

10

INTEGRATION TECHNIQUES

SYLLABUS

- Integration of
 - $f'(x)(f(x))^n$ (including $n = -1$)
 - $f'(x)e^{f(x)}$
 - $\sin^2 x, \cos^2 x, \tan^2 x$
 - $\frac{1}{a^2 + x^2}, \frac{1}{\sqrt{a^2 - x^2}}, \frac{1}{a^2 - x^2}, \frac{1}{x^2 - a^2}$
- Integration by a given substitution
- Integration by parts

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LECTURE 1

Lesson Outline

- Integration as a Reverse Operation to Differentiation
 - Integration of Standard Functions
 - $f'(x)(f(x))^n$
 - Exponential Functions
 - Trigonometric Functions
-

1 INTEGRATION AS A REVERSE OPERATION TO DIFFERENTIATION

Integration reverses the process of differentiation, i.e., for any constant C ,

$$\frac{d}{dx}(f(x)) = f'(x) \Leftrightarrow \int f'(x) dx = f(x) + C.$$

For example, $\frac{d}{dx}(e^{\sin x}) = \cos x e^{\sin x} \Leftrightarrow \int \cos x e^{\sin x} dx = e^{\sin x} + C.$

Notes

- $\frac{d}{dx}(f(x))$ means ‘differentiating the function $f(x)$ with respect to (w.r.t.) x ’.
- $\int f(x) dx$ means ‘integrating the function $f(x)$ w.r.t o x ’.
- Integration is distributive with respect to addition, subtraction and scalar multiplication, i.e.,

$$\int (a.f(x) \pm b.g(x)) dx = a \int f(x) dx \pm b \int g(x) dx.$$

2 INTEGRATION OF STANDARD FUNCTIONS

Recap integration of x^n and $(ax+b)^n$, $a \neq 0$ from ‘O’ level:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$	$\int x^{-1} dx = \int \frac{1}{x} dx = \ln x + C$
$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$	$\int (ax+b)^{-1} dx = \int \frac{1}{ax+b} dx$ $= \frac{\ln ax+b }{a} + C$

Example 1Integrate the following with respect to x .

(a) $5x^4 - \frac{4}{x} + \frac{7}{(2x+1)^3}$

(b) $\frac{(1-x)(2x-1)}{2\sqrt{x}}$

(c) $\frac{1}{\sqrt{x+2} - \sqrt{x}}$

Solution

$$\begin{aligned}
 \text{(a)} \quad \int 5x^4 - \frac{4}{x} + \frac{7}{(2x+1)^3} dx &= \frac{5x^5}{5} - 4\ln|x| + 7 \frac{(2x+1)^{-2}}{(2)(-2)} + C \\
 &= x^5 - 4\ln|x| - \frac{7}{4(2x+1)^2} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int \frac{(1-x)(2x-1)}{2\sqrt{x}} dx &= \frac{1}{2} \int \frac{-1+3x-2x^2}{\sqrt{x}} dx \\
 &= \frac{1}{2} \int -\frac{1}{\sqrt{x}} + 3\sqrt{x} - 2x^{\frac{3}{2}} dx \\
 &= \frac{1}{2} \left(-\frac{\sqrt{x}}{\frac{1}{2}} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{\frac{5}{2}} \right) + C \\
 &= -\sqrt{x} + x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int \frac{1}{\sqrt{x+2} - \sqrt{x}} dx &= \int \frac{1}{\sqrt{x+2} - \sqrt{x}} \cdot \frac{\sqrt{x+2} + \sqrt{x}}{\sqrt{x+2} + \sqrt{x}} dx \\
 &= \int \frac{\sqrt{x+2} + \sqrt{x}}{(x+2) - (x)} dx = \frac{1}{2} \int \sqrt{x+2} + \sqrt{x} dx \\
 &= \frac{1}{2} \left(\frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) + C = \frac{1}{3}(x+2)^{\frac{3}{2}} + \frac{1}{3}x^{\frac{3}{2}} + C
 \end{aligned}$$

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2.1 Integration of $f'(x)(f(x))^n$

From $\frac{d}{dx}(f(x))^{n+1} = (n+1)(f(x))^n f'(x)$, $n \neq -1$ and $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$,

integrate both sides with respect to x to get

For $n \neq -1$	For $n = -1$
$\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + C$	$\int f'(x)(f(x))^{-1} dx = \int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$

Examples of $\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + C$ where $n \neq -1$

$$(a) \quad \int \underbrace{(2x)}_{f'(x)} \underbrace{(x^2+1)^3}_{[f(x)]^n} dx = \frac{(x^2+1)^4}{4} + C$$

$$(b) \quad \int \underbrace{(2x)}_{f'(x)} \underbrace{(x^2+1)^{-3}}_{[f(x)]^n} dx = \boxed{}$$

$$(c) \quad \int \underbrace{\cos x}_{f'(x)} \underbrace{\sin^2 x}_{[f(x)]^n} dx = \frac{(\sin x)^3}{3} + C$$

$$(d) \quad \int \underbrace{\sec^2 x}_{f'(x)} \underbrace{\tan^2 x}_{[f(x)]^n} dx = \boxed{}$$

$$\left| \begin{array}{l} \sin^2 x = (\sin x)^2 \\ \tan^2 x = (\tan x)^2 \end{array} \right.$$

Examples of $\int f'(x)(f(x))^{-1} dx = \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

$$(a) \quad \int \frac{2x}{x^2+1} dx = \ln(x^2+1) + C$$

$$(b) \quad \int \frac{3x^2+2x}{x^3+x^2+1} dx = \ln|x^3+x^2+1| + C$$

$$(c) \quad \int \cot x dx = \boxed{} = \ln|\sin x| + C$$

$$(d) \quad \int \tan x dx = \boxed{}$$

$$= -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C$$

$$= \ln|\sec x| + C$$

Example 2

Find

$$(a) \int \frac{7x^2}{\sqrt{x^3+1}} dx \quad (b) \int \frac{1}{x \ln x} dx \quad (c) \int \frac{1}{1-4e^{-x}} dx$$

Solution

$$\begin{aligned}
 (a) \quad \int \frac{7x^2}{\sqrt{x^3+1}} dx &= 7 \int x^2 (x^3+1)^{-\frac{1}{2}} dx \\
 &= \frac{7}{\boxed{}} \int \underbrace{\boxed{}}_{f'(x)} x^2 \underbrace{(x^3+1)^{-\frac{1}{2}}}_{(f(x))^n} dx \\
 &= \frac{7}{3} \cdot \frac{(x^3+1)^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{14}{3} \sqrt{x^3+1} + C
 \end{aligned}$$

Use $\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1}$

$$\begin{aligned}
 (b) \quad \int \frac{1}{x \ln x} dx &= \int \frac{1}{x} \cdot \frac{1}{\ln x} dx \\
 &= \int \frac{\boxed{}_{f'(x)}}{\boxed{}_{f(x)}} dx = \ln |\ln x| + C
 \end{aligned}$$

Use $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$

$$\begin{aligned}
 (c) \quad \int \frac{1}{1-4e^{-x}} dx &= \int \frac{1}{e^{-x}(e^x-4)} dx \\
 &= \int \frac{\boxed{}_{f'(x)}}{\underbrace{e^x-4}_{f(x)}} dx = \ln |e^x-4| + C
 \end{aligned}$$

Use $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$

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2.2 Integration of Exponential Functions

Based on the differentiation of e^x and $a^x, a \neq 0$, we have the following results:

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C \quad \text{and} \quad \int a^x dx = \frac{1}{\ln a} a^x + C, a \neq 0, 1.$$

Differentiate $e^{f(x)}$ and $a^{f(x)}$ respectively, we have the following results:

$\int e^x dx = e^x + C$	$\int f'(x)e^{f(x)} dx = e^{f(x)} + C$
$\int a^x dx = \frac{1}{\ln a} a^x + C$	$\int f'(x)a^{f(x)} dx = \frac{1}{\ln a} a^{f(x)} + C$

Examples of $\int f'(x)e^{f(x)} dx = e^{f(x)} + C$

$$(a) \int_{f'(x)}^{f(x)} 2 e^{2x} dx = e^{2x} + C$$

$$(b) \int_{f'(x)}^{f(x)} 2x e^{x^2} dx = e^{x^2} + C$$

$$(c) \int_{f'(x)}^{f(x)} \cos x e^{\sin x} dx = e^{\sin x} + C$$

$$(d) \int_{f'(x)}^{f(x)} \underbrace{\sec^2 x}_{f'(x)} e^{\tan x} dx = e^{\tan x} + C$$

Example 3

Integrate the following with respect to x .

$$(a) 4^{5-3x}$$

$$(b) (\sin x + x \cos x)e^{x \sin x}$$

$$(c) \frac{e^{\tan x}}{(1 + \sin x)(1 - \sin x)}$$

Solution

$$(a) \int 4^{5-3x} dx = \frac{1}{-3} \int_{f'(x)} (-3) 4^{5-3x} dx$$

$$\left. \begin{aligned} \text{Use } \int f'(x) a^{f(x)} dx \\ = \frac{1}{\ln a} a^{f(x)} + C \end{aligned} \right|$$

=

=

$$\left. \begin{aligned} \text{Use } \int f'(x) e^{f(x)} dx \\ = e^{f(x)} + C \end{aligned} \right|$$

$$(b) \int \underbrace{(\sin x + x \cos x)}_{f'(x)} e^{x \sin x} dx =$$

$$(c) \int \frac{e^{\tan x}}{(1 + \sin x)(1 - \sin x)} dx =$$

$$\left. \begin{aligned} \text{Use } \int f'(x) e^{f(x)} dx \\ = e^{f(x)} + C \end{aligned} \right|$$

=

$$= e^{\tan x} + C$$

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2.3 Integration of Trigonometric Functions**2.3.1 Integration of Standard Trigonometric Functions**

Recap integration of trigonometric functions from O-level:

$\int \cos x dx = \sin x + C$	$\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C, a \neq 0$
$\int \sin x dx = -\cos x + C$	$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + C, a \neq 0$
$\int \sec^2 x dx = \tan x + C$	$\int \sec^2(ax + b) dx = \frac{1}{a} \tan(ax + b) + C, a \neq 0$

Note that the angle x is measured in radians.

Also recap differentiation of trigonometric functions from Chapter 5:

$\int \operatorname{cosec}^2 x dx = -\cot x + C$	$\int \operatorname{cosec}^2(ax + b) dx = -\frac{1}{a} \cot(ax + b) + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \sec(ax + b) \tan(ax + b) dx = \frac{1}{a} \sec(ax + b) + C$
$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$	$\int \operatorname{cosec}(ax + b) \cot(ax + b) dx = -\frac{1}{a} \operatorname{cosec}(ax + b) + C$

Note

Other integration formulae of trigonometric functions which can be found in MF 27, are given as follows without the modulus, you should add in the modulus sign where it is necessary:

$$\int \tan x \, dx = \ln(\sec x) + C \quad \left(|x| < \frac{1}{2}\pi\right)$$

$$\int \cot x \, dx = \ln(\sin x) + C \quad (0 < x < \pi)$$

$$\int \operatorname{cosec} x \, dx = -\ln(\operatorname{cosec} x + \cot x) + C \quad (0 < x < \pi)$$

$$\int \sec x \, dx = \ln(\sec x + \tan x) + C \quad \left(|x| < \frac{1}{2}\pi\right)$$

Example 4

Integrate the following with respect to x .

(a) $\cos\left(\frac{\pi}{4} - 2x\right)$

(b) $\tan x \sec^4 x$

(c) $\frac{2 + \sin x}{\cos^2 x}$

(d) $\frac{\operatorname{cosec}^2 x}{(2 + \cot x)^2}$

Solution

(a) $\int \cos\left(\frac{\pi}{4} - 2x\right) dx =$

(b) $\int \tan x \sec^4 x \, dx =$

$$= \frac{\sec^4 x}{4} + C$$

$$\begin{array}{l} f(x) = \sec x \text{ and} \\ f'(x) = \sec x \tan x \end{array}$$

(c) $\int \frac{2 + \sin x}{\cos^2 x} dx = \int \frac{2}{\cos^2 x} + \frac{\sin x}{\cos^2 x} dx$

$$=$$

$$=$$

$$= 2 \tan x + \sec x + C$$

(d) $\int \frac{\operatorname{cosec}^2 x}{(2 + \cot x)^2} dx =$

$$=$$

$$= \frac{1}{(2 + \cot x)} + C$$

$$\begin{array}{l} f(x) = 2 + \cot x \text{ and} \\ f'(x) = -\operatorname{cosec}^2 x \end{array}$$

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Extra Practice

Annex B: Practice Question 1 – 4

LECTURE 2

Lesson Outline

- Integration of Trigonometric Functions using Trigonometric Identities
 - Integration of $\frac{P(x)}{Q(x)}$ or $\frac{P(x)}{\sqrt{Q(x)}}$
 - Integration of $\frac{1}{a^2 + x^2}$ and $\frac{1}{\sqrt{a^2 - x^2}}, |x| < a$
-

2.3.2 Integration using Trigonometric Identities

The following trigonometric identities can be used to modify the integrands, making them easier to integrate.

Double Angle Formula

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$$

Pythagorean Identity

$$\sin^2 x + \cos^2 x \equiv 1$$

$$\tan^2 x + 1 \equiv \sec^2 x$$

$$1 + \cot^2 x \equiv \operatorname{cosec}^2 x$$

Example 5

$$\begin{aligned} \text{(a)} \quad \int \cos^2 x \, dx &= \int \frac{1 + \cos 2x}{2} \, dx \\ &= \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) + C = \frac{1}{2} x + \frac{\sin 2x}{4} + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) + C = \frac{1}{2} x - \frac{\sin 2x}{4} + C \end{aligned}$$

$$\text{(c)} \quad \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$$

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Example 6Find (a) $\int \sin^2 x \cos^2 x \, dx$ (b) $\int \sin^4 x \, dx$ *Solution*

$$\begin{aligned}
 \text{(a)} \quad & \int \sin^2 x \cos^2 x \, dx \\
 &= \int (\sin x \cos x)^2 \, dx \\
 &= \\
 &= \\
 &= \\
 &= \frac{1}{8} \left(x - \frac{\sin 4x}{4} \right) + C \\
 &= \frac{x}{8} - \frac{\sin 4x}{32} + C
 \end{aligned}$$

Alternatively

$$\begin{aligned}
 & \int \sin^2 x \cos^2 x \, dx \\
 &= \\
 &= \\
 &= \\
 &= \\
 &= \frac{1}{8} \left(x - \frac{\sin 4x}{4} \right) + C \\
 &= \frac{x}{8} - \frac{\sin 4x}{32} + C
 \end{aligned}$$

$$\text{(b)} \quad \int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$$

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$$\begin{aligned}
 &= \frac{1}{4} \left(\frac{3}{2} x - \frac{2 \sin 2x}{2} + \frac{\sin 4x}{8} \right) + C \\
 &= \frac{3}{8} x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C
 \end{aligned}$$

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Example 7Find (a) $\int \cos^3 x \, dx$ (b) $\int \cos^2 x \sin^3 x \, dx$ *Solution*

(a) $\int \cos^3 x \, dx = \int \cos x \cos^2 x \, dx$

$$=$$

$$=$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

For the second integral, let
 $f(x) = \sin x$ and $f'(x) = \cos x$

$$f(x) = \cos x \text{ and } f'(x) = -\sin x \quad \text{(b) } \int \cos^2 x \sin^3 x \, dx =$$

$$=$$

$$= \int \cos^2 x \sin x - \cos^4 x \sin x \, dx$$

$$=$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

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Example 8Find (a) $\int \tan^3 x \, dx$ (b) $\int \sec^4 x \, dx$ *Solution*

$$\int \tan x \, dx \text{ is in MF27.} \quad \text{(a) } \int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$

$$=$$

$$=$$

$$= \frac{(\tan x)^2}{2} - \ln |\sec x| + C$$

$$= \frac{1}{2} \tan^2 x - \ln |\sec x| + C$$

For the first integral, let
 $f(x) = \tan x$ and $f'(x) = \sec^2 x$

$$(b) \quad \int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$$

$$=$$

$$=$$

$$= \frac{(\tan x)^3}{3} + \tan x + C$$

$$= \frac{1}{3} \tan^3 x + \tan x + C$$

For the second integral, let
 $f(x) = \tan x$ and $f'(x) = \sec^2 x$

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2.4 Integration of $\frac{P(x)}{Q(x)}$ or $\frac{P(x)}{\sqrt{Q(x)}}$

2.4.1 Integration of $\frac{1}{a^2 + x^2}$ AND $\frac{1}{\sqrt{a^2 - x^2}}$, $|x| < a$

Using the derivatives of $\tan^{-1} x$ and $\sin^{-1} x$, we get the following, where a is a constant:

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left(\frac{x}{a} \right) + C$$

Replacing x by $x+h$, we get:

$$\int \frac{1}{a^2 + (x+h)^2} \, dx = \frac{1}{a} \tan^{-1} \left(\frac{x+h}{a} \right) + C$$

$$\int \frac{1}{\sqrt{a^2 - (x+h)^2}} \, dx = \sin^{-1} \left(\frac{x+h}{a} \right) + C$$

To apply the two
 formulae, the coefficient
 of x^2 must be 1.

Note

From MF 27, we have $\frac{d}{dx} \left(\sin^{-1} \frac{x}{a} \right) = \frac{1}{a \sqrt{1 - \left(\frac{x}{a} \right)^2}} = \frac{1}{\sqrt{a^2 - x^2}}$.

Hence, $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left(\frac{x}{a} \right) + C$.

Example 9

Find

(a) $\int \frac{1}{1+3x^2} dx$

(b) $\int \frac{3}{\sqrt{6-2x^2}} dx$

(c) $\int \frac{1}{3x^2-2x+1} dx$

(d) $\int \frac{1}{\sqrt{6+4x-2x^2}} dx$

Solution

(a) $\int \frac{1}{1+3x^2} dx =$

Coefficient of x^2 is not 1, hence we factorise 3 out of the denominator.

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{\frac{1}{3}}} \tan^{-1} \frac{x}{\sqrt{\frac{1}{3}}} + C$$

$$= \frac{\sqrt{3}}{3} \tan^{-1} \sqrt{3}x + C$$

Coefficient of x^2 is not 1. Take note of $\sqrt{}$

(b) $\int \frac{3}{\sqrt{6-2x^2}} dx =$ $= \frac{3}{\sqrt{2}} \sin^{-1} \frac{x}{\sqrt{3}} + C$

(c) $\int \frac{1}{3x^2-2x+1} dx =$

Factorise out the coefficient of x^2 and complete square

=

$$= \frac{1}{3} \left[\frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \frac{(x-\frac{1}{3})}{\frac{\sqrt{2}}{3}} \right] + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x-1}{\sqrt{2}} \right) + C$$

(d) $\int \frac{1}{\sqrt{6+4x-2x^2}} dx =$

Factorise out the coefficient of x^2 and complete square. Note that you cannot factorise out -2 .

=

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{4-(x-1)^2}} dx$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{x-1}{2} + C$$

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LECTURE 3

Lesson Outline

- Integration of $\frac{1}{x^2 - a^2}$ and $\frac{1}{a^2 - x^2}$
 - Integration of $\frac{px+q}{ax^2+bx+c}$, $a \neq 0$ and $\frac{px+q}{\sqrt{ax^2+bx+c}}$, $p \neq 0$, $a < 0$
-

2.4.2 Integration of $\frac{1}{x^2 - a^2}$ AND $\frac{1}{a^2 - x^2}$

For $a > 0$,

$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left \frac{x-a}{x+a} \right + C$	$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left \frac{a+x}{a-x} \right + C$
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Note

In MF 27, the above formulae are given as follows, without the modulus:

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left(\frac{x-a}{x+a} \right) + c \quad (x > a)$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left(\frac{a+x}{a-x} \right) + c \quad (|x| < a)$$

Proof

$$\begin{aligned} \int \frac{1}{x^2 - a^2} dx &= \int \frac{1}{(x-a)(x+a)} dx \\ &= \int \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)} dx \\ &= \frac{1}{2a} \ln|x-a| - \frac{1}{2a} \ln|x+a| + C \\ &= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \end{aligned}$$

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Example 10

Find (a) $\int \frac{5}{9-x^2} dx$ (b) $\int \frac{1}{2x^2-2x-10} dx$

Solution

$$\begin{aligned} \text{(a)} \quad \int \frac{5}{9-x^2} dx &= \\ &= 5 \left(\frac{1}{2(3)} \ln \left| \frac{3+x}{3-x} \right| \right) + C \\ &= \frac{5}{6} \ln \left| \frac{3+x}{3-x} \right| + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \frac{1}{2x^2-2x-10} dx &= \\ &= \\ &= \frac{1}{2} \left(\frac{1}{2\left(\frac{\sqrt{21}}{2}\right)} \ln \left| \frac{\left(x-\frac{1}{2}\right)-\frac{\sqrt{21}}{2}}{\left(x-\frac{1}{2}\right)+\frac{\sqrt{21}}{2}} \right| \right) + C \\ &= \frac{1}{2\sqrt{21}} \ln \left| \frac{(2x-1)-\sqrt{21}}{(2x-1)+\sqrt{21}} \right| + C \end{aligned}$$

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2.4.3 Integration of $\frac{px+q}{ax^2+bx+c}$, $a \neq 0$

Consider whether this formula $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$ can be applied. If

the integrand doesn't fit the formula, evaluate the following cases:

Case 1 If ax^2+bx+c **can** be factorised, resolve into partial fractions and continue integration using standard form.

Case 2 If ax^2+bx+c **cannot** be factorised, the numerator is to be expressed in the form $kf'(x)+m$, where m and k are constants to be determined.

Let $px+q = kf'(x)+m$ where $f(x) = ax^2+bx+c$

$$\begin{aligned} \text{i.e.} \quad \int \frac{px+q}{ax^2+bx+c} dx &= \int \frac{kf'(x)+m}{f(x)} dx \\ &= k \int \frac{f'(x)}{f(x)} dx + m \underbrace{\int \frac{1}{f(x)} dx}_{\substack{\text{Complete square} \\ \text{then use } \int \frac{1}{a^2+x^2} dx}} \end{aligned}$$

Example 11Integrate the following with respect to x .

(a) $\frac{1-x}{(1+x)(1+x^2)}$

(b) $\frac{x+2}{x^2+2x+3}$

Solution

$$\begin{aligned}
 \text{(a)} \quad \int \frac{1-x}{(1+x)(1+x^2)} dx &= \quad \text{(by partial fractions)} \\
 &= \int \frac{1}{1+x} dx - \frac{1}{2} \int \frac{2x}{1+x^2} dx \\
 &= \ln|1+x| - \frac{1}{2} \ln|1+x^2| + C
 \end{aligned}$$

For the second integral,
let $f(x)=1+x^2$ and
 $f'(x)=2x$

$$\begin{aligned}
 \text{(b)} \quad \int \frac{x+2}{x^2+2x+3} dx &= \\
 &= \frac{1}{2} \ln|x^2+2x+3| + C + \int \frac{1}{(x+1)^2+2} dx \\
 &= \frac{1}{2} \ln|x^2+2x+3| + \frac{1}{\sqrt{2}} \tan^{-1} \frac{x+1}{\sqrt{2}} + C
 \end{aligned}$$

For the first integral,
let $f(x)=x^2+2x+3$
and $f'(x)=2x+2$

■

2.4.4 Integration of $\frac{px+q}{\sqrt{ax^2+bx+c}}$, $p \neq 0$, $a < 0$ Let $px+q = kf'(x) + m$ where $f(x) = ax^2 + bx + c$

$$\begin{aligned}
 \text{i.e.} \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx &= \int \frac{kf'(x)+m}{\sqrt{f(x)}} dx \\
 &= k \int \frac{f'(x)}{\sqrt{f(x)}} dx + m \int \frac{1}{\sqrt{f(x)}} dx \\
 &\quad \text{Complete square} \\
 &\quad \text{then use } \int \frac{1}{\sqrt{a^2-x^2}} dx
 \end{aligned}$$

Example 12

Integrate $\frac{x}{\sqrt{2x-x^2}}$ with respect to x .

Solution

$$\begin{aligned}
 \int \frac{x}{\sqrt{2x-x^2}} dx &= \\
 &= -\frac{1}{2} \int \frac{2-2x}{\sqrt{2x-x^2}} dx + \int \frac{1}{\sqrt{-(x-1)^2+1}} dx \\
 &= -\frac{1}{2} \frac{(2x-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + \sin^{-1}(x-1) + C \\
 &= -\sqrt{2x-x^2} + \sin^{-1}(x-1) + C
 \end{aligned}$$

■

Example 13 (Miscellaneous)

Find

$$\begin{array}{lll}
 \text{(a)} \int \frac{1}{4+9x} dx & \text{(b)} \int \frac{x}{4+9x^2} dx & \text{(c)} \int \frac{1}{4+9x^2} dx \\
 \text{(d)} \int \frac{1}{\sqrt{4-9x}} dx & \text{(e)} \int \frac{x}{\sqrt{4-9x^2}} dx & \text{(f)} \int \frac{1}{\sqrt{4-9x^2}} dx
 \end{array}$$

Solution

$$\text{(a)} \quad \int \frac{1}{4+9x} dx = \frac{1}{9} \ln|4+9x| + C$$

$$\begin{aligned}
 \text{(b)} \quad \int \frac{x}{4+9x^2} dx &= \\
 &= \frac{1}{18} \ln|4+9x^2| + C \\
 &= \frac{1}{18} \ln(4+9x^2) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int \frac{1}{4+9x^2} dx &= \\
 &= \frac{1}{9} \cdot \frac{1}{\sqrt{\frac{4}{9}}} \tan^{-1}\left(\frac{x}{\sqrt{\frac{4}{9}}}\right) + C \\
 &= \frac{1}{6} \tan^{-1} \frac{3x}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int \frac{1}{\sqrt{4-9x}} \, dx &= \\
 &= \frac{(4-9x)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)(-9)} + C \\
 &= -\frac{2}{9}\sqrt{4-9x} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \int \frac{x}{\sqrt{4-9x^2}} \, dx &= \\
 &= -\frac{1}{18} \frac{\sqrt{4-9x^2}}{\frac{1}{2}} + C \\
 &= -\frac{1}{9}\sqrt{4-9x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad \int \frac{1}{\sqrt{4-9x^2}} \, dx &= \\
 &= \frac{1}{3} \int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}} \, dx \\
 &= \frac{1}{3} \sin^{-1} \frac{x}{\frac{2}{3}} + C \\
 &= \frac{1}{3} \sin^{-1} \frac{3x}{2} + C
 \end{aligned}$$

■

Extra Practice

Annex B: Practice Question 6

LECTURE 4

Lesson Outline

- Integration by Substitution
 - Integration by Parts
-

3 INTEGRATION BY SUBSTITUTION

If the given integrand is not in one of the standard forms, we can transform it into one of the standard forms by changing the variable involved in the integration with a given substitution.

For example, $\int 2x(1+2x)^5 \, dx$ using the given substitution $u = 1 + 2x$.

$$u = 1 + 2x \Rightarrow \frac{du}{dx} = 2$$

$$\begin{aligned} \int 2x(1+2x)^5 \, dx &= \int 2x(1+2x)^5 \frac{dx}{du} \, du \\ &= \int \underbrace{2x(1+2x)^5}_{u^{-1} \cdot u^5} \underbrace{\frac{dx}{du}}_{\frac{1}{2}} \, du \\ &= \int (u-1)(u)^5 \left(\frac{1}{2}\right) \, du \\ &= \frac{1}{2} \int u^6 - u^5 \, du \\ &= \frac{1}{2} \left(\frac{u^7}{7} - \frac{u^6}{6} \right) + C \\ &= \frac{1}{2} \left(\frac{(1+2x)^7}{7} - \frac{(1+2x)^6}{6} \right) + C \\ &= \frac{1}{14}(1+2x)^7 - \frac{1}{12}(1+2x)^6 + C \end{aligned}$$

After integration, replace back the substitution. The final expression should be in terms of x again.

Example 14

By means of suitable substitutions, find

$$(a) \int \frac{1}{(x+1)\sqrt{x^2+2x}} dx \quad \left(\text{let } x+1 = \frac{1}{u}\right)$$

$$(b) \int \sqrt{1-4x^2} dx \quad \left(\text{let } x = \frac{1}{2} \sin \theta\right)$$

Solution

$$(a) \quad x+1 = \frac{1}{u} \Rightarrow \frac{dx}{du} = -\frac{1}{u^2}$$

$$\int \frac{1}{(x+1)\sqrt{x^2+2x}} dx = \int \frac{1}{(x+1)\sqrt{(x+1)^2-1}} \frac{dx}{du} du$$

=

$$= \int \frac{u^2}{\sqrt{1-u^2}} \left(-\frac{1}{u^2}\right) du$$

$$= \int -\frac{1}{\sqrt{1-u^2}} du$$

$$= -\sin^{-1} u + C$$

$$= -\sin^{-1} \left(\frac{1}{x+1}\right) + C$$

Alternatively,

$$\int -\frac{1}{\sqrt{1-u^2}} du = \cos^{-1} u + C.$$

Thus $\cos^{-1} \frac{1}{x+1} + C$ is also an alternative answer.

$$(b) \quad x = \frac{1}{2} \sin \theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{2} \cos \theta$$

$$\int \sqrt{1-4x^2} dx =$$

$$= \int \sqrt{1-\sin^2 \theta} \cdot \frac{1}{2} \cos \theta d\theta$$

$$= \frac{1}{2} \int \cos^2 \theta d\theta$$

$$= \frac{1}{2} \int \frac{\cos 2\theta + 1}{2} d\theta$$

$$= \frac{1}{4} \left(\frac{\sin 2\theta}{2} + \theta \right) + C$$

$$= \frac{1}{4} (\sin \theta \cos \theta + \theta) + C$$

$$= \frac{1}{4} (2x\sqrt{1-4x^2} + \sin^{-1} 2x) + C$$

Note that the final expression must be in terms of x . Why

■ $\cos \theta = \sqrt{1-4x^2} ?$

4 INTEGRATION BY PARTS

From the product rule for differentiation,

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx},$$

where u and v are functions of x .

Integrate both sides with respect to x ,

$$uv = \int \left(u \frac{dv}{dx} \right) dx + \int \left(v \frac{du}{dx} \right) dx$$

Rearranging the equation, we get

$$\int \left(u \frac{dv}{dx} \right) dx = uv - \int \left(v \frac{du}{dx} \right) dx.$$

Notes

- This method is used when any of the previous methods fail. The integrand is a product of two functions, where one is to be **integrated**, while the other is to be **differentiated**.
- A useful guide for the **choice of u** is provided by **LIATE** in this order of priority

The following will serve as a guide (not a rule) in choosing the function to be differentiated.

Decreasing preference for the function to be differentiated 	L – Logarithmic functions	e.g. $\ln(x-2)$
	I – Inverse trigo functions	e.g. $\sin^{-1}(3x)$
	A – Algebraic functions	e.g. $\sqrt{1-5x^4}$, $x^2 + 3$
	T – Trigonometric functions	e.g. $\tan 4x$
	E – Exponential functions	e.g. e^{-3x+1}

For example

(a) $\int x \ln x \, dx$

(b) $\int \ln x \, dx$

Solution

(a) $\int x \ln x \, dx =$ $\begin{aligned} &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \end{aligned}$	<table border="0" style="width: 100%;"> <tr> <td style="width: 50%;">$u = \ln x$</td> <td style="width: 50%;">$\frac{dv}{dx} = x$</td> </tr> <tr> <td>$\frac{du}{dx} = \frac{1}{x}$</td> <td>$u = \frac{x^2}{2}$</td> </tr> </table>	$u = \ln x$	$\frac{dv}{dx} = x$	$\frac{du}{dx} = \frac{1}{x}$	$u = \frac{x^2}{2}$
$u = \ln x$	$\frac{dv}{dx} = x$				
$\frac{du}{dx} = \frac{1}{x}$	$u = \frac{x^2}{2}$				

(b) $\int \ln x \, dx =$ $\begin{aligned} &= (\ln x)(x) - \int (x) \left(\frac{1}{x} \right) dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C \end{aligned}$	<table border="0" style="width: 100%;"> <tr> <td style="width: 50%;">$u = \ln x$</td> <td style="width: 50%;">$\frac{dv}{dx} = 1$</td> </tr> <tr> <td>$\frac{du}{dx} =$</td> <td>$u =$</td> </tr> </table>	$u = \ln x$	$\frac{dv}{dx} = 1$	$\frac{du}{dx} =$	$u =$
$u = \ln x$	$\frac{dv}{dx} = 1$				
$\frac{du}{dx} =$	$u =$				

Example 15

Find

(a) $\int x \tan^{-1} x \, dx$

(b) $\int \cos^{-1} x \, dx$

(c) $\int e^{2x} \sin x \, dx$

Solution

(a) $\int x \underbrace{\tan^{-1} x}_u \, dx =$ $\begin{aligned} &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx \\ &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) \, dx \\ &= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + C \\ &= \frac{1}{2} (x^2 \tan^{-1} x - x + \tan^{-1} x) + C \end{aligned}$	<table border="0" style="width: 100%;"> <tr> <td style="width: 50%;">$u = \tan^{-1} x$</td> <td style="width: 50%;">$\frac{dv}{dx} = x$</td> </tr> <tr> <td>$\frac{du}{dx} =$</td> <td>$u =$</td> </tr> </table>	$u = \tan^{-1} x$	$\frac{dv}{dx} = x$	$\frac{du}{dx} =$	$u =$
$u = \tan^{-1} x$	$\frac{dv}{dx} = x$				
$\frac{du}{dx} =$	$u =$				

$$(b) \quad \int \cos^{-1} x \, dx = \int 1 \cdot \underbrace{\cos^{-1} x}_u \, dx$$

$$\begin{array}{lcl}
 u = \cos^{-1} x & \left. \begin{array}{l} \frac{dv}{dx} = 1 \\ u = \end{array} \right| & = \\
 \frac{du}{dx} = & & = x \cos^{-1} x - \frac{1}{2} \int (-2x) \cdot (1-x^2)^{-\frac{1}{2}} \, dx \\
 & & = x \cos^{-1} x - \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C \\
 & & = x \cos^{-1} x - \sqrt{1-x^2} + C
 \end{array}$$

$$\begin{array}{lcl}
 u = e^{2x} & \left. \begin{array}{l} \frac{dv}{dx} = \sin x \\ u = \end{array} \right| & (c) \quad \int e^{2x} \sin x \, dx = \\
 \frac{du}{dx} = & & \int u \frac{dv}{dx} \\
 & & = -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx \\
 & & =
 \end{array}$$

$$\begin{array}{lcl}
 u = e^{2x} & \left. \begin{array}{l} \frac{dv}{dx} = \cos x \\ u = \end{array} \right| & \therefore \int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx \\
 \frac{du}{dx} = & & \Rightarrow 5 \int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2e^{2x} \sin x + C \\
 & & \Rightarrow \int e^{2x} \sin x \, dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + k \quad \text{where } k = \frac{C}{5}
 \end{array}$$

■

ANNEX A: SUMMARY OF FORMULAE

Formulae Found in MF27

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c \quad (|x| < a)$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left(\frac{x - a}{x + a} \right) + c \quad (x > a)$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left(\frac{a + x}{a - x} \right) + c \quad (|x| < a)$$

$$\int \tan x dx = \ln(\sec x) + C \quad (|x| < \frac{1}{2}\pi)$$

$$\int \cot x dx = \ln(\sin x) + C \quad (0 < x < \pi)$$

$$\int \operatorname{cosec} x dx = -\ln(\operatorname{cosec} x + \cot x) + C \quad (0 < x < \pi)$$

$$\int \sec x dx = \ln(\sec x + \tan x) + C \quad (|x| < \frac{1}{2}\pi)$$

Formulae not in MF27

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$$

$$\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \frac{1}{ax + b} dx = \frac{\ln|ax + b|}{a} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int e^{ax+b} \, dx = \frac{e^{ax+b}}{a} + C$$

$$\int f'(x)e^{f(x)} \, dx = e^{f(x)} + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$

Integration by parts: $\int \left(u \frac{dv}{dx} \right) dx = uv - \int \left(v \frac{du}{dx} \right) dx$

ANNEX B**Practice Questions on Integration Techniques**

Find the following integrals.

- 1 (a) $\int \sqrt[5]{x^2} \, dx$ (b) $\int (2-5x)^9 \, dx$ (c) $\int \frac{4}{1-3x} \, dx$
 (d) $\int \frac{4}{1-3x} \, dx$
- 2 (a) $\int (e^x + e^{-x})^2 \, dx$ (b) $\int 5^{2x} \, dx$ (c) $\int x e^{3x^2} \, dx$
 (d) $\int e^{\cos 2x} \sin 2x \, dx$
- 3 (a) $\int \sin x \sqrt{\cos x} \, dx$ (b) $\int \frac{x}{(3x^2-4)^3} \, dx$ (c) $\int \frac{x^2-1}{\sqrt{x^3-3x}} \, dx$
- 4 (a) $\int \frac{x^2}{x^3+1} \, dx$ (b) $\int \tan 2x \, dx$ (c) $\int \cot\left(\frac{x}{3}\right) \, dx$
 (d) $\int \frac{e^x}{2e^x+3} \, dx$ (e) $\int \frac{e^{-x}}{1-4e^{-x}} \, dx$ (f) $\int \sin\left(4x-\frac{\pi}{4}\right) \, dx$
- 5 (a) $\int \cos^2 2x \, dx$ (b) $\int \cos^4 x \, dx$ (c) $\int \tan^2 5x \, dx$
- 6 Use the formulae in MF27 for this question.
 (a) $\int \frac{1}{5+4x^2} \, dx$ (b) $\int \frac{1}{\sqrt{5-4x^2}} \, dx$ (c) $\int \frac{1}{4x^2-5} \, dx$
 (d) $\int \frac{1}{5-4x^2} \, dx$

Answers

- 1 (a) $\frac{5}{7}x^{\frac{7}{5}} + c$ (b) $-\frac{1}{50}(2-5x)^{10} + c$ (c) $-\frac{4}{3}\ln|1-3x| + c$
 (d) $\frac{1}{4}x^4 + \frac{2}{3}x^3 + x^2 + 2x + \ln|x| + c$
- 2 (a) $\frac{1}{2}e^{2x} - \frac{1}{2e^{2x}} + 2x + c$ (b) $\frac{1}{2\ln 5}5^{2x} + c$ (c) $\frac{1}{6}e^{3x^2} + c$ (d) $-\frac{1}{2}e^{\cos 2x} + c$
- 3 (a) $-\frac{2}{3}(\cos x)^{\frac{3}{2}} + c$ (b) $-\frac{1}{12(3x^2-4)^2} + c$ (c) $\frac{2}{3}\sqrt{x^3-3x} + c$
- 4 (a) $\frac{1}{3}\ln|x^3+1| + c$ (b) $-\frac{1}{2}\ln|\cos 2x| + c = \frac{1}{2}\ln|\sec 2x| + c$ (c) $3\ln\left|\sin \frac{x}{3}\right| + c$
 (d) $\frac{1}{2}\ln(2e^x+3) + c$ (e) $\frac{1}{4}\ln|1-4e^{-x}| + c$ (f) $-\frac{1}{4}\cos\left(4x-\frac{\pi}{4}\right) + c$
- 5 (a) $\frac{1}{2}x + \frac{1}{8}\sin 4x + c$ (b) $\frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + c$ (c) $\frac{1}{5}\tan 5x - x + c$
- 6 (a) $\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2x}{\sqrt{5}} + c$ (b) $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{5}} + c$ (c) $\frac{1}{4\sqrt{5}}\ln\left|\frac{2x-\sqrt{5}}{2x+\sqrt{5}}\right| + c$ (d) $\frac{1}{4\sqrt{5}}\ln\left|\frac{\sqrt{5}+2x}{\sqrt{5}-2x}\right| + c$