from Newton's law of gravitation and the definition of gravitational field strength, the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass.

(e) recall and apply the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass to new situations or to solve related problems.

(f) show an understanding that near the surface of Earth g is approximately constant and equal to the acceleration of free fall.

(g) define the gravitational potential at a point as the work done per unit mass in bringing a small test mass from infinity to that point.

(h) solve problems using the equation $\phi = -\frac{GM}{r}$ for the gravitational potential in the field of a point mass.

(i) analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes.

(j) show an understanding of geostationary orbits and their application.

Nature of Science
A good theory should lead to new discoveries. Newton's theory of gravitation has enabled us to work out problems connected with space travel, leading to new knowledge about the solar system and the discovery of new planets.

Relating Science and Society
Explore how data from satellites is used in Singapore, making reference to the work of the Centre for Remote Imaging, Sensing and Processing (CRISP) and the Meteorological Service Singapore (MSS). Examples of how the data is used by Singapore include the monitoring of the haze problem and the prediction of the weather. By analysing satellites currently deployed around the globe, compare the functionality and purposes of various types of satellites (e.g. geostationary versus polar and low earth orbit), and weigh the costs and benefits of launching our own satellites.

60 ASMC, Regional Haze Situation. Retrieved from [http://asmc.asean.org/home/].
61 NEA, Haze updates. Retrieved from [http://www.haze.gov.sg/haze-updates].

Introduction

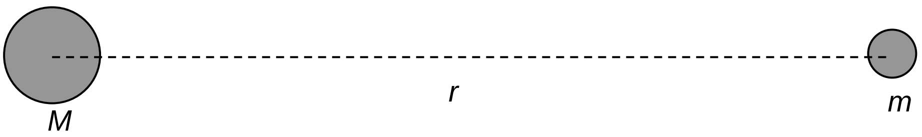
Before 1686, much data had been collected on the motions of the Moon and the planets, but scientists had yet to understand the forces which governed those motions. Isaac Newton knew that, from his laws of motion, a net force had to be acting on the Moon. Otherwise, the Moon would move in a straight-line path rather than in its almost circular orbit. Along with other reasons, this led Newton to postulate the Universal Law of Gravitation.

A.1 Newton's Law of Gravitation

Newton's Law of Gravitation:

The mutual force of attraction between any two point masses is proportional to the product of the masses and inversely proportional to the square of their separation.

- Mathematically, the magnitude of the gravitational force F between two masses M and m, separated by a distance r, is given as



$$F \propto \frac{Mm}{r^2}$$

$$F = G\frac{Mm}{r^2}$$

For spherical mass of uniform densities, r will be centre to centre distance.

where G is the universal gravitational constant. (G = 6.67 x 10^{-11} N m^2 kg^{-2})

- Note that:
  - When stating the law, you must specify that the masses are point masses. In practice, point masses are not possible, but the law applies to masses whose separation is much greater than their dimensions (separation >> radius). The law also applies to spheres of *uniform* densities, given their separation is much greater than their dimensions. In which case, the mass of the sphere is taken to be concentrated at its *centre*.

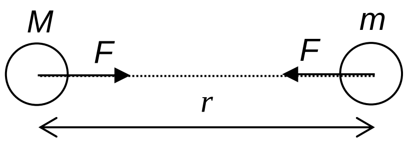
Additional Notes

- Gravitational forces are always <u>attractive</u>. Sometimes a negative sign is used to indicate the attractive nature of the force. The magnitude of the gravitational force between two masses is actually given by:

$$F = G\frac{Mm}{r^2}$$

- The force acts along the line joining the centres of the bodies.

- The mutual attractive forces between the two bodies form an action-reaction pair. *M is attracted towards m with a force F towards the right. At the same time, m is attracted by M with an equal force F but in the opposite direction.* (Newton's 3rd Law)



- Since the value of *G* is very small, gravitational forces are significant only when the bodies have significant masses.

Worked example 1

Two identical lead spheres of mass 3.00 kg each are placed such that the separation is 10.0 m from the centre of one sphere to another. They attract each other with a gravitational force *F*. Calculate the force *F*. (6.00 x 10^{-12} N)

Solution

$$F = G\frac{Mm}{r^2}$$

$$= 6.67 \times 10^{-11} \times \frac{3.00 \times 3.00}{(10.0)^2}$$

$$= 6.00 \times 10^{-12} \text{ N}$$

Example 1

Find the force between
(a) the Earth and a 50 kg girl
(b) a 60 kg boy and a 50 kg girl, one metre apart.

[mass of Earth = 6.0 $\times$ 10^{24} kg, radius of Earth = 6.4 $\times$ 10^6 m]

Example 2

The average distance separating the centre of Earth and the centre of Moon is 3.84×10^8 m. A spaceship of mass 3.00×10^4 kg travels along the line joining the centre of mass of the Earth and the Moon.

(a) Determine the net gravitational force on the spaceship due to the Earth and the Moon when it is located halfway between them.

(b) Determine the distance between the spaceship and the Earth where no net gravitational force acts on the spaceship.

[mass of Earth = 6.00×10^{24} kg, mass of Moon = 7.36×10^{22} kg]

Thinking Process

Key question on objective:

Is force a vector or scalar? So does the direction matter?

- Vector. Direction matters.

Key questions to solve the question:

(a) What is the question asking for?

- Net gravitational force on spaceship

What is the approach to take?

- Find gravitational force between spaceship and earth; between spaceship and moon
- Then take vector sum of the two forces to find the net force

(b) What is the net force at the required point? Where is the point?

- Need the point where $F_{ES} = F_{MS}$

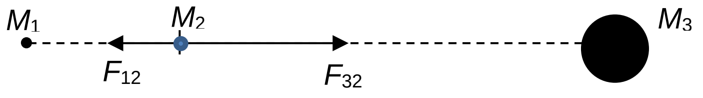
Express in terms of *r*, where distance from centre of earth to the spaceship is *r*.

- I will need a point, where distance from centre of earth to the spaceship is *r*, and the distance from the centre of the moon to the spaceship will be ($3.48 \times 10^8 - r$) m

Additional Notes

Learning Point:

$F_{12} = \dfrac{GM_1M_2}{r^2}$ is the gravitational force that M_1 exerts on M_2. If there is a third mass M_3, then M_3 also exerts a gravitational force on M_2. To find the net gravitational force acting on M_2, the resultant of the 2 forces must be found.



A.2 The Gravitational Field

- A *field* is a model used by physicists to help them understand how objects not in direct contact with each other (called *action-at-a-distance*) can affect each other's behaviour.
- An action-at-a-distance effect in which one body A exerts a force on another body B not in contact with it, can be regarded as due to a 'field of force' in the region around A.
- **Field of force** is a region of space where a body experiences a force.
- The gravitational field is an example of a field of force.
- A mass M creates a force field around itself. Any other mass m placed in this field would be attracted towards M. The force field round mass M is called the gravitational field of M.

A.2.1 Definition of a Gravitational Field

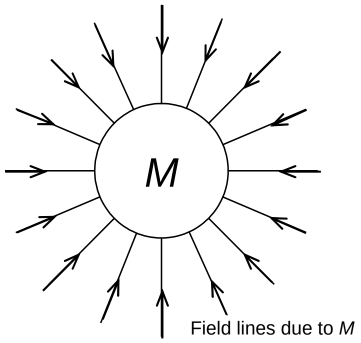
Memorize

- Gravitational field is a region of space where a mass experiences a force. The direction of the field is the direction of the force on the mass.

Representing gravitational field by field lines

Field lines can be drawn to visualize a gravitational field.

E.g. Gravitational field around the Earth can be represented as shown:



Note:

- Closer lines of force indicates stronger gravitational field.
- The field is actually three dimensional in nature.
- The field is directed towards the centre of the Earth and it gets stronger as it gets closer to the Earth's surface.

Additional Notes

A.2.2 Gravitational Field Strength, *g*

- To *quantify* the strength of a mass's gravitational field at a certain point, the term *gravitational field strength, g* is introduced.

- The gravitational field strength at a point is defined as the <u>gravitational force exerted per unit mass</u> placed at that point.

Memorize

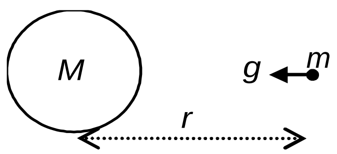
- $\boxed{g = \dfrac{F}{m}}$

  unit of g: **N kg^{-1}** or **m s^{-2}**
  g is a ***vector*** quantity.
  Its direction is that which a small test mass would move when placed in the field.

- Gravitational field strength is always associated with an attractive force only (compare with electric field, where the forces may be attractive or repulsive).

- Derivation of $g = \dfrac{GM}{r^2}$ for the gravitational field strength of a **point mass** M (or spherical mass)

  By definition,
  the gravitational field strength at any point r from a point mass (or uniform sphere) M is,
  $$g = \frac{F}{m}$$
  By Newton's law of gravitation,
  $$F = \frac{GMm}{r^2}$$
  therefore,
  $$g = \frac{F}{m} = \frac{GMm}{r^2 m} \Rightarrow \boxed{g = \frac{GM}{r^2}}$$



Note:

- When using the above equation $g = \dfrac{GM}{r^2}$ to calculate gravitational field strength, <u>the mass used should be that of the mass generating the field</u>.

- Gravitational field strength is a <u>vector</u> quantity. Its direction is towards the mass which generates the field as gravitational forces are always attractive.

Additional Notes

Worked example 2

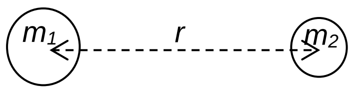
Two point masses m_1 and m_2 are a distance r apart. Determine the magnitude of the gravitational field strength caused by m_1 at position of m_2.

Thinking Process

- $g = \frac{F}{m} = \frac{GM}{r^2}$ (where G is the gravitational constant, 6.67 x 10^{-11} N m^2 kg^{-2}, M is the mass of the body <u>causing the field</u>, m is the mass of the body <u>in the field)</u>
- For this question, to get the field strength by m_1 at position of m_2, take note that m_1 is causing the field.

Solution

Force which m_1 exerts on m_2, $F_{12} = \frac{Gm_1m_2}{r^2}$



Force per unit mass on m_2 due to m_1's field

$$g_1 = {F_{12}}/{m_2} = \frac{Gm_1m_2}{r^2} \times \frac{1}{m_2} = \frac{Gm_1}{r^2}$$

A.2.3 Graph of Gravitational Field Strength due to a spherical mass *M*

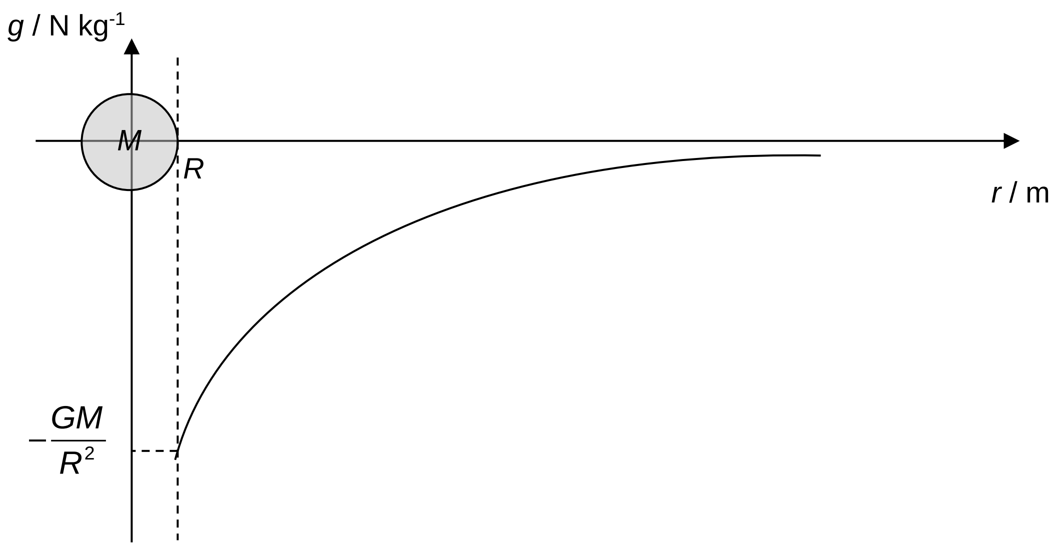
- $g = \frac{GM}{r^2}$ of a point mass M is known as an **inverse square law field** since $g \propto \frac{1}{r^2}$
- When <u>r gets larger</u>, the magnitude of the <u>gravitational field at that point decreases</u>.
- Gravitational field strength is largest at <u>the surface of M</u>.

Note that the gravitational field strength g is a vector quantity, hence when considering if g increases or decreases, the magnitude is considered.

Compare this with gravitational potential Φ on page 7-12.

Note:

- The equation $g = \frac{GM}{r^2}$ is only valid for $r \geq R$, i.e. from the surface of the mass and above.



Additional Notes

Worked example 3

An object weighs 270 N at the Earth's surface. Determine its weight at an **altitude equal to twice the radius** of the Earth. (30 N)

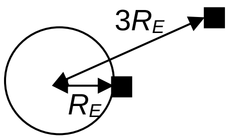
Thinking Process

- For this question, M will be mass of the earth (because earth is the body causing the field)
- The mass of the object is not known, thus it is necessary to express the weight of the object at the surface (which is known) first, then express the weight of the object at the desired point in terms of this weight.

Solution

$$W_{surface} = mg_{surface}$$

$$270 = m\frac{GM_E}{R_E^2}$$

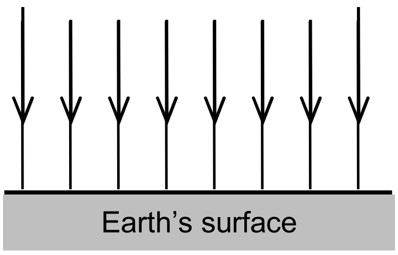


$$W \propto \frac{1}{r^2}$$

$$\frac{W_{3RE}}{W_{surface}} = \frac{{r_{surface}}^2}{{r_{3RE}}^2} = \frac{1}{9}$$

$$W_{3RE} = 30\text{ N}$$

A.2.4 Gravitational field strength near the Earth's surface



- Near the Earth's surface, the field strength does not change very much over small distances and so the field can be regarded as uniform and the field lines are parallel as shown.
- The gravitational field strength near the Earth's surface is approximately constant and has a numerical value of about 9.8 N kg^{-1} and this causes an acceleration due to gravity, so the acceleration of free fall is numerically equal to the gravitational field strength since $F=ma=mg$.

To show g is approximately constant near Earth:
[Radius of Earth= 6.4 x 10^6 m; Mass of Earth = 6.0 x 10^{24} kg]
At a height h (a few km) close to the surface of the Earth,

$$g = \frac{GM}{r^2}$$

$$= \frac{GM}{(R+h)^2} \quad \text{where } M = \text{mass of Earth}, R = \text{radius of Earth}$$

$$\approx \frac{GM}{R^2} \qquad (\therefore R >> h)$$

$$= \frac{(6.67\times10^{-11})(6.0\times10^{24})}{(6.4\times10^{6})^2} = 9.8\text{ m s}^{-2}\text{ or N kg}^{-1}$$

When solving questions related to gravitational field, take note of the height of the body above the surface of the earth.

Example 3

Compare the gravitational field strength, *g,* at the following points above the Earth's surface: 10 m, 1000 m, 10,000 m. Comment on the variation of the Earth's gravitational field strength 'near' its surface.
[radius of Earth= 6.4 x 10^6 m, mass of Earth = 6.0 x 10^{24} kg]

Example 4
What is the gravitational force acting on a 2000 kg spacecraft when it orbits two Earth radii from the Earth's centre?
(Given: gravitational field strength on Earth's surface = 9.81 N kg^{-1})

Note gravitational field strength (g_s) at the surface of a planet of mass *M* and radius *R* is

$$g_s = \frac{GM}{R^2}$$

$$\boxed{\Rightarrow g_s R^2 = GM}$$

(which is useful in problem solving when g_s and *R* are given but not the mass *M*)

A.3 Gravitational Potential and Potential Energy

- Because the gravitational field strength is a vector quantity it is difficult to make calculations of speed and energy of objects moving in the field. Therefore it is useful to define a scalar property of the field at each point, and this property is called the potential of the field.

Additional Notes

A.3.1 Gravitational Potential Energy *U*

- The gravitational potential energy <u>of a mass at a point</u> is the work done on the mass by an external force in moving it from infinity to that point.

Memorize

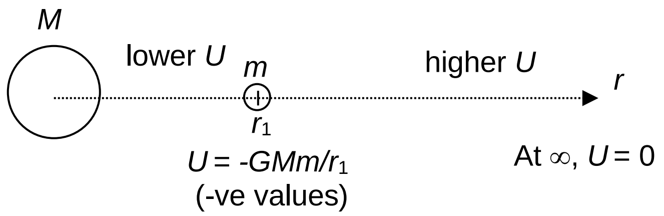
- In equation form,

$$\boxed{U = -\frac{GMm}{r}}$$

U is a **scalar**,
Unit of U is **joule (J)**.

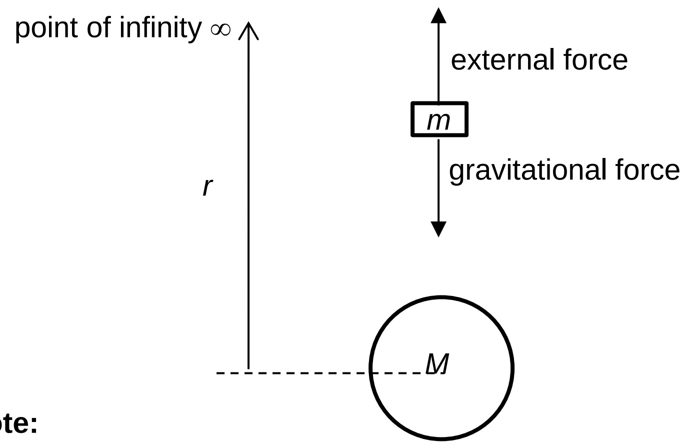
Note that there is no contradiction between this formula and the commonly used formula $\Delta U = mgh$. Their reference point taken is different.

ΔU is indeed equal to mgh near the surface of the Earth. Derivation can be found in Appendix.



Gravitational potential energy is **negative** because, by convention, the zero of gravitational potential energy is taken to be zero at infinity. Since all gravitational forces are attractive, work done by the external force on the mass in moving it from infinity is negative.

Memorize



- **Note:**
  - The work done by the external force does not accelerate the mass, so there is no increase in KE of the mass.
  - The negative sign here indicates that the external force is in opposite direction to the motion of the mass. (Recall work done = $f\,d\cos\theta$)
  - As mentioned in 'Work, Energy, Power', the choice of reference point for gravitational potential energy is arbitrary. The convention is to take gravitational potential energy to be zero when $r \to \infty$.

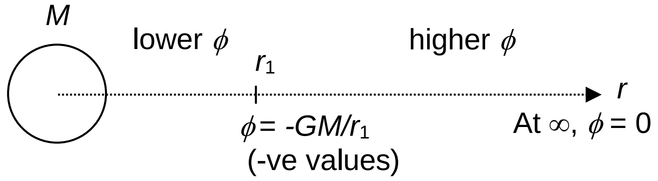
A.3.2 Gravitational Potential ϕ

Memorize

- The **gravitational potential** at a point is the work done per unit mass in bringing a small test mass from infinity to that point.

- In equation form,

$$\boxed{\phi = \frac{U}{m}}$$



ϕ is a **scalar**.
Unit of ϕ *is*: **J kg^{-1}**

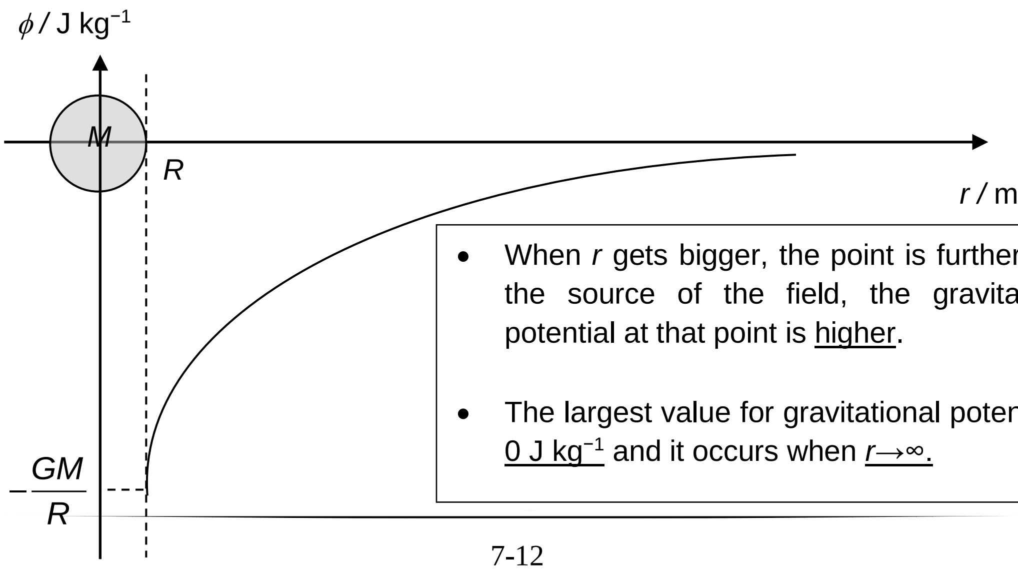
- **Note:**
  - The gravitational potential at infinity is zero.
  - Gravitational potential can also be understood as gravitational potential energy per unit mass, $\phi = \dfrac{U}{m} = \dfrac{-\frac{GMm}{r}}{m} = -\dfrac{GM}{r}$

$$\boxed{\phi = -\frac{GM}{r}}$$

Memorize

  - **All** values of gravitational potential are **negative**. Gravitational potential is always **negative** because the gravitational potential is taken to be zero at infinity and gravitational forces are attractive, work done by the external agent on the point mass moving it from infinity is negative.
  - A body placed in a gravitational field will tend to move from a point of higher potential (less negative) to a point of lower potential (more negative).
  - **Gravitational potential is a property of a point in the field** and does not depend on the mass at the point

- Consider the variation of gravitational potential ϕ due to a spherical uniform mass M with the distance r from its centre of mass:



- When r gets bigger, the point is further from the source of the field, the gravitational potential at that point is <u>higher</u>.
- The largest value for gravitational potential is <u>0 J kg^{-1}</u> and it occurs when <u>$r \to \infty$.</u>

When comparing potential, consider the negative sign as potential is a scalar quantity e.g. – 4 J kg^{-1} is larger than – 8 J kg^{-1}

Additional Notes

Example 5

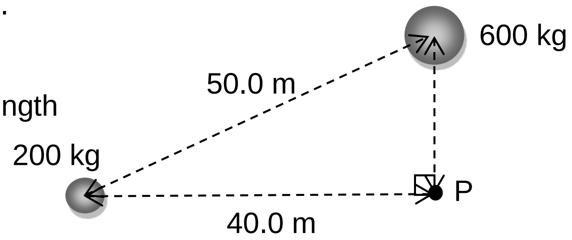
A satellite of mass 50 kg moves from a point where the gravitational potential due to the Earth is -20 MJ kg^{-1}, to another point where the gravitational potential is -60 MJ kg^{-1}.
In which direction does the satellite move and what is its change in potential energy?

A closer to the Earth and a loss of 2000 MJ of potential energy.
B closer to the Earth and a loss of 40 MJ of potential energy.
C further from the Earth and a gain of 2000 MJ of potential energy.
D further from the Earth and a gain of 40 MJ of potential energy.

Example 6

The two masses of 200 kg and 600 kg are rearranged such that they lie on the vertices of a right angle triangle, as shown in the diagram.

Calculate at point P:
(a) the magnitude of the gravitational field strength
(b) the gravitational potential



Learning Point:
Since gravitational field strength is a vector, the resultant gravitational field strength at a point is the **vector** sum of the gravitational field strength due to the different masses at that point. However, since gravitational potential is a scalar, the resultant gravitational potential at a point is the **algebraic** sum of the gravitational potential due to the different masses at that point.

A.3.3 Equipotentials

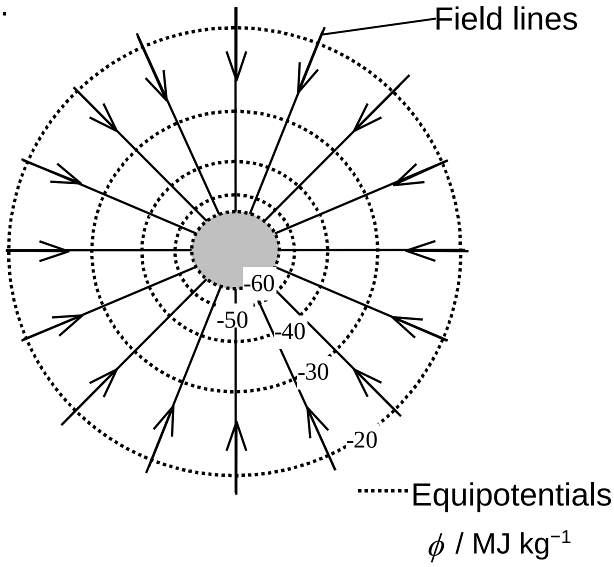
- Beside field lines, a gravitational field may also be visualised using equipotential lines (or surfaces).
- The line (or surface) where all points on it has the same gravitational potential is known as an **equipotential line (or surface)**.

For a *non-uniform* radial field such as that produced by a sphere like the Earth,

gravitational potential $\phi = -\dfrac{GM}{r}$

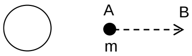
$\Rightarrow$ all points at the same distance from the centre of the earth have the same ϕ

The equipotential surfaces are just *spherical shells* as shown.



<u>**Note**</u>

1. Regular increments of equipotential lines are *not equally spaced* due to the non-uniform field.
2. *No work* is done along the same equipotential. Thus the equipotential lines are always *perpendicular* to the field lines.
3. The field lines always act in the direction of *decreasing* potential.
4. Near the Earth's surface (*uniform* field), the equipotentials are just planes *parallel* to the surface.

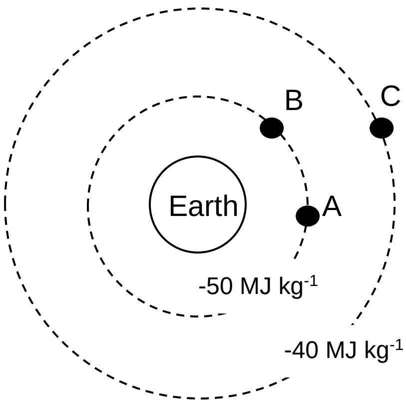


WD by external force in moving a mass m from point A to B:

$WD_{A\ to\ B}$
$= \Delta U$
$= U_B - U_A$
$= m\,\Delta\phi$
$= m\,(\phi_B - \phi_A)$
(since $U = m\,\phi$)

Example 7
The figure shows the Earth's gravitational field. The dashed lines represent equipotential lines. The gravitational potential is as shown for each of the equipotential lines.

Calculate the work done by the gravitational field in bringing a spacecraft of mass 5000 kg
(a) from A to C,
(b) from A to B.



Additional Notes

A.3.4 The relationship between *U* and *F*

The negative of the potential energy gradient at a point in the field is equal to the gravitational force at that point.

- Mathematically,

$$\boxed{F = -\frac{dU}{dr}}$$

- The potential energy gradient is numerically the gravitational force.
  The **minus sign** indicates that the direction of gravitation force F is the same as the direction of decreasing potential energy, i.e. direction of *F* is from higher GPE to lower GPE

A.3.5 The relationship between ϕ and *g*

- The negative of the potential gradient at a point in the field is equal to the gravitational field strength at that point.

- Mathematically,

$$\boxed{g = -\frac{d\phi}{dr}}$$

- The potential gradient is numerically the gravitational field strength.
  The **minus sign** indicates that the direction of the field is the same as the direction of decreasing potential, i.e. direction of *g* is from higher potential to lower potential.

Example 8

Values for the gravitational potential due to the Earth are given in the table below.

Distance from Earth's surface / m	Gravitational potential / MJ kg^{-1}
0	-62.72
390 000	-59.12
400 000	-59.03
410 000	-58.94
Infinity	0

(a) A satellite of mass 700 kg falls from a height 400 000 m to Earth's surface. Determine the loss in potential energy.

(b) Deduce a value for the Earth's gravitational field at a height of 400 000 m.

Thinking Process
How do I link gravitational potential to gravitational field?

- $g = -\dfrac{d\phi}{dr}$ could be used.

Key questions to solve the question:

(a) **What is the question asking for?**
- loss of potential energy

How to find loss of potential?
- change in potential = final potential – initial potential

How is potential related to potential energy?
- $U = m\phi$, where U is the potential energy, and ϕ is the gravitational potential
- $\Delta U = m\Delta\phi$

(b) **What is the question asking for?**
- g at height of 400 000 m ?

What formula to use?
- $g = -\dfrac{d\phi}{dr}$

How to use the information to calculate $-\dfrac{d\phi}{dr}$?
- The same method as finding gradient of tangent to a point on a curve

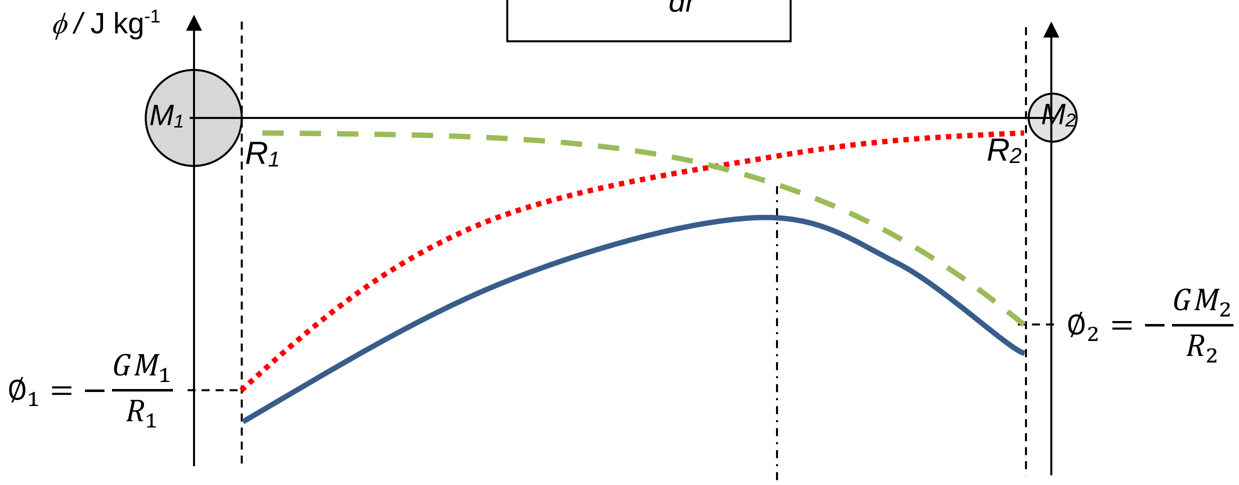
Worked example 4

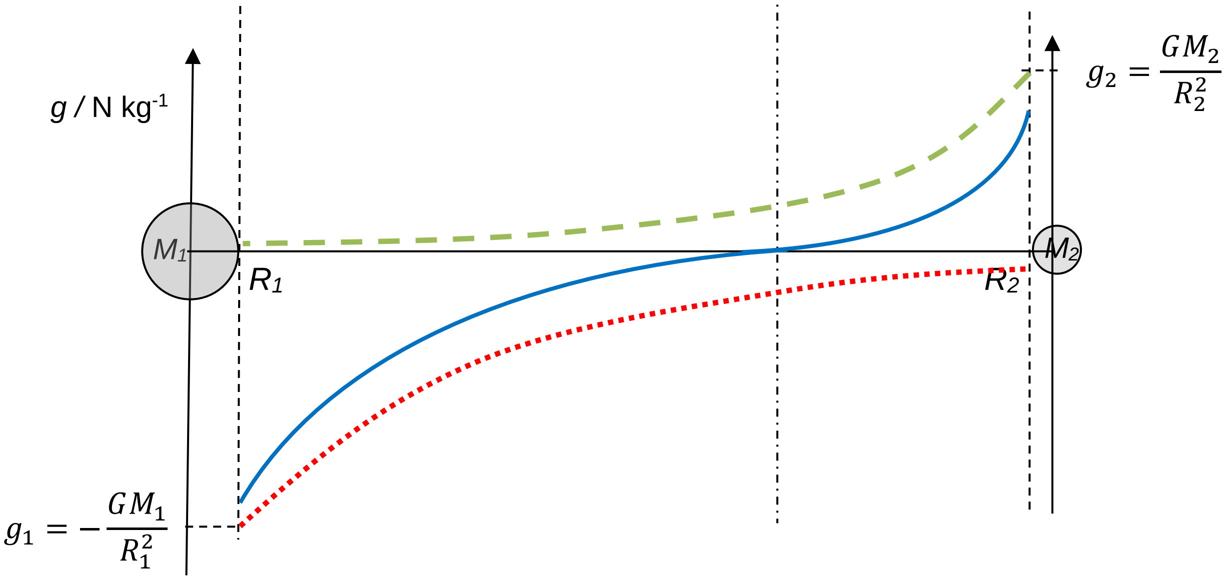
Sketch the graph of how gravitational potential, ϕ, varies along the line joining the centres of mass 1 and mass 2.
On a separate graph, sketch how the gravitational field strength, g, varies along the line joining the centre of mass 1 and mass 2.

Thinking Process
Is potential a scalar or vector? So should the net potential be a vector sum or algebraic sum of the component potentials?
- Scalar. The net potential (the thick line below) should be an algebraic sum of the component potentials.

Is gravitational field a scalar or vector? So should the net field be a vector sum or algebraic sum of the component field?
- Vector. The net field (the thick line below) should be a vector sum of the component fields.

$$g = -\frac{d\phi}{dr}$$





Additional Notes

Potential-distance graphs can never be positive.

Convention taken: rightward to be positive

Comparing the two graphs, since $g = -\frac{d\phi}{dr}$, the negative of the potential gradient at any point gives the gravitational field strength g at the same point.

Note that a bigger mass does not necessary have a larger g on its surface i.e. depends on the ratio of M to r^2

A.3.6 Relationship between *F, g,* ϕ & *U*

Gravitational force		Gravitational Potential Energy
$F = \frac{GMm}{r^2}$ (–ve sign is omitted if only magnitude is needed)	← $F = -\frac{dU}{dr}$ →	$U = -\frac{GMm}{r}$
↓ $g = \frac{F}{m}$ ↑ $F = mg$		↓ $\phi = \frac{U}{m}$ ↑ $U = m\,\phi$
Gravitational field $g = \frac{GM}{r^2}$ (–ve sign is omitted if only magnitude is needed)	← $g = -\frac{d\phi}{dr}$ →	Gravitational Potential $\phi = -\frac{GM}{r}$

B.1 Applications of the Law of Gravitation

In this final section of the topic, we will explore several practical applications of Newton's universal law of gravitation.

B.1.1 Conservation of Energy

A mass *m* that is moving further away from (or closer to) another mass *M*, leads to a change in its gravitational potential energy ($U = -\frac{GMm}{r}$). In order for the total mechanical energy (gravitational potential energy and kinetic energy) to be conserved, there is a corresponding change in its kinetic energy. In other words, a gain or loss in its GPE is equal to its respective loss or gain in KE.

Additional Notes

Example 9

The Earth may be assumed to be an isolated sphere of radius 6.4 x 10^3 km with its mass of 6.0 x 10^{24} kg concentrated at its centre. An object is projected vertically from the surface of the Earth so that it reaches an altitude of 1.3 x 10^4 km.
Calculate for this object,
(a) the change in gravitational potential, and
(b) the speed of projection from the Earth's surface assuming air resistance is negligible.

Thinking Process

Key question on objective:
Can I find the initial and final gravitational potential from what I have, and thus the change in gravitational potential?

- Yes. Using $\Delta\phi = \phi_{final} - \phi_{initial} = -GM\left(\dfrac{1}{r_{final}} - \dfrac{1}{r_{initial}}\right)$

How do I link the speed of an object to its energy?
- Use K.E. = ½ mv^2

Key questions to solve the question:
(a) **What is the question asking for?**
- Change in gravitational potential, which is final potential – initial potential

What is the formula I should be using?
- $\phi = -\dfrac{GM}{r}$ (where G is the gravitational constant, 6.67 x 10^{-11} N m^2 kg^{-2}, M is the mass of the body <u>causing the field</u>, r is the distance from the centre of one body to the centre of the other)

(b) **What is the question asking for?**
- Initial speed of the object

Is energy conserved?
- Yes. Loss in K.E. = Gain in G.P.E

What is the speed of the object at the highest point?
- 0 m s^{-1}

Escape Speed

- Escape speed is defined as the *minimum speed needed to project a mass m from the surface of a planet such that it would just come to rest at infinity.*
- In order to escape the influence of the field, the body must travel infinitely far from the gravitational source.
- Neglecting air resistance and by *conservation of energy*, the escape speed can be calculated.
- As we are interested in the minimum speed required, the speed of the body should be zero as it approaches infinity.

Consider a projectile of mass *m,* leaving the surface of a planet of mass *M* and radius *R* with an escape speed *v*.

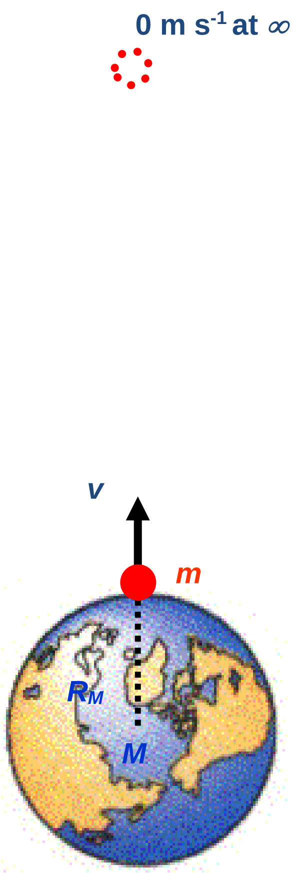
Neglecting air resistance and by *conservation of energy*,

Total initial GPE and KE = total final GPE and KE

$$\left(-\frac{GMm}{R_M}\right)+\frac{1}{2}mv^2=0+0$$

$$\frac{1}{2}mv^2=\frac{GMm}{R_M}$$

$$\Rightarrow \text{escape speed } v=\sqrt{\frac{2GM}{R_M}}$$



Note that the expression for escape speed needs to be derived and shown before using it to solve questions.

Additional Notes

Worked example 5

Given: M_E = mass of the Earth = 5.98 x 10^{24} kg
R_E = radius of the Earth = 6.37 x 10^6 m

(a) Calculate the escape speed of an oxygen molecule at the Earth's surface. (1.12 x 10^4 m s^{-1})

(b) Calculate the escape speed of the same oxygen molecule at a height 0.2 R_E above the Earth's surface. (1.02 x 10^4 m s^{-1})

Solution:

(a) By Conservation of Energy, initial total energy = final total energy
$KE_i + GPE_i = KE_f + GPE_f$

$$\left(-\frac{GM_E m_O}{R_E}\right)+\frac{1}{2}m_O v^2 = 0+0$$

$$\frac{1}{2}m_O v^2 = \frac{GM_E m_O}{R_E}$$

$$v=\sqrt{\frac{2GM_E}{R_E}}=\sqrt{\frac{2(6.67\times10^{-11})(5.98\times10^{24})}{6.37\times10^{6}}}=1.12\times10^4 \text{ m s}^{-1}$$

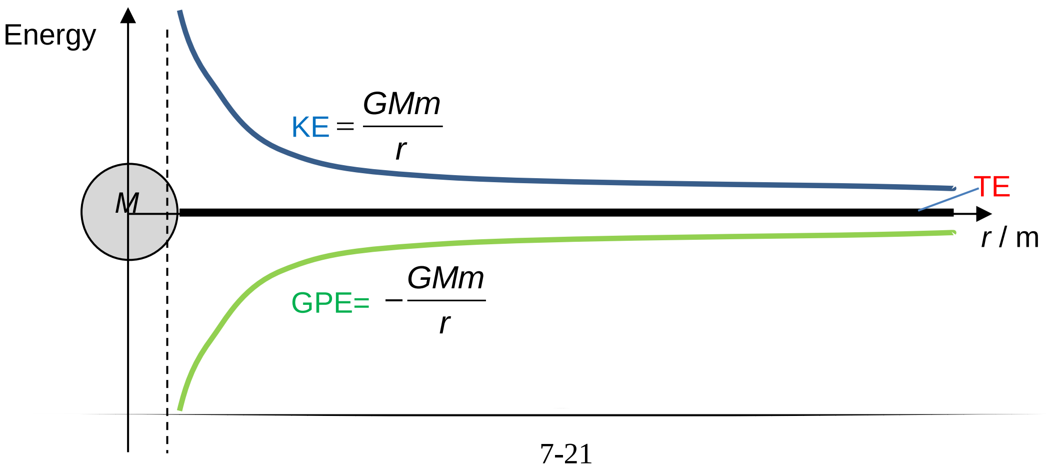
(b) At $r = R_E$, $v_{escape,1}=\sqrt{\frac{2GM_E}{R_E}}$

At $r = \underline{1.2R_E}$, $v_{escape,2}=\sqrt{\frac{2GM_E}{1.2R_E}}$

$$\frac{v_{escape,2}}{v_{escape,1}}=\sqrt{\frac{R_E}{1.2R_E}} \qquad \{G, M_E \text{ are constants}\}$$

$$v_{escape,2}=\sqrt{\frac{1}{1.2}}\left(1.12\times10^4\right)=1.02\times10^4 \text{ m s}^{-1}$$

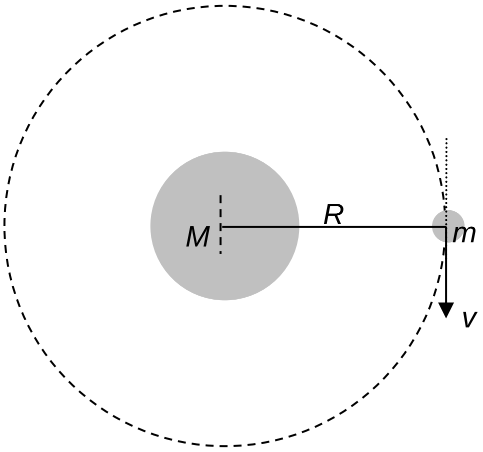
(c) The oxygen molecule of mass *m* projected from the Earth's surface with escape speed has a variation of GPE of the oxygen molecule with distance *r* from the centre of the Earth is shown below. Sketch how the total mechanical energy TE and kinetic energy KE of the oxygen molecule varies with *r*.



Note that these energy graphs are as shown because this is a case of conservation of total mechanical energy.

These graphs should be contrasted with the energy graphs of a satellite orbiting with a particular radius as shown in 7-23.

B.1.2 Satellites (Circular Orbits)



- Since the satellite m is undergoing uniform circular motion, there must be a resultant force acting on it.
- The only force that is experienced by m is due to the gravitational force which M exerts on m.

- Therefore, the gravitational force by the planet on the satellite provides the centripetal force necessary for the satellite's circular motion.

- The statement above in equation form:

Gravitational force provides centripetal force.

F_{net} on $m = \frac{mv^2}{r}$ or $mr\omega^2$

$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow$ linear velocity $v = \sqrt{\frac{GM}{r}}$

$\frac{GMm}{r^2} = mr\omega^2 \Rightarrow$ angular veocity $\omega = \sqrt{\frac{GM}{r^3}}$

period of orbit $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r^3}{GM}}$

There are 2 expressions for the centripetal force:

We use the form with v for finding out quantities like linear velocity and kinetic energy of the satellite.

We use the form with ω to find out the angular velocity of the satellite and thus, the period the satellite takes to orbit about the earth.

Additional Notes

B.1.3 Kinetic, Gravitational Potential and Total Energy of a Satellite

kinetic energy of satellite $= \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{GM}{r}\right) = \frac{GMm}{2r}$

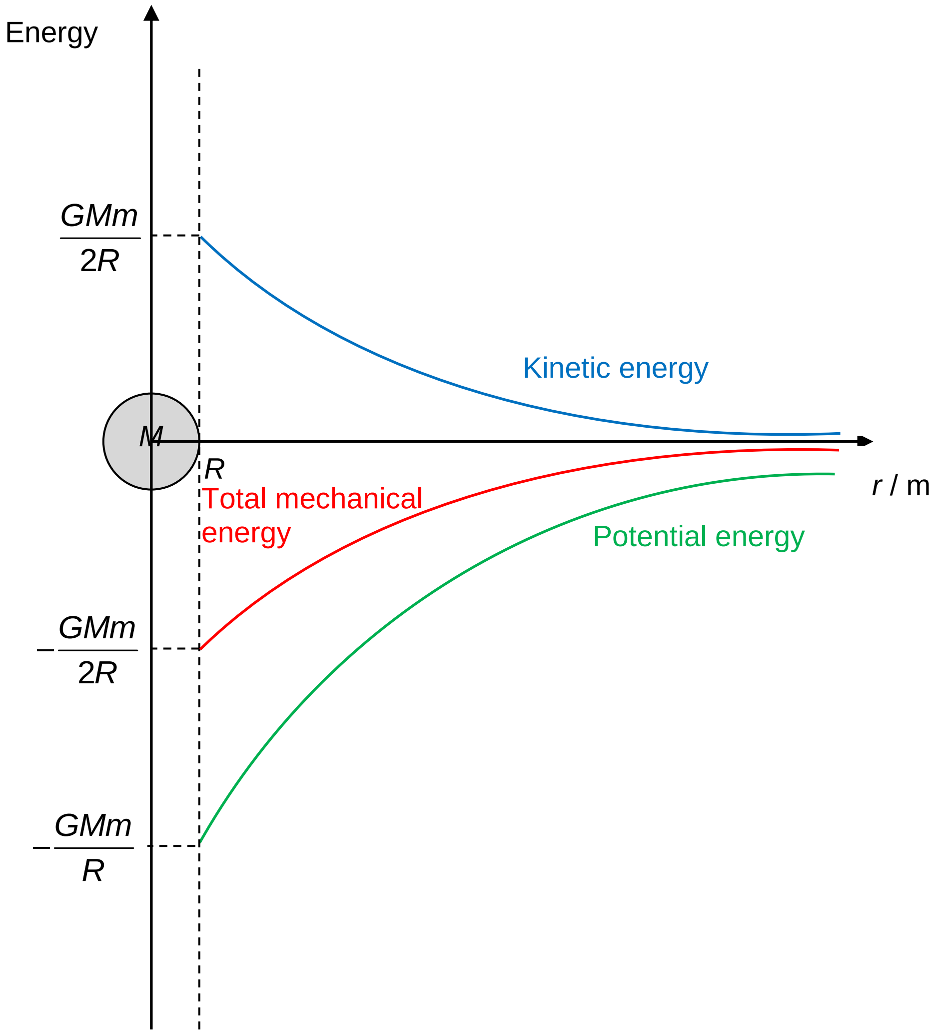
gravitational potential energy of satellite $= -\frac{GMm}{r}$

total energy of satellite $=$ kinetic energy + grav. potential energy

$$= \frac{GMm}{2r} + \left(-\frac{GMm}{r}\right)$$

$$= -\frac{GMm}{2r}$$

Graphically, the energies of the satellite can be represented as shown below.



Note that these energy graphs are to be interpreted in the following way:

If a satellite is orbiting at a radius of r from the centre of the planet, the satellite will have the indicated amount of KE, GPE and hence TE.

Note that the expressions for KE and TE for the satellite need to be derived and shown before using them to solve questions.

Note that the total mechanical energy for satellites is not constant, unlike the case for a mass with sufficient energy to escape to infinity in 7-21.

Note

1. The negative sign for E_T denotes that the satellite is *bounded* to the Earth by gravitational attraction.

   In other words, energy has to be supplied before the satellite can be dislodged from the gravitational pull of the Earth.

   If a satellite loses energy, it will fall towards Earth.

2. $E_T = -\dfrac{GMm}{2r}$ $\qquad E_k = \dfrac{GMm}{2r}$ $\qquad E_P = -\dfrac{GMm}{r}$

   By comparison,

   $$E_T = -E_k = \tfrac{1}{2} E_p$$

3. A satellite orbiting close to the Earth's atmosphere will lose energy (i.e. E_T) due to atmospheric friction and moves towards lower orbits (i.e. r decreases).

   Amazingly, the reduction of E_T is accompanied by an increase in E_k causing the satellite to spiral faster and faster towards the Earth until the satellite either hits the ground or burns up in the atmosphere.

   Thus, although E_T decreases, it is possible for E_k to increase because the loss in E_p is twice the gain in E_k.
   (Since $-E_k = \frac{1}{2} E_p \Rightarrow \Delta E_p = -2\Delta E_k$)

Additional Notes

B.1.4 Period of Satellites / Orbiting planets

Example 10

(a) Derive the relationship between, *r*, the radial distance between the satellite and the centre of the planet and, *T,* the period of orbit by satellite around the planet.
Hence, given that the orbital period of a satellite around the Earth is 24 hrs, calculate the radius of orbit of the satellite.
(Take the mass of the Earth to be 6.0 x 10^{24} kg)

Thinking Process

Key question on objective:
How do I link the gravitational force to the centripetal force?
- The gravitational force provides the centripetal force.

Key questions to solve the question:
What is the formula used in circular motion?
- $F = m r \omega^2$

What provides the centripetal force?
- The gravitational force

What is the formula for gravitational force?
- $F = \dfrac{GMm}{r^2}$

Worked example 6

An Earth satellite is moved from one stable circular orbit to another stable circular orbit at a greater distance from the Earth. Which one of the following quantities increases for the satellite as a result of the change?

A gravitational force
B gravitational potential energy
C angular velocity
D linear speed
E centripetal acceleration

Thinking Process

Key question on objective:
Is energy a vector or scalar? So if energy is getting less negative, is the energy increasing or decreasing?

- Energy is a scalar. So if energy is getting less negative, the energy is increasing.

Key questions to solve the question:
Is A correct?
- $F = \dfrac{GMm}{r^2}$, when r increases, *F* decreases (so A is wrong)

Is C correct?
- $\dfrac{GMm}{r^2} = m\omega^2 r \quad \Rightarrow \quad \omega = \sqrt{\dfrac{GM}{r^3}}$
- When *r* increases, ω decreases (so C is wrong)

Is D correct?
- $\dfrac{GMm}{r^2} = \dfrac{mv^2}{r} \quad \Rightarrow \quad v = \sqrt{\dfrac{GM}{r}}$
- When *r* increases, *v* decreases (so D is wrong)

Is E correct?
- *F* = *ma*, since *F* decreases, *a* decreases too (so E is wrong)

Is B correct?
- GPE of satellite = $-\dfrac{GMm}{r}$, when *r* increases, GPE becomes less negative (closer to 0 J, the largest possible GPE) (so B is the answer)

Solution

(Refer to Page 23)

Answer: B

Additional Notes

B.1.5 Geostationary Satellites

- Geostationary orbit refers to a circular orbit around the Earth in which a satellite would appear stationary to an observer on the Earth's surface because it revolves around the Earth with the same period and in the same direction as the rotation of the Earth.

- Geostationary satellites are satellites which always remain directly above the same spot on the Earth.

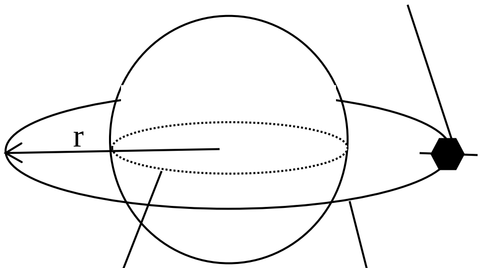
- In order for a satellite to be geostationary, the following must hold.

1. The period of a geostationary satellite orbiting the Earth is 24 hours.

2. A geostationary satellite must move from west to east because the earth rotates from west to east.

3. The radius of the geostationary orbit is 42300 km, based on Example 10.

4. A geostationary satellite must be placed vertically above the equator because the force of attraction to the Earth is to its centre so the circular orbit must be centred on the Earth's centre. If the orbit is not on the equator the satellite would have varying latitude and so not be geostationary.



Objects that orbit round the Earth are called *satellites*. Our Moon is a *natural* satellite whereas communication and weather satellites are *artificial* ones.

Artificial satellites can be classified as low earth orbit (LEO) or geostationary satellites (GEO).

A satellite circling close to Earth (say< 200 km) is termed to be in *low earth orbit*.

Main Advantages and Disadvantages of Geostationary Satellites

Advantages	Disadvantages
Since they are positioned at high altitudes of 42300 km, they can view the whole earth disk below them, rather than a small area, and they can scan the same area very frequently. ⇨ Ideal for meteorological applications and remote imaging	Since they are positioned at high altitudes, the spatial resolution of their images tends to be poorer than some low orbit polar satellites.
Since they remain stationary above the same point on Earth, they are ideal for use as communication satellites as they require no tracking to receive the downlink signals. ⇨ No need to adjust satellite dish	Since they are positioned directly above the equator, they cannot view the north and south poles, and the spatial resolution for areas at latitudes greater than 70 degrees is poorer than that nearer to the equator.

B.1.6 True Weight and Apparent Weight

True weight is defined to be the gravitational force acting on a mass placed in a gravitational field.

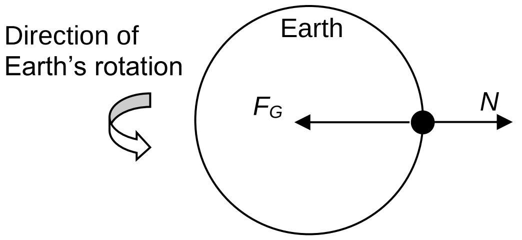
Apparent weight is perceived to be what a bathroom scale reads, i.e. the normal reaction force required to support the mass in position.

We are made aware of our weight because the ground (or whatever supports us) exerts an upward push on us as a result of the downward push our feet exert on the ground. It is this upward push which make us 'feel' the force of gravity.

In fact we judge our weight (apparent weight) from the upward push exerted on us by the floor. If our feet are completely unsupported we experience weightlessness. Passengers in a lift which has a continuous downward acceleration equal to g would get no support from the floor since they, too, would be falling with the same acceleration as the lift. There is no upward push on them and so no sensation of weight is felt.

Let F_G – Gravitational force on body at Equator, i.e. True weight
N_{eq} – Apparent weight at Equator
N_{pole} – Apparent weight at poles

(A) Consider a body placed at the Equator lying on the surface of the Earth.



Applying Newton's Second Law,

Net force = ma_c

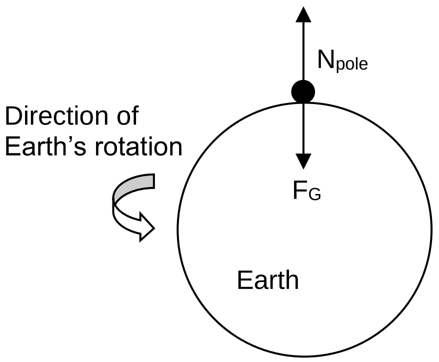
$F_G - N_{eq} = ma_c$ where $F_G = \frac{GMm}{R^2}$

$N_{eq} = F_G - ma_c$

$\therefore N_{eq} < F_G$ (1)

Eq. (1) shows that the body at the Equator would appear to be lighter than what it actually is (since apparent weight < true weight). Part of the true weight ($F_G - N_{eq}$) is used to provide the centripetal force to keep the body in circular motion.

(B) Consider a body placed at the Pole lying on the surface of the Earth.



At the pole the body is not rotating relative to the Earth. Hence bodies placed at the poles experiences no centripetal acceleration. Thus,

Applying Newton's Second Law,
Net force = ma_c
$F_G - N_{pole} = 0$
$N_{pole} = F_G$ (2)

Eq. (2) shows that the body at the Pole weighs as what it actually is (since apparent weight = true weight).

From Eq. (1) and (2), $N_{eq} < N_{pole}$
i.e. A body would appear lighter on a given weight scale at the Equator, as compared to at the Poles.

Note: *Its true weight* $F_G = \frac{GMm}{R^2}$ *remains the same at the Equator and poles*.

Additional Notes

APPENDIX

1. What is *weightlessness*?

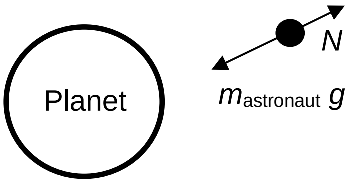
True weightlessness vs Weightlessness

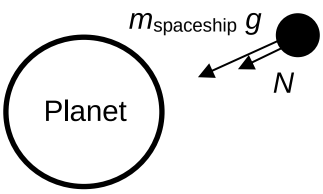
True weightlessness	Weightlessness
Occurs when **true weight is zero**. This will only occur if the mass is at a point where $\boldsymbol{g}$ **= 0 N kg^{-1}**. Examples: • Mass is outside the earth's gravitational field. • Mass at a point between two masses where *resultant g* is *zero*.	**Weightlessness** is the situation when a person experiences free fall and has zero contact force on him. Weightlessness occurs when weight is the only force acting on a body Some examples: • A body that is undergoing free-fall • An orbiting satellite

Why does an astronaut in orbit experience weightlessness?

Both the satellite and the astronaut are falling freely with the same acceleration, the force the satellite exerts on the astronaut is zero, hence the astronaut experiences weightlessness.

Note: If there is contact force N between them, the equations for the spaceship and the astronaut in the spaceship can be written as
$m_{astronaut}\, g - N = m_{astronaut}\, a_c$ and $m_{spaceship}\, g + N = m_{spaceship}\, a_c$.
Since g and a_c are the same for them, N has to be zero.



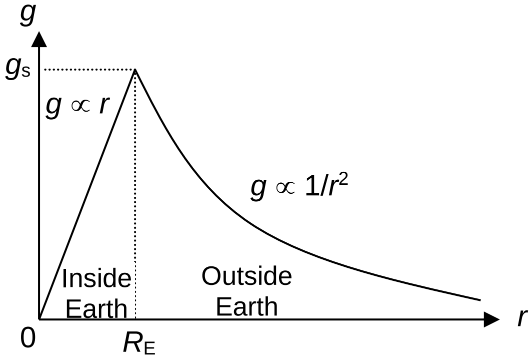


2. Assuming the Earth is a homogeneous sphere of radius R_E, show that for r < R_E, the gravitational field strength, g, is proportional to r. Sketch the variation of g with distance r from the centre of the Earth.

For $r < R_E$

$$g = \frac{GM}{r^2} = \frac{G(\rho V)}{r^2} = \frac{G}{r^2}\rho\left(\frac{4}{3}\pi r^3\right) = \frac{4G\pi\rho}{3}r$$

If density of earth, ρ is constant (homogeneous)
$\Rightarrow$ $g \propto r$ where $4/3G\pi\rho$ is a constant (shown)



3. Change in Gravitational Potential Energy, Δ*U* near the surface of the Earth

The change in gravitational potential energy near the surface of the Earth, Δ*U* is given by *mgh* where *g* is the gravitational field strength at the surface of the Earth and *h* is the height above the surface of the Earth.

Since $U_{RE} = -\frac{GM_E m}{R_E}$ and $U_{RE+h} = -\frac{GM_E m}{R_E+h}$

$$\Delta U = -\frac{GM_E m}{R_E+h} - \left(-\frac{GM_E m}{R_E}\right) = \frac{-GM_E mR_E + GM_E mR_E + GM_E mh}{(R_E+h)R_E}$$

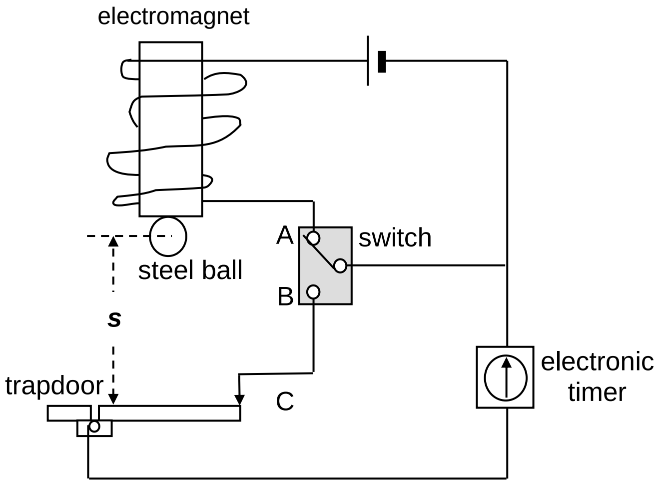
$$= \frac{GM_E mh}{(R_E+h)R_E}$$

Near the surface of the Earth, $h \ll R_E$,

$$\Delta U = \frac{GM_E mh}{R_E^2} = \left(\frac{GM_E}{R_E^2}\right)mh = gmh = mgh$$

4. Experiment to determine the acceleration of free fall, *g* using a falling body

The experimental set up is as shown:



i. Electronic timer is used to measure the time taken, *t*, for a small steel ball to fall through a known distance of *s*.
ii. A small heavy object with smooth surfaces, such as a small steel ball, is usually used in order to minimise the effects of air resistance.
iii. When the switch is in position A, the electromagnet holds up a small steel ball.
iv. When the switch is moved quickly to position B, the electromagnet releases the ball and the timer starts timing.
v. When the ball hits the trapdoor, the timer circuit is opened at C and the timer stops.
vi. Using the equation of motion

$$s = u\,t + ½\,g\,t^2$$

Since $u = 0$ ⇒ $s = ½\,g\,t^2$

vii. The experiment can be repeated a number of times with different *s*.
Plot a graph of *s* against t^2,
Gradient = ½ *g*
***g* = 2 × (gradient)**

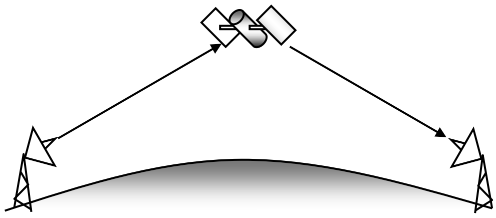
This approach helps to reduce the effects of random errors.

Additional Notes

5. Application of Geostationary Satellites

Telecommunications

Communication satellites are placed in the geostationary orbits for relaying telephone, radio, and television signals round the curve surface of the Earth. Geostationary communication satellites could remain at the same relative location, in constant electronic line-of-sight, to the stations on the Earth's surface, which facilitates telecommunications.



Summary

Concept	Definition	Formulas or important concepts
Newton's Law of Gravitation (F)	The mutual force of attraction F between any two point masses is proportional to the product of the masses and inversely proportional to the square of their separation.	$F \propto \frac{M_1 M_2}{r_{12}^2}$ $F = G\frac{M_1 M_2}{r_{12}^2}$
Gravitational Field	A gravitational field is a region of space where a mass experiences a force. The direction of the field is the direction of the force on the mass.	
Gravitational field strength at a point (g)	g is defined as <u>gravitational force exerted per unit mass</u> placed at that point.	$g = \frac{F}{m} = \frac{GM}{r^2}$
Gravitational potential energy of a mass at a point (U)	The gravitational potential energy <u>of a mass at a point</u> is the work done on the mass by an external force in moving it from infinity to that point.	$U = -\frac{GMm}{r}$
Gravitational potential at a point (ϕ)	The gravitational potential at a point is the work done per unit mass in bringing a small test mass from infinity to that point.	$\phi = \frac{U}{m} = -\frac{GM}{r}$
Relationship between U and F		$F = -\frac{dU}{dr}$ $\frac{dU}{dr}$: potential energy gradient
Relationship ϕ and g		$g = -\frac{d\phi}{dr}$ $\frac{d\phi}{dr}$: potential gradient