RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 7C: Maclaurin Series

SYLLABUS INCLUDES

- Standard series expansion of $(1+x)^n$ for any rational n, e^x, $\sin x$, $\cos x$ and $\ln(1+x)$
- Derivation of the first few terms of the Maclaurin series by
  (i) repeated differentiation, e.g. $\sec x$
  (ii) repeated implicit differentiation, e.g. $y^3+y^2+y=x^2-2x$
  (iii) using standard series, e.g. $\mathrm{e}^x(\cos 2x)$, $\ln\left(\dfrac{1+x}{1-x}\right)$
- Range of values of x for which a standard series converges
- Concept of Maclaurin's series as an approximation of a function
- Small angle approximations: $\sin x \approx x$, $\cos x \approx 1-\dfrac{x^2}{2}$, $\tan x \approx x$

PRE-REQUISITES

- Differentiation Techniques
- Use of the Binomial Theorem for positive integer n
- Use of the notations $n!$ and $\dbinom{n}{r}$
- Partial Fractions

CONTENT

1 Maclaurin Series

For any polynomial of the form $\mathrm{f}(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n$, there is a relationship between the coefficients $a_0, a_1, ..., a_n$ and derivatives at $x=0$, i.e. $\mathrm{f}(0), \mathrm{f}'(0), ..., \mathrm{f}^{(n)}(0)$.

For example, for the cubic function $\mathrm{f}(x)=\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{24}$, where $a_0=0,\ a_1=\frac{1}{2},\ a_2=-\frac{1}{8},\ a_3=\frac{1}{24}$,

we observe that $\quad \mathrm{f}(0)=0=a_0$

$\mathrm{f}'(x)=\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}, \qquad \mathrm{f}'(0)=\frac{1}{2}=a_1$

$\mathrm{f}''(x)=-\frac{1}{4}+\frac{x}{4}, \qquad \mathrm{f}''(0)=-\frac{1}{4}=2\left(-\frac{1}{8}\right)=2a_2$

$\mathrm{f}^{(3)}(x)=\frac{1}{4}, \qquad \mathrm{f}^{(3)}(0)=\frac{1}{4}=6\left(\frac{1}{24}\right)=6a_3$

So $a_0=\mathrm{f}(0),\ a_1=\mathrm{f}'(0),\ a_2=\frac{1}{2}\mathrm{f}''(0),\ a_3=\frac{1}{3!}\mathrm{f}^{(3)}(0)$.

We shall now find a polynomial to **approximate** standard functions (for x in an interval about 0).

1.1 Derivation of Maclaurin Series by Differentiation

Given a function f, for x in an interval around 0,
if $\mathrm{f}(x)$ can be expressed as

$$\mathrm{f}(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n+\ldots,$$

then $\quad \mathrm{f}(0)=a_0$

$\mathrm{f}'(x)=a_1+2a_2x+3a_3x^2+\ldots+na_nx^{n-1}+\ldots \quad \Rightarrow \mathrm{f}'(0)=a_1$

$\mathrm{f}''(x)=2a_2+3\cdot 2\cdot a_3x+\ldots+n(n-1)a_nx^{n-2}+\ldots \quad \Rightarrow \mathrm{f}''(0)=2a_2$

$\mathrm{f}^{(3)}(x)=3\cdot 2\cdot a_3+\ldots+n(n-1)(n-2)a_nx^{n-3}+\ldots \Rightarrow \mathrm{f}^{(3)}(0)=3\cdot 2\cdot a_3$

a_0	$\mathrm{f}(0)$
a_1	$\mathrm{f}'(0)$
a_2	$\frac{\mathrm{f}''(0)}{2!}$
a_3	$\frac{\mathrm{f}^{(3)}(0)}{3!}$
…	…
a_n	$\frac{\mathrm{f}^{(n)}(0)}{n!}$

Hence we deduce that, after differentiating function f n times, we will get $a_n=\frac{\mathrm{f}^{(n)}(0)}{n!}$.

Thus, if $\mathrm{f}(x)$ can be expanded as a power series (where all terms are non-negative powers of x), for a given range of x including zero, then

$$\boxed{\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2!}x^2 + \frac{\mathrm{f}^{(3)}(0)}{3!}x^3 + \ldots + \frac{\mathrm{f}^{(n)}(0)}{n!}x^n + \ldots}$$

Note: the expression on the RHS of $\mathrm{f}(x)$ is known as the **Maclaurin Series**, where $\mathrm{f}^{(n)}(0)$ is the nth derivative of f w.r.t x, evaluated at $x=0$.

$y=\mathrm{f}(x)$

$\mathrm{f}(0)$	$\mathrm{f}'(0)$	$\mathrm{f}''(0)$	$\mathrm{f}^{(3)}(0)$	$\cdots$	$\mathrm{f}^{(n)}(0)$
$y\big\rvert_{x=0}$	$\frac{\mathrm{d}y}{\mathrm{d}x}\Big\rvert_{x=0}$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\Big\rvert_{x=0}$	$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\Big\rvert_{x=0}$	$\cdots$	$\frac{\mathrm{d}^ny}{\mathrm{d}x^n}\Big\rvert_{x=0}$

Example 1

Use the Maclaurin Theorem $\mathrm{f}(x)=\mathrm{f}(0)+\mathrm{f}'(0)x+\dfrac{\mathrm{f}''(0)}{2!}x^2+\dfrac{\mathrm{f}^{(3)}(0)}{3!}x^3+\ldots+\dfrac{\mathrm{f}^{(n)}(0)}{n!}x^n+\ldots$

to find the first four non-zero terms of **(a)** e^x, **(b)** $(1+x)^{\frac{1}{2}}$, and **(c)** $\cos 2x$.

Solution:

(a) Let $\mathrm{f}(x)=\mathrm{e}^x$ $\mathrm{f}(x)=\mathrm{e}^x \Rightarrow \mathrm{f}(0)=1$ $\mathrm{f}'(x)=\mathrm{e}^x \Rightarrow \mathrm{f}'(0)=1$ $\mathrm{f}''(x)=\mathrm{e}^x \Rightarrow \mathrm{f}''(0)=1$ $\mathrm{f}^{(3)}(x)=\mathrm{e}^x \Rightarrow \mathrm{f}^{(3)}(0)=1$ By Maclaurin Theorem, $\mathrm{e}^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots$ $=1+x+\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\cdots$	**(b)** Let $\mathrm{f}(x)=(1+x)^{\frac{1}{2}} \Rightarrow \mathrm{f}(0)=1$ $\mathrm{f}'(x)=\dfrac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow \mathrm{f}'(0)=\dfrac{1}{2}$ $\mathrm{f}''(x)=-\dfrac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow \mathrm{f}''(0)=-\dfrac{1}{4}$ $\mathrm{f}^{(3)}(x)=\dfrac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow \mathrm{f}^{(3)}(0)=\dfrac{3}{8}$ By Maclaurin Theorem, $(1+x)^{\frac{1}{2}}=1+\dfrac{1}{2}x+\left(-\dfrac{1}{4}\right)\dfrac{x^2}{2}+\left(\dfrac{3}{8}\right)\dfrac{x^3}{3!}+\cdots$ $=1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+\dfrac{1}{16}x^3+\cdots$
(c) Let $\mathrm{f}(x)=\cos 2x \Rightarrow \mathrm{f}(0)=1$ $\mathrm{f}'(x)=-2\sin 2x \Rightarrow \mathrm{f}'(0)=0$ $\mathrm{f}''(x)=-4\cos 2x \Rightarrow \mathrm{f}''(0)=-4$ $\mathrm{f}^{(3)}(x)=8\sin 2x \Rightarrow \mathrm{f}^{(3)}(0)=0$ $\mathrm{f}^{(4)}(x)=16\cos 2x \Rightarrow \mathrm{f}^{(4)}(0)=16$ $\mathrm{f}^{(5)}(x)=-32\sin 2x \Rightarrow \mathrm{f}^{(5)}(0)=0$ $\mathrm{f}^{(6)}(x)=-64\cos 2x \Rightarrow \mathrm{f}^{(6)}(0)=-64$ By Maclaurin Theorem, $\cos 2x=1+0x+\dfrac{(-4)}{2!}x^2+\dfrac{0}{3!}x^3+\dfrac{16}{4!}x^4+\dfrac{0}{5!}x^5+\dfrac{(-64)}{6!}x^6+\cdots$ $=1-2x^2+\dfrac{2}{3}x^4-\dfrac{4}{45}x^6+\cdots$	

Example 2

Given that $y=\sqrt{\sin x+\mathrm{e}^{x}}$, show that $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}-\mathrm{e}^{x}=\cos x$.

By further differentiation of this result, obtain the first four terms in the Maclaurin series for y.

Solution:

$$y=\sqrt{\sin x+\mathrm{e}^{x}} \qquad \cdots (1)$$
$$y^{2}=\sin x+\mathrm{e}^{x}$$

Differentiate w.r.t x,

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x+\mathrm{e}^{x}$$
$$2y\frac{\mathrm{d}y}{\mathrm{d}x}-\mathrm{e}^{x}=\cos x \qquad \cdots (2)$$

Differentiate (2) w.r.t x,

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{2}+2y\frac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}-\mathrm{e}^{x}=-\sin x \qquad \cdots (3)$$

Differentiate (3) w.r.t x,

$$4\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}+\left(2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}+2y\frac{\mathrm{d}^{3}y}{\mathrm{d}x^{3}}\right)-\mathrm{e}^{x}=-\cos x \qquad \cdots (4)$$

Substitute $x=0$,

(1): $y=1$

(2): $2(1)\dfrac{\mathrm{d}y}{\mathrm{d}x}-1=1 \quad\Rightarrow\quad \dfrac{\mathrm{d}y}{\mathrm{d}x}=1$

(3): $2(1)+2(1)\dfrac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}-1=0 \quad\Rightarrow\quad \dfrac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}=-\dfrac{1}{2}$

(4): $4\left(-\dfrac{1}{2}\right)+2\left(-\dfrac{1}{2}\right)+2\dfrac{\mathrm{d}^{3}y}{\mathrm{d}x^{3}}-1=-1 \quad\Rightarrow\quad \dfrac{\mathrm{d}^{3}y}{\mathrm{d}x^{3}}=\dfrac{3}{2}$

By Maclaurin Theorem,

$$y=1+1x+\frac{\left(-\frac{1}{2}\right)}{2!}x^{2}+\frac{\left(\frac{3}{2}\right)}{3!}x^{3}+\cdots$$
$$=1+x-\frac{1}{4}x^{2}+\frac{1}{4}x^{3}+\ldots$$

Some problems are more efficiently solved when implicit differentiation is applied.

Example 3

Given the curve $y^3 + y - x = 0$, obtain the Maclaurin series for y, in ascending powers of x, up to and including the term in x^3.

Solution:

$y^3 + y - x = 0 \qquad \cdots (1)$

Differentiate (1) w.r.t x, $3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} - 1 = 0$

$$\left(3y^2 + 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \qquad \cdots (2)$$

Differentiate (2) w.r.t x,

$$\left(3y^2 + 1\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0 \qquad \cdots (3)$$

Differentiate (3) w.r.t x,

$$\left(3y^2 + 1\right)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 6y\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 12y\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = 0 \qquad \cdots (4)$$

Substitute $x = 0$,

(1): $y(y^2 + 1) = 0 \quad \Rightarrow y = 0$ since $y^2 + 1 > 0$

(2): $\left(3(0) + 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1$

(3): $\left(3(0)^2 + 1\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6(0)(1)^2 = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$

(4): $\left(3(0)^2 + 1\right)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 6(0) + 12(0) + 6(1)^3 = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -6$

By Maclaurin Theorem, $y = x - \frac{6}{3!}x^3 + \ldots = x - x^3 + \ldots$

Example 4

Given that y is a function of x such that $(1 - 9x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + mx\frac{\mathrm{d}y}{\mathrm{d}x} = 9y$ and the Maclaurin series for y is $1 + 3x + nx^2 + 9x^3 + \ldots$, where m and n are some constants, find the values of m and n.

Solution:

$$(1 - 9x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + mx\frac{\mathrm{d}y}{\mathrm{d}x} = 9y \qquad \cdots (1)$$

Differentiate w.r.t x,

$$\left(1 - 9x^2\right)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \left(-18x\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + m\frac{\mathrm{d}y}{\mathrm{d}x} + mx\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 9\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \cdots (2)$$

Given Maclaurin series $y = 1 + 3x + nx^2 + 9x^3 + \ldots$, let $y=\mathrm{f}(x)$, then $\mathrm{f}(0)=1,\quad \mathrm{f}'(0)=3$

$$\frac{\mathrm{f}''(0)}{2!}=n \quad\Rightarrow\quad \mathrm{f}''(0)=2n, \qquad \frac{\mathrm{f}^{(3)}(0)}{3!}=9 \quad\Rightarrow\quad \mathrm{f}^{(3)}(0)=54$$

Substitute $x=0$ in:

(1): $(1-0)2n+0=9(1) \quad\Rightarrow\quad n=\frac{9}{2}$

(2): $54+0+3m+0=9(3) \quad\Rightarrow\quad m=-9$

1.2 Standard Series Expansion of e^x, $\sin x$, $\cos x$, $\ln(1+x)$

Using Maclaurin Theorem, we obtain the following expansions:

(a) $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\cdots(n-r+1)}{r!}x^r+\cdots$ for $|x|<1$

(b) $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ for all x

(c) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$ for all x

(d) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$ for all x

(e) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots$ for $-1<x\le 1$

Note:

(i) The above standard series expansions can be quoted unless their derivations are required. **They can be found in Formula List**.

(ii) The series obtained may converge to the respective functions for all values of x, or only within a limited range of values of x. The range of validity of the expansions may vary depending on the type of function.

For example, $x=1$ is in the range of validity for the expansion of e^x in part **(b)**, and $\ln(1+x)$ in part **(e)**, so we can substitute $x=1$ to obtain

$$\mathrm{e} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

(iii) The more terms used in the expansion, the better the approximation (see Appendix 1).

Example 5

Using the standard series expansions for e^x, $\sin x$, $\cos x$ and $\ln(1+x)$, obtain the expansion of the following functions in ascending powers of x, up to and including the term in x^3.

(a) $\sin 3x$ **(b)** $\mathrm{e}^x \cos x$ **(c)** $\ln(2+x)$

Find the range of validity of the expansion in part **(c)**.

Solution:

(a) $\sin 3x = 3x - \dfrac{(3x)^3}{3!} + \cdots = 3x - \dfrac{9}{2}x^3 + \cdots$

(b) $\mathrm{e}^x \cos x = \left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\right)\left(1 - \dfrac{x^2}{2!} + \cdots\right)$

$$= 1 + x + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{6} - \frac{1}{2}\right)x^3 + \cdots = 1 + x - \frac{1}{3}x^3 + \cdots$$

(c) $\ln(2+x) = \ln\left[2\left(1 + \dfrac{x}{2}\right)\right] = \ln 2 + \ln\left(1 + \dfrac{x}{2}\right)$

$$= \ln 2 + \left(\frac{x}{2} - \frac{\left(\frac{x}{2}\right)^2}{2} + \frac{\left(\frac{x}{2}\right)^3}{3} - \cdots\right)$$

$$= \ln 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} + \cdots$$

Expansion is valid for $-1 < \dfrac{x}{2} \le 1 \iff -2 < x \le 2$.

Example 6

Given that $y = \mathrm{e}^{\tan^{-1} x}$, where $\tan^{-1} x$ denotes the principal value, show that $\left(1 + x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = y$.

By repeated differentiation of this result, show also that

$$\left(1 + x^2\right)\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 6x\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 6\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$$

Hence, or otherwise, obtain the expansion of $\mathrm{e}^{\tan^{-1} x}$, in ascending powers of x up to and including the term in x^4, and show that, when x is sufficiently small for x^5 and higher powers of x to be neglected, $\mathrm{e}^{\tan^{-1} x} \approx \mathrm{e}^x - \dfrac{1}{3}\left(x^3 + x^4\right)$.

Solution:

$$y = \mathrm{e}^{\tan^{-1} x} \quad \cdots (1)$$

$$\ln y = \tan^{-1} x$$

Differentiate w.r.t x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2} \quad \Rightarrow \quad \left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = y \quad \cdots (2)$$

Differentiate (2) w.r.t x,

$$\left(1+x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} \quad \cdots (3)$$

Differentiate (3) w.r.t x,

$$\left(1+x^2\right)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

$$\left(1+x^2\right)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 4x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \quad \cdots (4)$$

Differentiate (4) w.r.t x,

$$\left(1+x^2\right)\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 2x\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 4x\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 4\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$$

$$\left(1+x^2\right)\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 6x\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 6\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} \quad \cdots (5)$$

Substitute $x = 0$,

(1): $y = \mathrm{e}^0 = 1$ (2): $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ (3): $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 1$

(4): $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2(1) = 1 \Rightarrow \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -1$ (5): $\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 6(1) = -1 \Rightarrow \frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -7$

By Maclaurin Theorem,

$$\mathrm{e}^{\tan^{-1} x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \cdots = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{7}{24}x^4 + \cdots$$

$$\mathrm{e}^x - \frac{1}{3}\left(x^3 + x^4\right) \approx \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) - \frac{1}{3}x^3 - \frac{1}{3}x^4$$

$$= 1 + x + \frac{1}{2}x^2 + \left(\frac{1}{6} - \frac{1}{3}\right)x^3 + \left(\frac{1}{24} - \frac{1}{3}\right)x^4 = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{7}{24}x^4$$

Hence when x is sufficiently small for x^5 and higher powers of x to be neglected,

$$\mathrm{e}^{\tan^{-1} x} \approx \mathrm{e}^x - \frac{1}{3}\left(x^3 + x^4\right)$$

Example 7

If $y=\ln(1+\sin x)$, find $\dfrac{dy}{dx}$ and show that $(1+\sin x)\dfrac{d^2y}{dx^2}=-1$.

Hence obtain the Maclaurin series for y, in ascending powers of x, up to and including the term in x^4. Using this series expansion, deduce that, if x^5 and higher powers of x may be neglected,

$$\ln\left(\frac{1-x}{1+\sin x}\right)\approx -\frac{x}{6}(12+3x^2+x^3).$$

Solution:

$$y=\ln(1+\sin x) \quad \cdots (1)$$

Differentiating (1) w.r.t x, $\dfrac{dy}{dx}=\dfrac{\cos x}{1+\sin x} \Rightarrow (1+\sin x)\dfrac{dy}{dx}=\cos x \quad \cdots (2)$

Differentiating (2) w.r.t x, $\cos x\dfrac{dy}{dx}+(1+\sin x)\dfrac{d^2y}{dx^2}=-\sin x$

$$\frac{\cos^2 x}{1+\sin x}+(1+\sin x)\frac{d^2y}{dx^2}=-\sin x$$

$$(1+\sin x)\frac{d^2y}{dx^2}=\frac{-\cos^2 x-\sin^2 x-\sin x}{1+\sin x}$$

$$(1+\sin x)\frac{d^2y}{dx^2}=\frac{-1-\sin x}{1+\sin x}$$

$$(1+\sin x)\frac{d^2y}{dx^2}=-1 \quad \cdots (3)$$

Differentiating (3) w.r.t x, $\cos x\dfrac{d^2y}{dx^2}+(1+\sin x)\dfrac{d^3y}{dx^3}=0 \quad \cdots (4)$

Differentiating (4) w.r.t x, $-\sin x\dfrac{d^2y}{dx^2}+\cos x\dfrac{d^3y}{dx^3}+(1+\sin x)\dfrac{d^4y}{dx^4}+\cos x\dfrac{d^3y}{dx^3}=0 \quad \cdots (5)$

Substitute $x=0$,

(1): $y=\ln(1+0)=0$ (2): $\dfrac{dy}{dx}=\dfrac{1}{1+0}=1$ (3): $\dfrac{d^2y}{dx^2}=-\dfrac{1}{1+0}=-1$

(4): $\dfrac{d^2y}{dx^2}+\dfrac{d^3y}{dx^3}=0\Rightarrow\dfrac{d^3y}{dx^3}=1$ (5): $2\dfrac{d^3y}{dx^3}+\dfrac{d^4y}{dx^4}=0\Rightarrow\dfrac{d^4y}{dx^4}=-2$

By Maclaurin Theorem,

$$\ln(1+\sin x)=0+x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{2x^4}{4!}+\cdots=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\cdots.$$

$$\ln\left(\frac{1-x}{1+\sin x}\right)=\ln(1-x)-\ln(1+\sin x)$$

$$\approx -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\left(x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}\right)$$

$$=-2x-\frac{x^3}{2}-\frac{x^4}{6}=-\frac{x}{6}\left(12+3x^2+x^3\right)$$

2 Binomial Expansion

2.1 Binomial Expansion for Positive Integral Index (Self-Reading)

In Secondary School, you have learnt the Binomial Expansion for **positive integral values of *n***.
(See Appendix 3 for a recap of the Binomial Theorem and the factorial and $\binom{n}{r}$ notations).

For $n \in \mathbb{Z}^+$ (set of positive integers),

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n,$$

where $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$. **[in Formula List]**

Note : Using summation notation: $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$.

Notice that

- The binomial expansion is in **descending powers of *a*** and **ascending powers of *b***.
- The **sum of powers** of a and b in each term of the expansion is always equal to $\boldsymbol{n}$.
- The expansion of $(a+b)^n$, $n \in \mathbb{Z}^+$ is a finite series with $\boldsymbol{n+1}$ terms.
- The general term of the expansion is $\binom{n}{r}a^{n-r}b^r$.

Example 8

(a) Expand $(2-3x)^4$, simplifying the coefficients.

(b) Find the first three (non-zero) terms and the last term in the expansion, in ascending powers of x, of $(1-2x-3x^2)^8$.

(c) Find the term independent of x in the expansion of $\left(x+\dfrac{2}{x^3}\right)^8$.

Solution:

(a)
$$\begin{aligned}(2-3x)^4 &= 2^4+\binom{4}{1}2^3(-3x)+\binom{4}{2}2^2(-3x)^2+\binom{4}{3}2^1(-3x)^3+(-3x)^4\\ &=16-96x+216x^2-216x^3+81x^4\end{aligned}$$

(b)
$$\begin{aligned}&(1-2x-3x^2)^8\\ &=\left[(1-3x)(1+x)\right]^8\\ &=(1-3x)^8(1+x)^8\\ &=\left[1+\binom{8}{1}(-3x)+\binom{8}{2}(-3x)^2+\ldots+\binom{8}{8}(-3x)^8\right]\left[1+\binom{8}{1}x+\binom{8}{2}x^2+\ldots+\binom{8}{8}x^8\right]\\ &=\left[1+8(-3)x+28(-3)^2x^2+\ldots+(-3)^8x^8\right]\left[1+8x+28x^2+\ldots+x^8\right]\\ &=1+\left[8+8(-3)\right]x+\left[28+8\cdot 8(-3)+28(-3)^2\right]x^2+\ldots+(-3)^8x^{16}\\ &=1-16x+88x^2+\ldots+6561x^{16}\end{aligned}$$

(c) General term $=\binom{8}{r}x^{8-r}\left(\dfrac{2}{x^3}\right)^r=\binom{8}{r}2^r x^{8-r-3r}=\binom{8}{r}2^r x^{8-4r}$

For the term independent of x, $8-4r=0 \Rightarrow r=2$

$\therefore$ Term independent of x $=\binom{8}{2}2^2=112$

2.2 Standard Series Expansion of $(1+x)^n$ for any rational n

The Binomial Expansion of $(1+x)^n$ for **<u>positive integral values of *n*</u>** is

$$(1+x)^n=1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+\binom{n}{r}x^r+\ldots+x^n.$$

We can rewrite the Binomial coefficient $\binom{n}{r}=\dfrac{n!}{r!(n-r)!}=\dfrac{n(n-1)\ldots(n-r+1)}{r!}$ to obtain

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots+x^n.$$

For example, $\quad (1+x)^4=1+4x+\dfrac{4(4-1)}{2!}x^2+\dfrac{4(4-1)(4-2)}{3!}x^3+x^4$
$\qquad\qquad\qquad\quad =1+4x+6x^2+4x^3+x^4$

The Binomial Expansion of $(1+x)^n$ for **<u>negative and non-integral values of *n*</u>** may be obtained using the Maclaurin Theorem.

Let $\mathrm{f}(x)=(1+x)^n$. We have $\mathrm{f}(0)=1$.

$$\mathrm{f}'(x)=n(1+x)^{n-1} \quad \Rightarrow \mathrm{f}'(0)=n$$
$$\mathrm{f}''(x)=n(n-1)(1+x)^{n-2} \quad \Rightarrow \mathrm{f}''(0)=n(n-1)$$
$$\mathrm{f}^{(3)}(x)=n(n-1)(n-2)(1+x)^{n-3} \quad \Rightarrow \mathrm{f}^{(3)}(0)=n(n-1)(n-2)$$

By Maclaurin Theorem, $$y=\mathrm{f}(0)+\mathrm{f}'(0)(x)+\frac{\mathrm{f}''(0)}{2!}(x^2)+\frac{\mathrm{f}^{(3)}(0)}{3!}(x^3)+\ldots$$
$$\therefore (1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

For $n\in\mathbb{Q}\setminus\mathbb{Z}_0^+$ (set of all rational numbers excluding 0 and all positive integers),

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots$$

The binomial expansion is **valid for** $\boldsymbol{|x|<1}$, i.e. $\boldsymbol{-1<x<1}$. **[in Formula List]**
This is also known as the **range of validity** of the binomial expansion.

Notice that

- The binomial expansion for $n\in\mathbb{Q}\setminus\mathbb{Z}_0^+$ has an infinite number of terms.

- The first term of the binomial is 1.
  If the expansion of $(a+x)^n$ is needed, where $n\in\mathbb{Q}\setminus\mathbb{Z}^+$, we can do it in 2 ways:

  (i) For ascending powers of x, we rewrite
  $$(a+x)^n=a^n\left(1+\frac{x}{a}\right)^n$$
  before applying the binomial expansion.
  In this case, the expansion is valid for $\left|\frac{x}{a}\right|<1$ (i.e. small x)

  (ii) For descending powers of x, we rewrite
  $$(a+x)^n=(x+a)^n=x^n\left(1+\frac{a}{x}\right)^n$$
  before applying the binomial expansion.
  In this case, the expansion is valid for $\left|\frac{a}{x}\right|<1$ (i.e. large x)

Special Binomial Expansions

Recall that the infinite geometric series $a+ar+ar^2+\ldots=\dfrac{a}{1-r}$ for $|r|<1$

Therefore, with $a=1,\ r=-x$,

$$\frac{1}{1+x}=(1+x)^{-1}=1-x+x^2-x^3+\ldots \text{ for } |x|<1 \quad \text{--- (I)}$$

Similarly, with $a=1,\ r=x$,

$$\frac{1}{1-x}=(1-x)^{-1}=1+x+x^2+x^3+\ldots \text{ for } |x|<1 \quad \text{--- (II)}$$

And with $a=1,\ r=-x^2$,

$$\frac{1}{1-\left(-x^2\right)}=\frac{1}{1+x^2}=(1+x^2)^{-1}=1-x^2+x^4-x^6+\ldots \text{ for } |x|<1 \quad \text{---(III)}$$

Example 9

Expand the following in ascending powers of x, up to and including the term in x^3.
In each case **(a)** to **(c)**, state the range of values of x for which the expansion is valid.

(a) $(1-2x)^{\frac{1}{3}}$ **(b)** $(2+x)^{-2}$ **(c)** $\dfrac{1-x^2}{\sqrt[3]{1+3x}}$ **(d)** $\tan x$

Solution:

(a) $$(1-2x)^{\frac{1}{3}}=1+\frac{1}{3}(-2x)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(-2x)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(-2x)^3+\cdots$$
$$=1-\frac{2}{3}x-\frac{4}{9}x^2-\frac{40}{81}x^3+\cdots$$

Expansion is valid for $|2x|<1 \iff |x|<\dfrac{1}{2} \iff -\dfrac{1}{2}<x<\dfrac{1}{2}$

(b) $$(2+x)^{-2}=\left[2\left(1+\frac{x}{2}\right)\right]^{-2}$$
$$=\frac{1}{4}\left(1+\frac{x}{2}\right)^{-2}$$
$$=\frac{1}{4}\left[1-2\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$
$$=\frac{1}{4}-\frac{1}{4}x+\frac{3}{16}x^2-\frac{1}{8}x^3+\cdots$$

Expansion is valid for $\left|\dfrac{x}{2}\right|<1 \iff |x|<2 \iff -2<x<2$.

(c) $\dfrac{1-x^2}{\sqrt[3]{1+3x}} = (1-x^2)(1+3x)^{-\frac{1}{3}}$

$$= (1-x^2)\left[1+\left(-\frac{1}{3}\right)(3x)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}(3x)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}(3x)^3+\cdots\right]$$

$$= (1-x^2)\left[1-x+2x^2-\frac{14}{3}x^3+\cdots\right]$$

$$= 1-x+(2-1)x^2+\left(-\frac{14}{3}+1\right)x^3+\cdots$$

$$= 1-x+x^2-\frac{11}{3}x^3+\cdots$$

Expansion is valid for $|3x|<1 \iff |x|<\frac{1}{3} \iff -\frac{1}{3}<x<\frac{1}{3}$

(d) $\tan x = \dfrac{\sin x}{\cos x}$

$$= \frac{x - \frac{x^3}{3!} + \cdots}{1 - \frac{x^2}{2!} + \cdots}$$

$$= \left(x - \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \cdots\right)^{-1}$$

$$= \left(x - \frac{x^3}{3!} + \cdots\right)\left(1 + (-1)\left(-\frac{x^2}{2!}\right) + \cdots\right)$$

$$= \left(x - \frac{x^3}{6} + \cdots\right)\left(1 + \frac{x^2}{2} + \cdots\right)$$

$$= x + \frac{x^3}{2} - \frac{x^3}{6} + \cdots$$

$$= x + \frac{x^3}{3} + \cdots$$

Example 10

Expand $\dfrac{5}{(1+3x)(1-2x)}$ as a series of ascending powers of x, as far as the term in x^3.

State the range of values of x for which the expansion is valid.

Solution:

Let $\dfrac{5}{(1+3x)(1-2x)}=\dfrac{A}{1+3x}+\dfrac{B}{1-2x}$

Then $\quad 5=A(1-2x)+B(1+3x)$

Substitute $x=\dfrac{1}{2}$, $B=2$

Substitute $x=-\dfrac{1}{3}$, $A=3$

$$\begin{aligned}
\frac{5}{(1+3x)(1-2x)} &= \frac{3}{1+3x}+\frac{2}{1-2x} \\
&= 3\left(1+3x\right)^{-1}+2\left(1-2x\right)^{-1} \\
&= 3\left[1-3x+\left(3x\right)^2-\left(3x\right)^3+\cdots\right]+2\left[1+2x+\left(2x\right)^2+\left(2x\right)^3+\cdots\right] \\
&= \left(3+2\right)+\left(-9+4\right)x+\left(27+8\right)x^2+\left(-81+16\right)x^3+\cdots \\
&= 5-5x+35x^2-65x^3+\cdots
\end{aligned}$$

Expansion is valid for $\quad |3x|<1 \quad \text{and} \quad |2x|<1$

$$|x|<\frac{1}{3} \quad \text{and} \quad |x|<\frac{1}{2}$$

$$-\frac{1}{3}<x<\frac{1}{3} \quad \text{and} \quad -\frac{1}{2}<x<\frac{1}{2}$$

i.e. $\quad -\dfrac{1}{3}<x<\dfrac{1}{3}$

Example 11 [9205/1998/01/Q12]

Express $\mathrm{f}(x)=\dfrac{7-3x-x^2}{(1-x)^2(2+x)}$ in partial fractions. Hence or otherwise prove that, if x^3 and higher powers of x may be neglected, then $\mathrm{f}(x)\approx\dfrac{1}{8}(28+30x+41x^2)$.

Solution:

Let $\mathrm{f}(x)=\dfrac{7-3x-x^2}{(1-x)^2(2+x)}=\dfrac{A}{1-x}+\dfrac{B}{(1-x)^2}+\dfrac{C}{2+x}$

$\Rightarrow\ 7-3x-x^2=A(1-x)(2+x)+B(2+x)+C(1-x)^2$

Substitute $x=1$, $B=1$

Substitute $x=-2$, $C=1$

Substitute $x=0$, $2A+2B+C=7\ \Rightarrow\ A=2$

$\therefore\ \mathrm{f}(x)=\dfrac{2}{1-x}+\dfrac{1}{(1-x)^2}+\dfrac{1}{2+x}$

Note : $(1-x)^{-2}=\dfrac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-1}=\dfrac{\mathrm{d}}{\mathrm{d}x}1+\dfrac{\mathrm{d}}{\mathrm{d}x}x+\dfrac{\mathrm{d}}{\mathrm{d}x}x^2+\ldots+\dfrac{\mathrm{d}}{\mathrm{d}x}x^{r+1}+\ldots$

$=0+1+2x+\ldots+(r+1)x^r+\ldots$

i.e. $(1-x)^{-2}=1+2x+3x^2+\ldots+(r+1)x^r+\ldots$ for $|x|<1$

$$\begin{aligned}\mathrm{f}(x)&=2(1-x)^{-1}+(1-x)^{-2}+(2+x)^{-1}\\&=2(1+x+x^2+\ldots)+(1+2x+3x^2+\ldots)+\left[2\left(1+\frac{x}{2}\right)\right]^{-1}\\&=2(1+x+x^2+\ldots)+(1+2x+3x^2+\ldots)+\frac{1}{2}\left[1-\frac{x}{2}+\left(\frac{x}{2}\right)^2+\ldots\right]\\&=\left(2+1+\frac{1}{2}\right)+\left(2+2-\frac{1}{4}\right)x+\left(2+3+\frac{1}{8}\right)x^2+\ldots\\&=\frac{7}{2}+\frac{15}{4}x+\frac{41}{8}x^2+\ldots\\&\approx\frac{1}{8}(28+30x+41x^2)\text{ (shown)}\end{aligned}$$

3 Applications of Series Expansion of a Function

Series expansion is a powerful mathematical technique with wide-ranging applications in various fields. By expressing a function as an infinite sum of terms, one can gain valuable insights into the behavior of functions and their approximations.

3.1 Estimation

Example 12

By using the first 3 terms in the expansion of $\sqrt{1+x}$, and substituting $x=0.01$, show that $\sqrt{101}\approx\dfrac{80399}{8000}$.

Solution:

$$\sqrt{1+x}=(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\cdots=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$$

Substitute $x=0.01$ into above expansion, we get

$$\sqrt{1+0.01}\approx 1+\frac{1}{2}(0.01)-\frac{1}{8}(0.01)^2$$

$$\sqrt{\frac{101}{100}}\approx\frac{80000+400-1}{80000}$$

$$\frac{1}{10}\sqrt{101}\approx\frac{80399}{80000}$$

$$\sqrt{101}\approx\frac{80399}{8000}\text{ (shown)}$$

Example 13

Given that $\tan y=x$, show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$. Hence express $\tan^{-1}x$ as a series in ascending powers of x up to the term in x^3. Use the fact that $\dfrac{\pi}{4}=\tan^{-1}\dfrac{1}{2}+\tan^{-1}\dfrac{1}{3}$ to deduce an estimate of π to 3 significant figures.

Solution:

$\tan y=x \quad\Rightarrow\quad y=\tan^{-1}x$

Differentiate w.r.t. x, $\quad\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{1+x^2}$

Differentiate w.r.t. x, $\quad\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\left(\dfrac{1}{1+x^2}\right)^2(2x)\quad\Rightarrow\quad\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ (shown)

Differentiate w.r.t. x, $\quad\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-4x\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$

Substitute $x=0$, $y=0$, $\frac{dy}{dx}=1$, $\frac{d^2y}{dx^2}=0$, $\frac{d^3y}{dx^3}=-2$

By Maclaurin Theorem, $\tan^{-1}x = x-\frac{1}{3}x^3+\cdots$

$$\frac{\pi}{4}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$$

$$\frac{\pi}{4}\approx\left[\frac{1}{2}-\frac{1}{3}\left(\frac{1}{2}\right)^3\right]+\left[\frac{1}{3}-\frac{1}{3}\left(\frac{1}{3}\right)^3\right]$$

$$\pi\approx 3.12$$

Example 14 [AJC Prelims 9740/2009/01/Q2]

(i) Find the expansion of $(x-8)^{\frac{1}{3}}$ in ascending powers of x up to and including the term in x^2. State the set of values of x for which the expansion is valid.

(ii) Using the expansion found in **(i)** and $x=-\frac{17}{8}$, obtain an estimate for $\sqrt[3]{3}$, giving your answer to three decimal places.

Solution:

(i) $(x-8)^{\frac{1}{3}}=(-8+x)^{\frac{1}{3}}$

$$=\left[-8\left(1-\frac{x}{8}\right)\right]^{\frac{1}{3}}=(-8)^{\frac{1}{3}}\left(1-\frac{x}{8}\right)^{\frac{1}{3}}$$

$$=-2\left[1+\frac{1}{3}\left(-\frac{x}{8}\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\frac{x}{8}\right)^2+...\right]$$

$$=-2\left(1-\frac{1}{24}x-\frac{1}{576}x^2+...\right)$$

Expansion is valid for $\left|\frac{x}{8}\right|<1 \iff |x|<8 \iff -8<x<8$

(ii) Substitute $x=-\frac{17}{8}$ into expansion in **(i)**, we get

$$\left(-\frac{17}{8}-8\right)^{\frac{1}{3}}\approx-2\left[1-\frac{1}{24}\left(-\frac{17}{8}\right)-\frac{1}{576}\left(-\frac{17}{8}\right)^2\right]$$

$$\left(-\frac{81}{8}\right)^{\frac{1}{3}}\approx-2.16140$$

$$-\frac{3}{2}\sqrt[3]{3}\approx-2.16140$$

$$\sqrt[3]{3}\approx1.441$$

3.2 Linear Approximations

Given a function $y = \mathrm{f}(x)$, the Maclaurin series expansion is

$$y = \mathrm{f}(0) + \mathrm{f}'(0)x + \frac{\mathrm{f}''(0)}{2!}x^2 + \frac{\mathrm{f}^{(3)}(0)}{3!}x^3 + \ldots + \frac{\mathrm{f}^{(n)}(0)}{n!}x^n + \ldots$$

The graph of $y = \mathrm{f}(x)$ passes through $\left(0, \mathrm{f}\left(0\right)\right)$ and the gradient of $y = \mathrm{f}(x)$ at $\left(0, \mathrm{f}\left(0\right)\right)$ is $\mathrm{f}'(0)$. Thus, the equation of the tangent to $y = \mathrm{f}(x)$ at $x = 0$ is $y = \mathrm{f}(0) + \mathrm{f}'(0)x$.

Notice that this equation corresponds to the Maclaurin series for $y = \mathrm{f}(x)$ up to and including the term in x.

For example, $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$, so $y = 1 + \frac{1}{2}x$ is a tangent to the graph of $y = \sqrt{1+x}$ at $x = 0$.

3.3 Trigonometric Series – Small Angle Approximations

One of the most important applications of the power series for trigonometric functions is in situations involving very small angles (angles close to zero, and not negative angles with very large magnitude).

Recall:

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\tan x = x + \frac{1}{3}x^3 + \cdots$$ (refer to **Example 9d**)

When x is small, x^3 and higher power of x are very small and therefore the trigonometric functions can be approximated by truncating the respective series expansions from the term in x^3. This gives rise to the following small angle approximations (see Appendix 3):

> For all small angles, positive or negative, we have
>
> (1) $\sin x \approx x$
>
> (2) $\cos x \approx 1 - \frac{x^2}{2}$
>
> (3) $\tan x \approx x$, where x is measured in **radians**.

Note: In using a small angle approximation, the "error" is as shown:

$$\left|\sin x - x\right| \le \frac{|x|^3}{3!}$$

$$\left|\cos x - \left(1 - \frac{x^2}{2!}\right)\right| \le \frac{|x|^4}{4!}$$

We observe that the smaller the value of x, the smaller the error and thus, the better the approximation.
The above result can also be extended to the general case for $k \in \mathbb{R}$:

(a) $\sin kx \approx kx$ (b) $\cos kx \approx 1-\dfrac{(kx)^2}{2}$ (c) $\tan kx \approx kx$

i.e. if x is small, kx is also considered small.

Examples of the use of the small angle approximations are in the calculation of the period of a simple pendulum, and in the calculation of the intensity minima in single slit diffraction. This approximation is used in most of the common expressions of geometrical optics which are built on the concept of surface power for lenses.

Example 15

Given that θ is sufficiently small for θ^3 and higher powers of θ to be neglected, obtain the series expansion for the following expressions:

(a) $\dfrac{\sin 3\theta}{1+\cos 2\theta}$ **(b)** $\sin\left(\alpha+\dfrac{\theta}{2}\right)\tan\dfrac{\theta}{2}$

Solution:

(a) $\dfrac{\sin 3\theta}{1+\cos 2\theta}$ $\approx \dfrac{3\theta}{1+\left(1-\frac{1}{2}(2\theta)^2\right)}$ since θ is small $=\dfrac{3\theta}{2-2\theta^2}$ $=\dfrac{3}{2}\theta\left(1-\theta^2\right)^{-1}$ $=\dfrac{3}{2}\theta\left(1+\theta^2+\cdots\right)=\dfrac{3}{2}\theta+\cdots$	**(b)** $\sin\left(\alpha+\dfrac{\theta}{2}\right)\tan\dfrac{\theta}{2}$ $=\left(\sin\alpha\cos\dfrac{\theta}{2}+\cos\alpha\sin\dfrac{\theta}{2}\right)\tan\dfrac{\theta}{2}$ $\approx\left(\sin\alpha\left(1-\dfrac{1}{2}\left(\dfrac{\theta}{2}\right)^2\right)+(\cos\alpha)\left(\dfrac{\theta}{2}\right)\right)\dfrac{\theta}{2}$ since θ is small $=\dfrac{\sin\alpha}{2}\theta+\dfrac{\cos\alpha}{4}\theta^2+\cdots$

Example 16 [9205/1997/01/Q8]

Given that x is sufficiently small for x^3 and higher powers of x to be neglected, and that $\cos x-4\sin x=6x$, show that a quadratic equation for x is $x^2+20x-2\approx 0$.
Hence find an approximate value for x, giving your answer to 3 significant figures.

Solution:

$$\cos x-4\sin x=6x$$

Since x is small, $\left(1-\dfrac{x^2}{2}\right)-4x\approx 6x$

$$2-x^2-8x\approx 12x$$

$$x^2+20x-2\approx 0 \quad \text{(shown)}$$

$$x\approx\frac{-20\pm\sqrt{400-4(-2)}}{2}\approx 0.0995 \text{ or } -20.1$$

Since x is close to zero, $x\approx 0.0995$ (3sf).

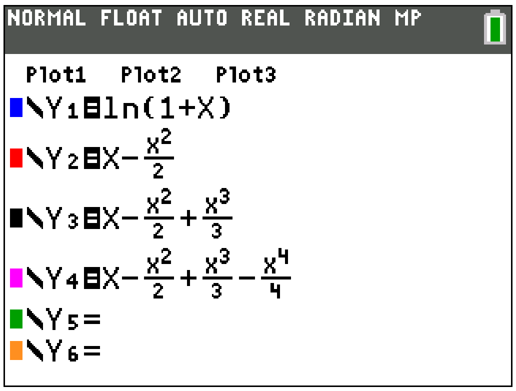
APPENDIX 1

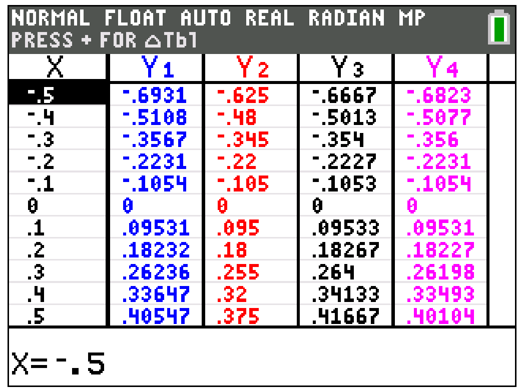
A tableau representation (using the GC) of the effects of the number of terms used in the Maclaurin series for $\ln(1+x)$.

Consider the power series $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$ for $-1 < x \le 1$.

By observing the table of values, we can see that the more terms we take on the right hand side of the equation, the better is the approximation.

Let Y_1 be $\ln(1+x)$. Let Y_2, Y_3 and Y_4 represent the first two, three and four terms of the power series for $\ln(1+x)$ respectively.

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X	Y1	Y2	Y3	Y4
-.5	-.6931	-.625	-.6667	-.6823
-.4	-.5108	-.48	-.5013	-.5077
-.3	-.3567	-.345	-.354	-.356
-.2	-.2231	-.22	-.2227	-.2231
-.1	-.1054	-.105	-.1053	-.1054
0	0	0	0	0
.1	.09531	.095	.09533	.09531
.2	.18232	.18	.18267	.18227
.3	.26236	.255	.264	.26198
.4	.33647	.32	.34133	.33493
.5	.40547	.375	.41667	.40104

X=-.5

APPENDIX 2

Binomial Theorem

A **binomial** is an algebraic expression with two terms, e.g. $x-y$, $3a+2b$, m^2-n.

Pascal's Triangle

We may be interested to expand the powers of a binomial as follows:

$$(1+x)^2=(1+x)(1+x)=1+2x+x^2$$
$$(1+x)^3=(1+x)^2(1+x)=(1+2x+x^2)(1+x)=1+3x+3x^2+x^3$$
$$(1+x)^4=(1+x)^3(1+x)=(1+3x+3x^2+x^3)(1+x)=1+4x+6x^2+4x^3+x^4$$

Arranging the coefficients of the powers of x in a triangular array yields the following:

$$\begin{array}{ll}
(1+x)^0 & \qquad\qquad\qquad 1 \\
(1+x)^1 & \qquad\qquad\quad 1 \quad 1 \\
(1+x)^2 & \qquad\qquad 1 \quad 2 \quad 1 \\
(1+x)^3 & \qquad\quad 1 \quad 3 \quad 3 \quad 1 \\
(1+x)^4 & \qquad 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
(1+x)^5 & \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
 & \qquad\qquad\quad \cdots\cdots
\end{array}$$

This is known as the **Pascal's Triangle**.

However if the power is large, forming the Pascal's Triangle to obtain the coefficients becomes tedious.

The Factorial and $\binom{n}{r}$ notations

Recall that for any $r \in \mathbb{Z}^+$, we define $r!$ (read as r factorial) as

$$r! = r(r-1)(r-2)...(3)(2)(1), \text{ where } 0!=1.$$

We'll now define another notation $\binom{n}{r}$ (read as n choose r), also written as ${}^nC_r$, as

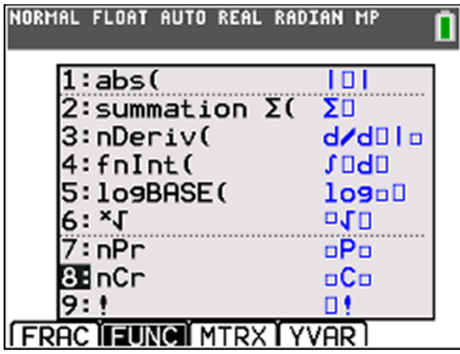
$$\binom{n}{r}=\frac{n!}{r!(n-r)!}, \text{ where } n\in\mathbb{Z}^+,\ r\in\mathbb{Z}_0^+,\ 0\le r\le n.$$

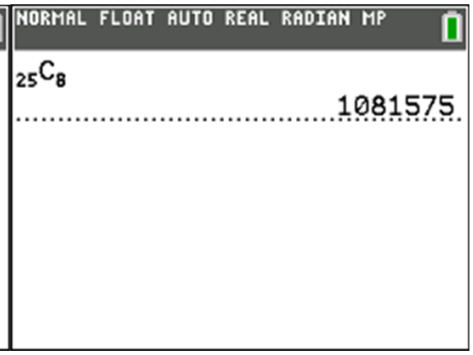
Note : $\binom{n}{r}=\binom{n}{n-r}$

This notation has a physical meaning attached to it; it represents the number of ways of choosing r objects from n distinct objects without replacement. Sounds interesting? You'll learn more about it in the topic of Permutations and Combinations.

How to use a GC to find the value of $\binom{n}{r}$?

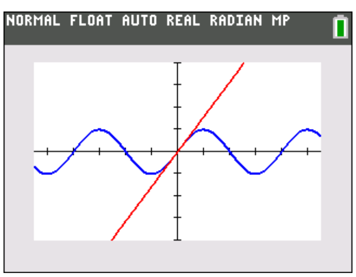
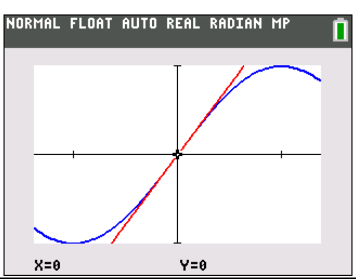
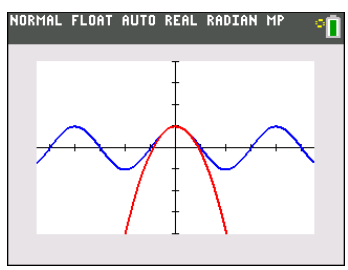
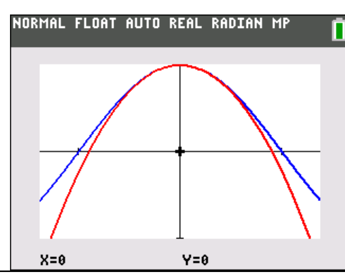
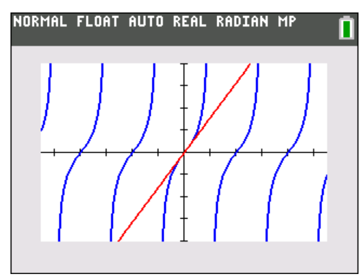
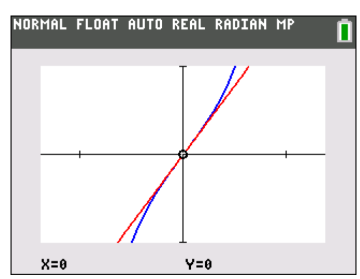
For example, if you want to find the value of $\binom{25}{8}$,





APPENDIX 3

Using a Graphing Calculator to illustrate Small Angle Approximations

Key in $Y_1 = \sin x$ and $Y_2 = x$ It can be seen from the diagram that $y = \sin x$ and $y = x$ take the same values for very small values of x.  	Key in $Y_1 = \cos x$ and $Y_2 = 1 - \frac{x^2}{2}$ It can be seen from the diagram that $y = \cos x$ and $y = 1 - \frac{x^2}{2}$ take the same values for very small values of x.  	Key in $Y_1 = \tan x$ and $Y_2 = x$ It can be seen from the diagram that $y = \tan x$ and $y = x$ take the same values for very small values of x.  

SUMMARY