2025 H2MA Prelim Paper 1

1	Solution [4] Complex Number	
(a)	Method 1 $$\mathrm{f}(-1)=0$$ $$a(-1)^3+b(-1)^2+c(-1)+d=0$$ $$-a+b-c+d=0$$ $b-c+d-a=0$ ------------------- (1) $$\mathrm{f}(4+i)=0$$ $$a(4+i)^3+b(4+i)^2+c(4+i)+d=0$$ $$a(52+47i)+b(15+8i)+c(4+i)+d=0$$ Comparing real parts, $15b+4c+d+52a=0$ ----------------- (2) Comparing imaginary parts, $8b+c+47a=0$ --------------------(3) Solving (1), (2) and (3) using the GC, $b=-7a$, $c=9a$ and $d=17a$ Method 2 Since all the coefficients of $\mathrm{f}(x)$ are real, $4+\mathrm{i}$ and $4-\mathrm{i}$ are conjugate roots of $\mathrm{f}(x)=0$. $\mathrm{f}(x)=ax^3+bx^2+cx+d$ $=a\left[x-(4+\mathrm{i})\right]\left[x-(4-\mathrm{i})\right][x+1]$ $=a\left[x^2-8x+17\right][x+1]$ $=a\left[x^3-7x^2+9x+17\right]$ $=ax^3-7ax^2+9ax+17a$ $b=-7a$, $c=9a$ and $d=17a$	

<table>
<tr><td>2</td><td>Solution [6] Inequality</td><td></td></tr>
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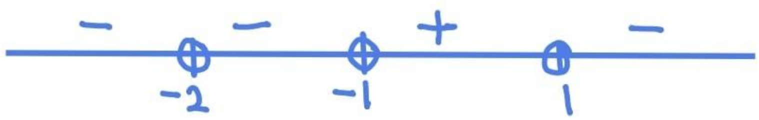
$$\frac{9}{1-x^2} < \frac{x+5}{x+1}$$

$$\frac{9-(x+5)(1-x)}{(x+1)(1-x)} < 0$$

$$\frac{9-(-x^2-4x+5)}{(x+1)(1-x)} < 0$$

$$\frac{x^2+4x+4}{(x+1)(1-x)} < 0$$

$$\frac{(x+2)^2}{(x+1)(1-x)} < 0$$



$x < -1$ or $x > 1$, $x \neq -2$

(Alternatively, $x < -2$ or $-2 < x < -1$ or $x > 1$)

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$$\frac{9}{1-e^{2x}} < \frac{e^x+5}{e^x+1}.$$

Replace x with e^x in the previous inequality,
$e^x < -1$ or $e^x > 1$
(no solution since $e^x > 0$ for $x \in \mathbb{R}$) $x > 0$
(Note $e^x \neq -2$ is always true)
Thus $x > 0$.

<u>Alternative presentation:</u>
$e^x < -2$ or $-2 < e^x < -1$ or $e^x > 1$
(no solution since $e^x > 0$ for $x \in \mathbb{R}$) $x > 0$
Thus $x > 0$.

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3	Solution [6] Complex Numbers	
	$z^2-(1+2i)z+1+7i=0$ In the above equation, $a=1,\ b=-1-2\text{i},\ c=1+7\text{i}$ $z=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $=\dfrac{1+2\text{i}\pm\sqrt{(-1-2\text{i})^2-4(1)(1+7\text{i})}}{2(1)}$ $=\dfrac{1+2\text{i}\pm\sqrt{4\text{i}-3-4-28\text{i}}}{2}$ $=\dfrac{1+2\text{i}\pm\sqrt{-7-24\text{i}}}{2}$ We want to find $\sqrt{-7-24\text{i}}$. Let $\sqrt{-7-24\text{i}}=x+\text{i}y,\ x,y\in\mathbb{R}$ $-7-24\text{i}=(x+\text{i}y)^2$ ---(*) $-7-24\text{i}=(x^2-y^2)+(2xy)\text{i}$ Comparing real and imaginary parts, $x^2-y^2=-7$ ---(1) $2xy=-24$ $x=\dfrac{-12}{y}$ ---(2) Substitute (2) into (1): $\left(\dfrac{-12}{y}\right)^2-y^2=-7$ $\dfrac{144}{y^2}-y^2=-7$ $144-y^4=-7y^2$ $y^4-7y^2-144=0$ $(y^2-16)(y^2+9)=0$ $y^2-16=0$ or $y^2+9=0$ $y^2=16$ $\qquad$ $y^2=-9$ $y=\pm4$ $\qquad$ (reject as $y\in\mathbb{R}$) When $y=4,\ x=-3$. When $y=-4,\ x=3$	

	The square roots of $-7-24\mathrm{i}$ are $3-4\mathrm{i}$ and $-3+4\mathrm{i}$. Hence, $z=\dfrac{1+2\mathrm{i}\pm(3-4\mathrm{i})}{2}$ $z=\dfrac{1+2\mathrm{i}+3-4\mathrm{i}}{2}$ or $z=\dfrac{1+2\mathrm{i}-3+4\mathrm{i}}{2}$ $z=2-\mathrm{i}$ or $z=-1+3\mathrm{i}$	

4	Solution [6] Summation	
(a)	$\sum_{r=1}^{N}\frac{1}{(r+1)(r+2)}$ $=\sum_{k-1=1}^{k-1=N}\frac{1}{((k-1)+1)((k-1)+2)}$ $=\sum_{k=2}^{N+1}\frac{1}{k(k+1)}$ $=\left(1-\frac{1}{(N+1)+1}\right)-\frac{1}{(1)(1+1)}$ $=1-\frac{1}{N+2}-\frac{1}{2}$ $=\frac{1}{2}-\frac{1}{N+2}$	
(b)	$8\sum_{r=k+1}^{\infty}\frac{1}{r(r+1)}=\sum_{r=1}^{k}\frac{1}{r(r+1)}$ $8\left(\sum_{r=1}^{\infty}\frac{1}{r(r+1)}-\sum_{r=1}^{k}\frac{1}{r(r+1)}\right)=\sum_{r=1}^{k}\frac{1}{r(r+1)}$ $8\sum_{r=1}^{\infty}\frac{1}{r(r+1)}=9\sum_{r=1}^{k}\frac{1}{r(r+1)}$ As $n\to\infty$, $\frac{1}{n+1}\to 0$. Hence $\sum_{r=1}^{\infty}\frac{1}{r(r+1)}=1$. $8(1)=9\sum_{r=1}^{k}\frac{1}{r(r+1)}$ $\sum_{r=1}^{k}\frac{1}{r(r+1)}=\frac{8}{9}$ $1-\frac{1}{k+1}=\frac{8}{9}$ $\frac{k}{k+1}=\frac{8}{9}$ $k=8$	

5	Solution [7] Integration by parts	
(a)	$\int \frac{x}{\sqrt{16-x^4}}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{\sqrt{(4)^2-(x^2)^2}}\,\mathrm{d}x$ $= \frac{1}{2}\sin^{-1}\left(\frac{x^2}{4}\right)+c$	
(b)	Let $u = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{4}\right)$, $\frac{\mathrm{d}v}{\mathrm{d}x} = x$ $\frac{\mathrm{du}}{\mathrm{d}x} = \frac{x}{\sqrt{16-x^4}}$ $\quad v = \frac{1}{2}x^2$ $\int \frac{x}{2}\sin^{-1}\left(\frac{x^2}{4}\right)\mathrm{d}x = \frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right) - \int \frac{x^2}{2}\cdot\frac{x}{\sqrt{16-x^4}}\,\mathrm{d}x$ $= \frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right) - \frac{1}{2}\int \frac{x^3}{\sqrt{16-x^4}}\,\mathrm{d}x$ $= \frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right) - \frac{1}{2}\left(-\frac{1}{4}\right)\int \frac{-4x^3}{\sqrt{16-x^4}}\,\mathrm{d}x$ $= \frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right) - \frac{\frac{1}{2}\left(-\frac{1}{4}\right)}{\frac{1}{2}}\sqrt{16-x^4}+c$ $= \frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right) + \frac{1}{4}\sqrt{16-x^4}+c$ <u>Alternative 1</u>	

	Let $u=\sin^{-1}\left(\frac{x^2}{4}\right)$, $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{x}{2}$ $\frac{\mathrm{du}}{\mathrm{d}x}=\frac{\frac{1}{2}x}{\sqrt{1-\frac{x^4}{16}}}$ $\quad v=\frac{x^2}{4}$ $\int\frac{x}{2}\sin^{-1}\left(\frac{x^2}{4}\right)\mathrm{d}x$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)-\int\frac{x^2}{4}\cdot\frac{\frac{1}{2}x}{\sqrt{1-\frac{x^4}{16}}}\mathrm{d}x$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\frac{1}{2}\int\frac{-\frac{1}{4}x^3}{\sqrt{1-\frac{x^4}{16}}}\mathrm{d}x$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\frac{1}{2}\frac{\sqrt{1-\frac{x^4}{16}}}{\frac{1}{2}}+C$ $=\frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right)+\sqrt{1-\frac{x^4}{16}}+C$		
	<u>Alternative #2</u> Consider $\int\sin^{-1}(x)\mathrm{d}x$ Let $u=\sin^{-1}(x)$, $\quad\frac{\mathrm{d}v}{\mathrm{d}x}=1$ $\frac{\mathrm{du}}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$ $\quad v=x$		

	$\int \sin^{-1}(x)\,\mathrm{d}x = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$ $= x\sin^{-1}(x) - \left(-\frac{1}{2}\right)\int \frac{-2x}{\sqrt{1-x^2}}\,\mathrm{d}x$ $= x\sin^{-1}(x) + \frac{\frac{1}{2}}{\frac{1}{2}}\left(1-x^2\right)^{-\frac{1}{2}+1} + c$ $= x\sin^{-1}(x) + \sqrt{1-x^2} + c$ Then $\int \sin^{-1}(f)\cdot f'\,\mathrm{d}x = f\sin^{-1}(f) + \sqrt{1-f^2} + c$ where f is a function in x. $\int \frac{x}{2}\sin^{-1}\left(\frac{x^2}{4}\right)\mathrm{d}x = \frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right) + \sqrt{1-\left(\frac{x^2}{4}\right)^2} + c$ $= \frac{x^2}{4}\sin^{-1}\left(\frac{x^2}{4}\right) + \frac{1}{4}\sqrt{16-x^4} + c$	

6	Solution [6] Abstract Vectors	
	$\mathbf{m}\times\mathbf{n}$ is a vector perpendicular to both $\mathbf{m}$ and $\mathbf{n}$. Thus, $\mathbf{m}\cdot(\mathbf{m}\times\mathbf{n}) = 0.$	
(a)	$\overrightarrow{OM} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$ *MR* is perpendicular to *p*, so *MR* is parallel to $\mathbf{a}\times\mathbf{b}$. $\overrightarrow{MR} = \lambda(\mathbf{a}\times\mathbf{b})$, for $\lambda \in \mathbb{R}$ $\overrightarrow{OR} - \overrightarrow{OM} = \lambda(\mathbf{a}\times\mathbf{b})$ $\overrightarrow{OR} = \frac{1}{2}(\mathbf{a}+\mathbf{c}) + \lambda(\mathbf{a}\times\mathbf{b})$ Thus, *R* is a set of points on the line ℓ given by the equation $\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{c}) + \lambda(\mathbf{a}\times\mathbf{b}),\ \lambda \in \mathbb{R}.$	
(b)	$p: \mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b},\ \alpha, \beta \in \mathbb{R}.$ At point of intersection,	

	$\frac{1}{2}(\mathbf{a}+\mathbf{c})+\lambda(\mathbf{a}\times\mathbf{b})=\alpha\mathbf{a}+\beta\mathbf{b} \quad (*)$ Scalar product (*) with **a**, $\frac{1}{2}\|\mathbf{a}\|^2+\frac{1}{2}\mathbf{a}\cdot\mathbf{c}+\lambda\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=\alpha\|\mathbf{a}\|^2+\beta\mathbf{a}\cdot\mathbf{b}$ $\frac{1}{2}+\frac{1}{2}(-2)=\alpha$ $\alpha=-\frac{1}{2}$ Scalar product (*) with **b**, $\frac{1}{2}\mathbf{b}\cdot\mathbf{a}+\frac{1}{2}\mathbf{b}\cdot\mathbf{c}+\lambda\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=\alpha\mathbf{b}\cdot\mathbf{a}+\beta\|\mathbf{b}\|^2$ $\frac{1}{2}(4)=\beta$ $\beta=2$ Position vector of the point of intersection is $-\frac{1}{2}\mathbf{a}+2\mathbf{b}$.	
	<u>Alternative</u> Since **a**, **b** are perpendicular unit vectors parallel to plane p, $\mathbf{a}\times\mathbf{b}$ would be perpendicular to **a** and **b** (and thus to p). Thus, $\overrightarrow{OM}=\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{a}\times\mathbf{b}$ for some $\alpha,\beta,\gamma\in\mathbb{R}$. Since X is the foot of perpendicular from M to p, $\overrightarrow{OX}=\alpha\mathbf{a}+\beta\mathbf{b}$ where $\alpha=\overrightarrow{OM}\bullet\mathbf{a}$ $=\frac{1}{2}(\mathbf{a}+\mathbf{c})\bullet\mathbf{a}$ $=\frac{1}{2}\mathbf{a}\bullet\mathbf{a}+\frac{1}{2}\mathbf{c}\bullet\mathbf{a}$ $=\frac{1}{2}(1)+\frac{1}{2}(-2)$ $=-\frac{1}{2}$	

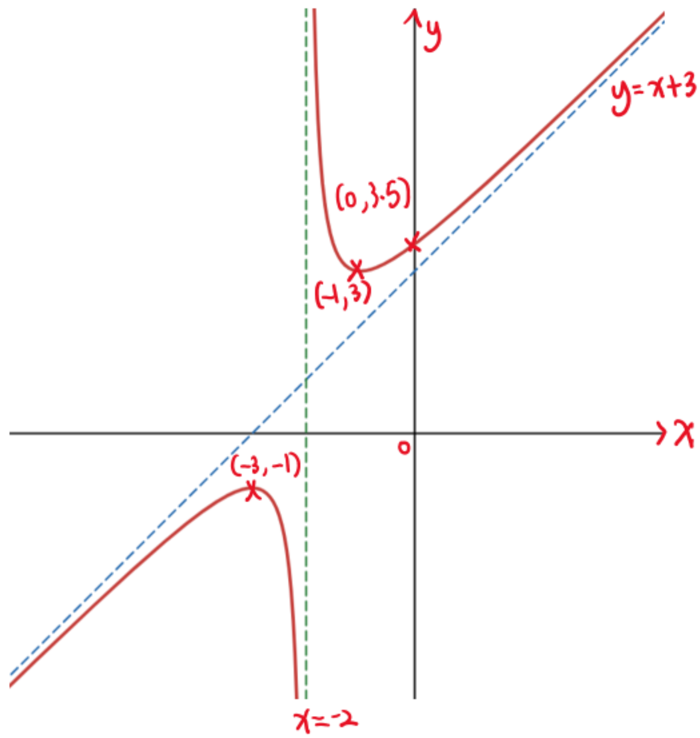
	$\beta = \overrightarrow{OM} \bullet \mathbf{b}$ $= \frac{1}{2}(\mathbf{a}+\mathbf{c}) \bullet \mathbf{b}$ $= \frac{1}{2}\mathbf{a}\bullet\mathbf{b} + \frac{1}{2}\mathbf{c}\bullet\mathbf{b}$ $= \frac{1}{2}(0) + \frac{1}{2}(4)$ $= 2$ Position vector of the point of intersection is $-\frac{1}{2}\mathbf{a} + 2\mathbf{b}$.	

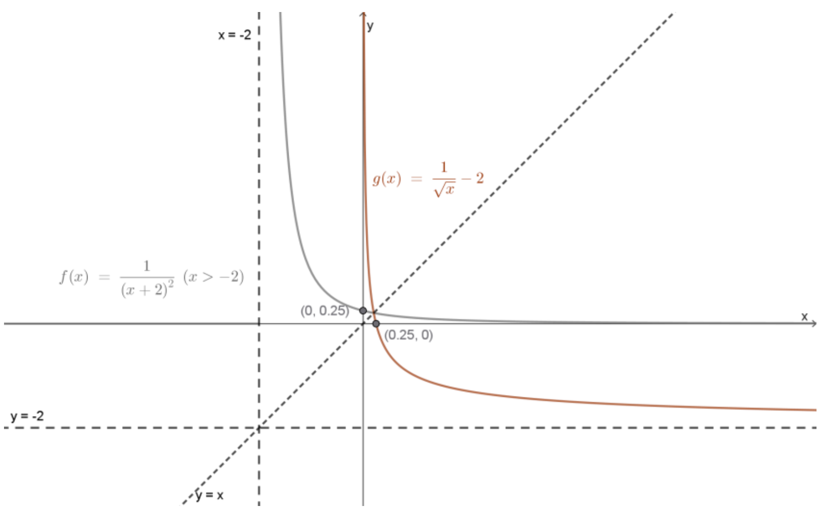
7	Solution [7] Differentiation and Applications	
(a)	$x^2 + y^2 - 4y = xy$ $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 4\frac{\mathrm{d}y}{\mathrm{d}x} = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$ $(2y - x - 4)\frac{\mathrm{d}y}{\mathrm{d}x} + 2x - y = 0$ $(2y - x - 4)\frac{\mathrm{d}y}{\mathrm{d}x} = y - 2x$	
(b)	Area of triangle *OMN*, *A* $= \frac{1}{2}(3)(y) = 1.5y$	
(c)	$A = 1.5y$ $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}A}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$ $= 1.5\frac{\mathrm{d}y}{\mathrm{d}x}$ $= 1.5\left(\frac{y-2x}{2y-x-4}\right)$	
(d)	At stationary point, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ $1.5\left(\frac{y-2x}{2y-x-4}\right) = 0$ $y = 2x - - - (*)$ $x^2 + (2x)^2 - 4(2x) = x(2x)$ $3x^2 - 8x = 0$	

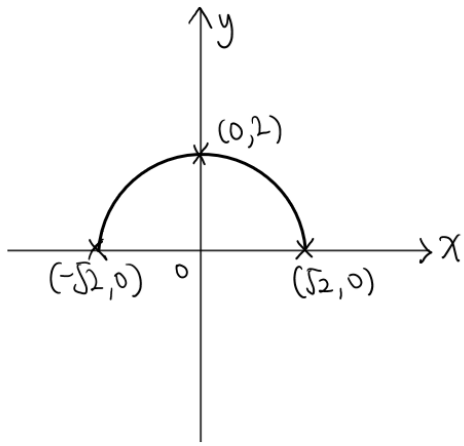
	Solving, $x=\frac{8}{3}$ or $x=0$ (Rejected since $x=0 \Rightarrow y=0$, using (*)) Note: GC skills to check your answer. $x^2+y^2-4y=xy$ $y^2-(x+4)y+x^2=0$ $y=\frac{(x+4)\pm\sqrt{(x+4)^2-4x^2}}{2}$ $y=\frac{(x+4)\pm\sqrt{(-x+4)(3x+4)}}{2}$ This implies that the maximal set of values of x for $y=\frac{(x+4)\pm\sqrt{(-x+4)(3x+4)}}{2}$ to be valid is $-\frac{3}{4}\le x\le 4$. To find the max value of $A=1.5y$ is equivalent to finding the maximum value of y. Use GC to graph $y=\frac{(x+4)+\sqrt{(-x+4)(3x+4)}}{2}$, to find the coordinates of the maximum point.	

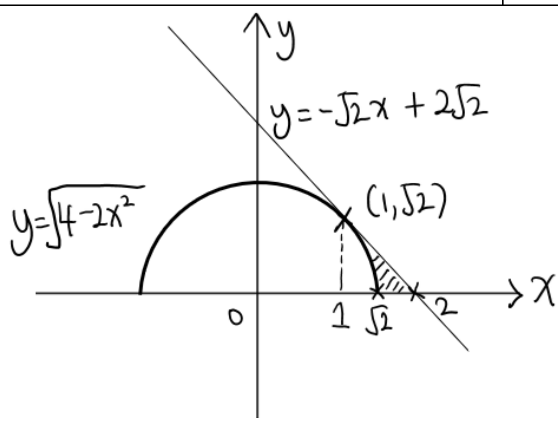
8	Solution [8] Maclaurin's Series	
(a)	$$(1+9x^2)\frac{\mathrm{d}y}{\mathrm{d}x}=3y$$ $$(1+9x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+18x\frac{\mathrm{d}y}{\mathrm{d}x}=3\frac{\mathrm{d}y}{\mathrm{d}x}$$ $$(1+9x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(18x-3)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$	
(b)	$$(1+9x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(18x-3)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$ $$(1+9x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+18x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(18x-3)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+18\frac{\mathrm{d}y}{\mathrm{d}x}=0$$ When $x=0$, $y=\mathrm{e}^2$, $\frac{\mathrm{d}y}{\mathrm{d}x}=3\mathrm{e}^2$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=9\mathrm{e}^2$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-27\mathrm{e}^2$ $$y=\mathrm{e}^2+3\mathrm{e}^2x+\frac{9\mathrm{e}^2}{2}x^2-\frac{27\mathrm{e}^2}{6}x^3+\cdots$$ $$=\mathrm{e}^2+3\mathrm{e}^2x+\frac{9\mathrm{e}^2}{2}x^2-\frac{9\mathrm{e}^2}{2}x^3+\cdots$$	
(c)	$$\ln y=2+\tan^{-1}3x$$ $$y=\mathrm{e}^{2+\tan^{-1}3x}=\mathrm{e}^2\mathrm{e}^{\tan^{-1}3x}$$ $$\mathrm{e}^{\tan^{-1}3x}=\frac{y}{\mathrm{e}^2}$$ $$=1+3x+\frac{9}{2}x^2-\frac{9}{2}x^3+\cdots$$	

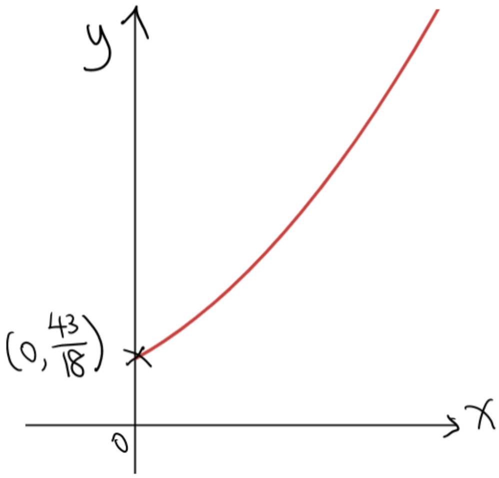
9	Solution [8] Graphing techniques	
(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=a-\frac{b-2a}{(x+2)^2}$ At stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $a-\frac{b-2a}{(x+2)^2}=0$ $a=\frac{b-2a}{(x+2)^2}$ $(x+2)^2=\frac{b-2a}{a}$ Note that $(x+2)^2\geq 0$ for all x. Hence, $\frac{b-2a}{a}\geq 0$ Since $a>0$ (given in question), $b-2a\geq 0$ Since $a\neq\frac{1}{2}b$ (given in question), $b-2a>0$ (or $b>2a$)	
(b)	$y=x+3+\frac{1}{x+2}$ $y(x+2)=(x+3)(x+2)+1$ $xy+2y=x^2+5x+7$ $x^2+5x-xy+7-2y=0$ $x^2+x(5-y)+7-2y=0$ Find values of y where $b^2-4ac<0$: $b^2-4ac=(5-y)^2-4(1)(7-2y)$ $=25-10y+y^2-28+8y$ $=y^2-2y-3$ $=(y+1)(y-3)$ $(y+1)(y-3)<0$ $-1<y<3$ Hence y cannot lie between -1 and 3 (shown).	

(c)		

10	Solution [9] [Functions[	
(a)	$a=-2$	
(b)	$y=\dfrac{1}{(x+2)^2}$ $\pm\dfrac{1}{\sqrt{y}}=x+2$ $\dfrac{1}{\sqrt{y}}=x+2$ since $x>-2$ $x=\dfrac{1}{\sqrt{y}}-2$ $f^{-1}(x)=\dfrac{1}{\sqrt{x}}-2$ $D_{f^{-1}}=R_f=(0,\infty)$	
(c)		
(d)	$R_f=(0,\infty)$, $D_g=[-1,\infty)$ Since $R_f \subseteq D_g$, gf exists. $D_{gf}=(-2,\infty)\xrightarrow{f}(0,\infty)\xrightarrow{g}[a,b)=R_{gf}$ Therefore $R_{gf}=[a,b)$	

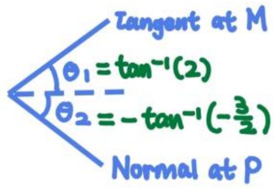
11	Solution [10] Integration (Area)		
(a)			
(b)	$y=\sqrt{4-2x^2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}\left(4-2x^2\right)^{-\frac{1}{2}}\left(-4x\right)$ $=\dfrac{-2x}{\sqrt{4-2x^2}}$ When $x=1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{2}$ $y-\sqrt{2}=-\sqrt{2}\left(x-1\right)$ $y=-\sqrt{2}x+2\sqrt{2}$		
(c)	$\int_1^{\sqrt{2}}\sqrt{4-2x^2}\ \mathrm{d}x$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{4-2\left(\sqrt{2}\sin\theta\right)^2}\sqrt{2}\cos\theta\ \mathrm{d}\theta$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{4\cos^2\theta}\sqrt{2}\cos\theta\ \mathrm{d}\theta$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}2\cos\theta\sqrt{2}\cos\theta\ \mathrm{d}\theta$ $=2\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\ \mathrm{d}\theta$ $=2\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\dfrac{1+\cos 2\theta}{2}\ \mathrm{d}\theta$	$x=\sqrt{2}\sin\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\sqrt{2}\cos\theta$ When $x=1,\ \theta=\dfrac{\pi}{4}$ When $x=\sqrt{2},\ \theta=\dfrac{\pi}{2}$	

	$=\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1+\cos 2\theta \ \mathrm{d}\theta$ $=\sqrt{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $=\sqrt{2}\left[\frac{\pi}{2}+0-\frac{\pi}{4}-\frac{1}{2}\right]$ $=\sqrt{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$		
(d)	 Area of region $=\frac{1}{2}\left(\sqrt{2}\right)(1)-\sqrt{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$ $=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{4}\pi+\frac{\sqrt{2}}{2}$ $=\sqrt{2}-\frac{\sqrt{2}}{4}\pi$ units^2		

12	Solution [11] Normal to parametric curve	
(a)		
(b)	$x = 6t - 5 \qquad y = 2t^2 + 1 \quad \text{where } t \geq \frac{5}{6}$ $\frac{dx}{dt} = 6 \qquad \frac{dy}{dt} = 4t$ $\frac{dy}{dx} = \frac{4t}{6} = \frac{2}{3}t$ When $t = 1$, $\frac{dy}{dx} = \frac{2}{3} \Rightarrow$ gradient of normal at $t = 1$ is $-\frac{3}{2}$ When $t = 1$, $x = 1$ and $y = 3$ $y - 3 = -\frac{3}{2}(x - 1)$ $y - 3 = -\frac{3}{2}x + \frac{3}{2}$ $y = -\frac{3}{2}x + \frac{9}{2}$ Alternatively $x = 6t - 5 \qquad y = 2t^2 + 1 \quad \text{where } t \geq \frac{5}{6}$ Then $y = \frac{1}{18}(x+5)^2 + 1$, $\frac{dy}{dx} = \frac{1}{9}(x+5)$ where x ≥ 0 When $t = 1$, then $x = 1$, $y = 3$, $\frac{dy}{dx} = \frac{1}{9}(1+5) = \frac{2}{3}$ Eqn of Normal: $y - 3 = -\frac{3}{2}(x-1) \Rightarrow y = -\frac{3}{2}x + \frac{9}{2}$	

(c) Tangent to C at point M, gradient is

$$\frac{2}{3}t = \frac{2}{3}(3) = 2$$



Note:
- $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$
- $\tan^{-1}(-\frac{3}{2})$ gives a negative value.

$$\theta_1 + \theta_2$$
$$= \tan^{-1}(2) - \tan^{-1}\left(-\frac{3}{2}\right)$$
$$= 2.089942 \text{ or } 119.7^\circ$$

Required acute angle

$$= \pi - 2.089942 \text{ or } 180^\circ - 119.7^\circ$$
$$= 1.05 \text{ or } 60.3^\circ$$

(d) D has parametric form $x - 5 = 6t - 5$ and $y + 3 = 2t^2 + 1$.

Consider

$$x - 5 = 6t - 5$$
$$x = 6t$$

Consider

$$y + 3 = 2t^2 + 1$$
$$y = 2t^2 - 2$$

D has parametric form: $x = 6t$, $y = 2t^2 - 2$.

(e) Curve C: $x = 6t - 5$ and $y = 2t^2 + 1$, $t \geq \frac{5}{6}$

Curve E: $x = 7u$ and $y = \frac{9}{u}$, $u \neq 0$

Since curves C and E intersect, we equate the x- and y-coordinates respectively.

Equating $x = 6t - 5$ and $x = 7u$:

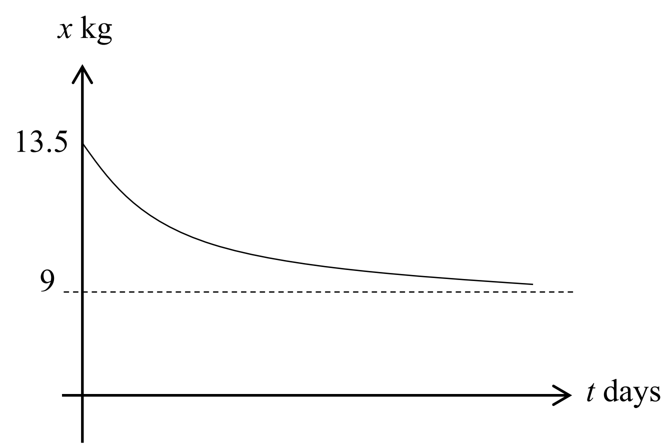
$$6t - 5 = 7u$$
$$u = \frac{6t - 5}{7} \quad \text{---(1)}$$

	Equating $y=2t^2+1$ and $y=\frac{9}{u}$: $2t^2+1=\frac{9}{u}$ ---(2) Substitute (1) into (2): $2t^2+1=\frac{9}{\frac{6t-5}{7}}$ $2t^2+1=\frac{63}{6t-5}$ $(2t^2+1)(6t-5)=63$ $12t^3-10t^2+6t-5=63$ $12t^3-10t^2+6t-68=0$ $6t^3-5t^2+3t-34=0$ (Shown) $(t-2)(6t^2+bt+17)=0$ Compare t term: $17-2b=3$ $b=7$ $(t-2)(6t^2+7t+17)=0$ $t-2=0$ or $6t^2+7t+17=0$ $t=2$ **(Not Applicable., justification below)** **<u>EITHER (discriminant)</u>** For $6t^2+7t+17$, discriminant $=b^2-4ac$ $=(7)^2-4(6)(17)$ $=-359<0$ and coefficient of t^2 is positive $\therefore$ $t=2$ is the only real root and this implies there is no other point of intersection.	

	<u>OR (completing the square)</u> $6t^2+7t+17=6\left(t^2+\frac{7}{6}t+\frac{17}{6}\right)$ $=6\left[\left(t+\frac{7}{12}\right)^2-\left(\frac{7}{12}\right)^2+\frac{17}{6}\right]$ $=6\left(t+\frac{7}{12}\right)^2+\frac{359}{24}>0$ When $t=2$, $x=6(2)-5$ and $y=2(2)^2+1$ $x=7$ and $y=9$ The point of intersection is $A(7,9)$.	

13	Solution [12] Differential equations	
(a)	Let $\frac{\mathrm{d}A}{\mathrm{d}t}$ and $\frac{\mathrm{d}B}{\mathrm{d}t}$ be the rate of roasting of coffee beans and rate of coffee beans sold respectively. $\frac{\mathrm{d}A}{\mathrm{d}t}=10$, $\frac{\mathrm{d}B}{\mathrm{d}t}\propto x^2 \Rightarrow \frac{\mathrm{d}B}{\mathrm{d}t}=kx^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}t}-\frac{\mathrm{d}B}{\mathrm{d}t}=10-kx^2$ When $x=5$, $\frac{\mathrm{d}x}{\mathrm{d}t}=0$: $0=10-k(5)^2$ $25k=10$ $k=\frac{2}{5}$ $\therefore \frac{\mathrm{d}x}{\mathrm{d}t}=10-\frac{2}{5}x^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=10-\frac{2}{5}x^2$ $=-\frac{2}{5}\left(x^2-25\right)$ $\int \frac{1}{x^2-25}\,\mathrm{d}x=\int -\frac{2}{5}\,\mathrm{d}t$ $\frac{1}{2(5)}\ln\left\|\frac{x-5}{x+5}\right\|=-\frac{2}{5}t+c$	
(b)	Let $\frac{\mathrm{d}C}{\mathrm{d}t}$ and $\frac{\mathrm{d}B}{\mathrm{d}t}$ be the new rate of roasting of coffee beans and rate of coffee beans sold respectively. $\frac{\mathrm{d}C}{\mathrm{d}t}=3x$, $\frac{\mathrm{d}B}{\mathrm{d}t}\propto x^2 \Rightarrow \frac{\mathrm{d}B}{\mathrm{d}t}=kx^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}C}{\mathrm{d}t}-\frac{\mathrm{d}B}{\mathrm{d}t}=3x-kx^2$ When $x=6$, $\frac{\mathrm{d}x}{\mathrm{d}t}=6$:	

	$6=3(6)-k(6)^2$ $6=18-36k$ $36k=12$ $k=\frac{1}{3}$ $\therefore \frac{\mathrm{d}x}{\mathrm{d}t}=3x-\frac{1}{3}x^2$	
(c)	$\frac{\mathrm{d}x}{\mathrm{d}t}=3x-\frac{1}{3}x^2$ $=\frac{1}{3}x(9-x)$ $\int \frac{1}{x(9-x)}\,\mathrm{d}x=\int \frac{1}{3}\,\mathrm{d}t$ $\int \frac{1}{9}\left(\frac{1}{x}+\frac{1}{9-x}\right)\mathrm{d}x=\int \frac{1}{3}\,\mathrm{d}t$ $\int \frac{1}{x}+\frac{1}{9-x}\,\mathrm{d}x=\int 3\ \mathrm{d}t$ $\ln\lvert x\rvert-\ln\lvert 9-x\rvert=3t+c$ $\ln\left\lvert\frac{x}{9-x}\right\rvert=3t+c$ $\left\lvert\frac{x}{9-x}\right\rvert=\mathrm{e}^{3t+c}$ $\frac{x}{9-x}=A\mathrm{e}^{3t}$, where $A=\pm e^{c}$ $x=A\mathrm{e}^{3t}(9-x)$ $x=9A\mathrm{e}^{3t}-xA\mathrm{e}^{3t}$ $x+xA\mathrm{e}^{3t}=9A\mathrm{e}^{3t}$ $x\left(1+A\mathrm{e}^{3t}\right)=9A\mathrm{e}^{3t}$ $x=\frac{9A\mathrm{e}^{3t}}{1+A\mathrm{e}^{3t}}$	

	When $t=0,\ x=13.5$: $x=\dfrac{9Ae^{3t}}{1+Ae^{3t}}$ $13.5=\dfrac{9A}{1+A}$ $13.5(1+A)=9A$ $13.5+13.5A=9A$ $4.5A=-13.5$ $A=-3$ $x=\dfrac{9(-3)e^{3t}}{1-3e^{3t}}$ $x=\dfrac{-27e^{3t}}{1-3e^{3t}}$ $x=\dfrac{27e^{3t}}{3e^{3t}-1}$ or $x=\dfrac{27}{3-e^{-3t}}$	
	As $t\to\infty$, $x=\dfrac{27e^{3t}}{3e^{3t}-1}=\dfrac{27}{3-e^{-3t}}\to 9$ In the long term, the amount of roasted coffee beans remaining in the cafe will stabilise at 9 kg/approaches 9 kg. 	
(d)	There is always roasted coffee beans left in the inventory for selling. OR The amount of roasted coffee beans stabilizing at a particular value allows the owner to plan for storage of adequate capacity.	