

SUMMARY: Vectors

A. Basic Results

Equal vectors: $\mathbf{a} = \mathbf{b} \Leftrightarrow |\mathbf{a}| = |\mathbf{b}|$ and $\mathbf{a}, \mathbf{b}$ are in the same direction

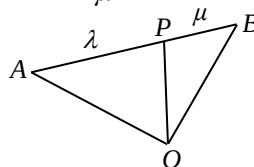
Parallel vectors: $\mathbf{a} // \mathbf{b} \Leftrightarrow \mathbf{a} = \lambda \mathbf{b}$ for some $\lambda \in \mathbb{R}, \lambda \neq 0$

Collinear points: A, B and C are collinear $\Leftrightarrow \overrightarrow{AB} = \lambda \overrightarrow{AC}$ for some $\lambda \in \mathbb{R}, \lambda \neq 0$

Ratio Theorem: If P divides AB in the ratio $\lambda : \mu$ ($AP : PB = \lambda : \mu$ or $\frac{AP}{PB} = \frac{\lambda}{\mu}$),

$$\text{then } \overrightarrow{OP} = \frac{\mu \overrightarrow{OA} + \lambda \overrightarrow{OB}}{\lambda + \mu}. \quad (\text{in MF27})$$

If P is the midpoint of AB , then $\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.



B. 3-D Vectors

Position vector of point P (relative to O) is $\overrightarrow{OP}$.

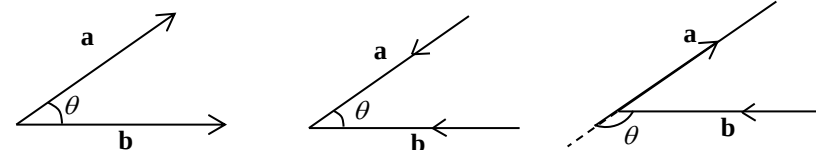
If $\overrightarrow{OP}$ is given by $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

Modulus of $\overrightarrow{OP}$, $|\mathbf{r}| = |\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$

Unit vector $//$ to $\overrightarrow{OP}$, $\hat{\mathbf{r}} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

C. Scalar Product (Dot Product)

Definition: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$

(i) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

(ii) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

(iii) $\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$

VECTORS

E. Some Applications

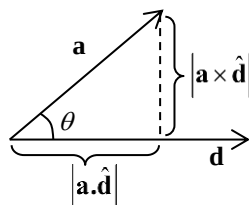
1. **Angle between $\mathbf{a}$ and $\mathbf{b}$:** $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$

Eg: $\cos \angle ABC = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|}$

2. **Projection/Perpendicular:**

Length of projection of $\mathbf{a}$ on $\mathbf{d} = |\mathbf{a} \cdot \hat{\mathbf{d}}|$

Length of component of $\mathbf{a} \perp \mathbf{d} = |\mathbf{a} \times \hat{\mathbf{d}}|$

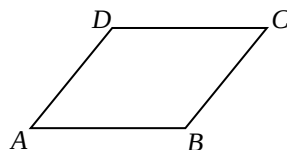


Vector component of $\mathbf{a}$ in the direction of $\mathbf{d} = (\mathbf{a} \cdot \hat{\mathbf{d}}) \hat{\mathbf{d}}$

3. **Areas:**

Area of parallelogram $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$

Area of $\triangle ABD = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AD}|$



D. Vector Product (Cross Product)

Definition: $\mathbf{a} \times \mathbf{b} = (|\mathbf{a}| |\mathbf{b}| \sin \theta) \hat{\mathbf{n}}$ and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$

θ is the angle measured from $\mathbf{a}$ to $\mathbf{b}$.

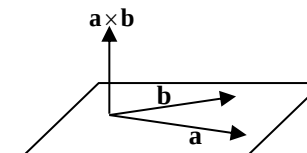
i.e. $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$ (in MF27)

(i) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

(ii) $\mathbf{a} // \mathbf{b} \Leftrightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$

$\therefore \mathbf{a} \times \mathbf{a} = \mathbf{0}$

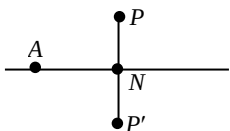


H Reflection of Point in (a) Line (b) Plane

(a) Reflection of point in Line

Given: $L: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$

p.v. of point P is $\mathbf{p}$



Step 1: Find $\overrightarrow{ON}$ where N is foot of perpendicular from P to L or Π

Step 2: M1: Since $\overrightarrow{ON} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OP'})$,

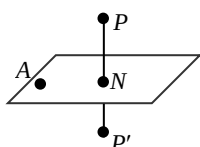
$$\overrightarrow{OP'} = 2\overrightarrow{ON} - \overrightarrow{OP}$$

M2: $\overrightarrow{PN} = \overrightarrow{NP'} \Rightarrow$ Find $\overrightarrow{OP'}$

(b) Reflection of point in Plane

Given: $\Pi: \mathbf{r} \cdot \mathbf{n} = d$

p.v. of point P is $\mathbf{p}$



A Equations of Lines

Vector form: $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}, \lambda \in \mathbb{R}$

Parametric form: $x = a_1 + \lambda b_1$

$$y = a_2 + \lambda b_2$$

$$z = a_3 + \lambda b_3$$

Cartesian form: $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$

Equations of Planes

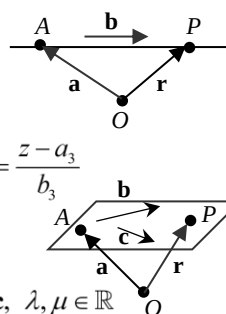
Parametric form: $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}, \lambda, \mu \in \mathbb{R}$

Scalar Product form: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\text{or } \mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}}$$

$\perp$ distance from O to plane

Cartesian form: $ax + by + cz = d$ (from $\mathbf{r} \cdot \mathbf{n} = d$)



B To show: A Point lies on a (a) Line (b) Plane

(a) Point lies on a Line

Given: $L: \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

To show $(2, 2, 4)$ lies on L ,

Sub. $\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$ and show that

λ is consistent (same value) for all 3 equations.

(b) Point lies on a Plane

Given: $\Pi: \mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 3$

To show $(1, 1, 1)$ lies on Π ,

Sub. $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

and show that LHS = RHS.

G Perpendicular Distance from Point to (a) Line (b) Plane

(a) $\perp$ Distance from Point to Line

Given: $L: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$

p.v. of point P is $\mathbf{p}$

M1: Find p.v. of N , foot of $\perp$ from point P to line L .

$$\therefore \perp \text{ distance} = |\overrightarrow{PN}|$$

M2: $\perp \text{ distance} = \left| \frac{\overrightarrow{AP} \times \hat{\mathbf{b}}}{|\hat{\mathbf{b}}|} \right|$
(See diagram below)

(b) $\perp$ Distance from Point to Plane

Given: $\Pi: \mathbf{r} \cdot \mathbf{n} = d$

p.v. of point P is $\mathbf{p}$

M1: Find p.v. of N , foot of $\perp$ from point P to plane Π .

$$\therefore \perp \text{ distance} = |\overrightarrow{PN}|$$

M2: $\perp \text{ distance} = \left| \frac{\overrightarrow{AP} \cdot \hat{\mathbf{n}}}{|\hat{\mathbf{n}}|} \right|$
(See diagram below)

VECTORS (LINES & PLANES)

E Relationship between Line & Plane

$$L: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b} \quad \text{--(1)}$$

$$\Pi: \mathbf{r} \cdot \mathbf{n} = d \quad \text{--(2)}$$

(a) Parallel

$$\mathbf{b} \cdot \mathbf{n} = 0 \text{ and } \mathbf{a} \cdot \mathbf{n} \neq d$$

(b) Line on plane

$$\mathbf{b} \cdot \mathbf{n} = 0 \text{ and } \mathbf{a} \cdot \mathbf{n} = d$$

(c) Line intersect plane at 1 point

$$\mathbf{b} \cdot \mathbf{n} \neq 0$$

D Intersection between (a) 2 Lines (b) Line & Plane (c) 2 Planes

(a) Between 2 Lines

$$L_1: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b} \quad \text{--(1)}$$

$$L_2: \mathbf{r} = \mathbf{c} + \mu \mathbf{d} \quad \text{--(2)}$$

Put (1) = (2) & solve 2 eqns for λ & μ .

Check that λ & μ satisfy the 3rd eqn.

Sub. λ into (1)

[or μ into (2)]
to obtain p.v. of point.

(b) Between Line & Plane

$$L: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b} \quad \text{--(1)}$$

$$\Pi: \mathbf{r} \cdot \mathbf{n} = d \quad \text{--(2)}$$

Sub. (1) into (2) to find λ .

Then put λ into (1)
to obtain p.v. of point.

(c) Between 2 Planes

$$\Pi_1: \mathbf{r} \cdot \mathbf{n}_1 = d_1$$

$$\Pi_2: \mathbf{r} \cdot \mathbf{n}_2 = d_2$$

M1: Equation of line of intersection is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{n}_1 \times \mathbf{n}_2)$$

[$\mathbf{a}$ is p.v. of any point on Π_1 & Π_2 (e.g. set $z = 0$ and solve eqns of Π_1 & Π_2)]

M2: Convert eqns of Π_1 & Π_2 to Cartesian form. Solve using GC.

$$\cos \theta = \frac{\mathbf{b} \cdot \mathbf{d}}{|\mathbf{b}| |\mathbf{d}|}$$

(b) $\angle$ bet. Line & Plane
Given: $L: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$
 $\Pi: \mathbf{r} \cdot \mathbf{n} = d$
Let ϕ be the angle bet. L & $\mathbf{n}$ of Π .

$$\cos \phi = \frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}| |\mathbf{n}|}$$

$$\therefore \theta = 90^\circ - \phi$$

or

$$\sin \theta = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}| |\mathbf{n}|}$$

(c) $\angle$ between 2 Planes
Given: $\Pi_1: \mathbf{r} \cdot \mathbf{n}_1 = d_1$
 $\Pi_2: \mathbf{r} \cdot \mathbf{n}_2 = d_2$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1| |\mathbf{n}_2|}$$

B**To show: A Point lies on a (a) Line (b) Plane****(a) Point lies on a Line**

Given: $L: \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

To show (2, 2, 4) lies on L , sub.

$\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$ and show that λ is consistent

(same value) for all 3 eqns.

(b) Point lies on a Plane

Given: $\Pi: \mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 3$

To show (1, 1, 1) lies on Π ,

sub. $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and show that

LHS = RHS.

A**Equations of Lines**

Eqn of line passing through $A(2, -1, 3)$ and $B(5, 3, 3)$:

Vector form: $\mathbf{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$

i.e. $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$

Parametric form: $x = 2 + 3\lambda$
 $y = -1 + 4\lambda$

$z = 3$
Cartesian form: $\frac{x-2}{3} = \frac{y+1}{4}, z = 3$

Equations of Planes

Eqn of plane passing through $A(3, -2, 0)$, $B(2, 0, 3)$ and $C(1, -1, 1)$:

Parametric form: $\mathbf{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}, \lambda, \mu \in \mathbb{R}$

$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \lambda, \mu \in \mathbb{R}$

Scalar Product form:

A vector normal to plane is $\mathbf{n} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix}$

Equation of plane is

$\mathbf{r} \cdot \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix}$ i.e. $\mathbf{r} \cdot \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} = 7$

Cartesian form: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -5 \\ 3 \end{pmatrix} = 7$ i.e. $-x - 5y + 3z = 7$

C**Angle θ between (a) 2 Lines (b) Line & Plane (c) 2 Planes****(a) Angle between 2 Lines**

$L_1: \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$L_2: \mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\cos \theta = \frac{\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{2^2 + 1^2 + 2^2} \sqrt{2^2 + 1^2 + 1^2}}$
 $\theta = 114.1^\circ$

(b) Angle between Line & Plane

$L: \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

$\Pi: \mathbf{r} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 1$

$\sin \theta = \frac{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}}{\sqrt{1^2 + 2^2 + 2^2} \sqrt{2^2 + 4^2 + 3^2}}$
 $\theta = 48.0^\circ$

(c) Angle between 2 Planes

$\Pi_1: \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 1$

$\Pi_2: \mathbf{r} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 2$

$\cos \theta = \frac{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{1^2 + 4^2 + 9^2} \sqrt{9^2 + 1^2 + 4^2}}$
 $\theta = 60^\circ$

**EXAMPLES
(LINES & PLANES)****D****Intersection between (a) 2 Lines (b) Line & Plane (c) 2 Planes****(a) Between 2 Lines**

$L_1: \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$L_2: \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

Put $\begin{pmatrix} 1+\lambda \\ -1-\lambda \\ 3+\lambda \end{pmatrix} = \begin{pmatrix} 2+2\mu \\ 4+\mu \\ 6+3\mu \end{pmatrix}$

Solving $\begin{cases} 1+\lambda = 2+2\mu \\ -1-\lambda = 4+\mu \end{cases}$

$\lambda = -3, \mu = -2$

Sub. $\lambda = -3, \mu = -2$ into 3rd

eqn:

LHS = $3 + \lambda = 3 - 3 = 0$

RHS = $6 + 3\mu = 6 - 6 = 0a$

Since LHS = RHS,

L_1 and L_2 intersect.

p.v. of point of intersection is

$\begin{pmatrix} 1+\lambda \\ -1-\lambda \\ 3+\lambda \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$

(b) Between Line & Plane

$L: \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{--(1)}$

$\Pi: \mathbf{r} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \quad \text{--(2)}$

Sub. (1) into (2),

$\begin{pmatrix} 1+\lambda \\ 1+2\lambda \\ 1+\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow \lambda = -1$

Then put $\lambda = -1$ into (1)

p.v. of point of intersection

is $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

(c) Between 2 Planes

$\Pi_1: \mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 2 \quad \text{--(1)}$

$\Pi_2: \mathbf{r} \cdot \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = 14 \quad \text{--(2)}$

M1: A vector parallel to line of intersection is

$\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 16 \end{pmatrix}$

p.v. of point that lies on both

planes is $\mathbf{a} = \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ 2 \end{pmatrix}$

[set $z = 0$ and solve Cartesian eqns of Π_1 & Π_2 for x & y .]

$\therefore$ Eqn of line of intersection

is $\mathbf{r} = \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$

M2: Convert eqns of

Π_1 & Π_2 to Cartesian form.

Solve using GC.

M3:

$\Pi_1: \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} \quad \text{--(1)}$

$\Pi_2: \mathbf{r} \cdot \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = -4 \quad \text{--(2)}$

Sub. (1) into (2):

$\begin{pmatrix} 2-2\mu \\ 1+4\mu \\ -1+5\lambda+3\mu \end{pmatrix} \cdot \begin{pmatrix} -7 \\ -2 \\ 3 \end{pmatrix} = -4$

$\Rightarrow \lambda = 1 - \mu$

$\therefore$ Eqn of line of intersection is

$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + (1-\mu) \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}, \mu \in \mathbb{R}$

E Relationship between Line and Plane

Given $L: \mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

(a) Parallel: $\Pi_a: \mathbf{r}: \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} = 4$

$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} = 0$ and $\begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} = 28 \neq 4$

(b) Line on Plane: $\Pi_b: \mathbf{r}: \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} = 28$

$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} = 0$ and $\begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} = 28 = \text{RHS}$

(c) Line intersects Plane at 1 point:

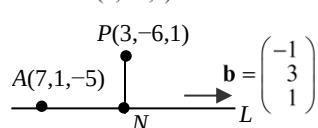
$\Pi_c: \mathbf{r}: \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 5$, $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 5 \neq 0$

F Foot of Perpendicular from Point to (a) Line (b) Plane

(a) Foot of $\perp$ from Point to Line

Given: $L: \mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

Point $P(3, -6, 1)$



M1:

(i) N lies on line $L \Rightarrow \overrightarrow{ON} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

for some real value of λ

(ii) $\overrightarrow{PN} \cdot \mathbf{b} = 0 \Rightarrow \left[\begin{pmatrix} 7-\lambda \\ 1+3\lambda \\ -5+\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = 0$

$\lambda = -1$

$\therefore \overrightarrow{ON} = \begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix}$

M2: $\overrightarrow{AN} = (\overrightarrow{AP} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}$ where $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

$= \left[\begin{pmatrix} -4 \\ -7 \\ 6 \end{pmatrix} \cdot \frac{1}{\sqrt{11}} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right] \frac{1}{\sqrt{11}} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

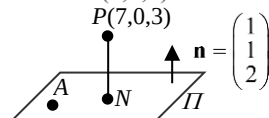
$= \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$

$\therefore \overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix}$

(b) Foot of $\perp$ from Point to Plane

Given: $\Pi: \mathbf{r}: \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1$

Point $P(7, 0, 3)$



M1: N is point of intersection of l_{PN} & Π .

$l_{PN}: \mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{--(1)}$

$\Pi: \mathbf{r}: \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1 \quad \text{--(2)}$

Sub. (1) into (2): $\lambda = -2$

$\therefore \overrightarrow{ON} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$

M2: $A(1, 0, 0)$ lies on plane.

$\overrightarrow{PN} = (\overrightarrow{PA} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}}$ where $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$= \left[\begin{pmatrix} -6 \\ 0 \\ -3 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right] \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ -2 \\ -4 \end{pmatrix}$

$\therefore \overrightarrow{ON} = \overrightarrow{OP} + \overrightarrow{PN} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$

G Perpendicular Distance from Point to (a) Line (b) Plane

(a) $\perp$ Distance from Point to Line

Given: $L: \mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

Point $P(3, -6, 1)$

M1: Let N be foot of $\perp$ from point P to line L .

From (F), $\overrightarrow{ON} = \begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix}$

$\therefore \perp \text{ distance} = |\overrightarrow{PN}| = \left| \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix} \right| = \sqrt{90}$

M2: $\perp \text{ distance} = |\overrightarrow{AP} \times \hat{\mathbf{b}}|$

$= \left| \begin{pmatrix} -4 \\ -7 \\ 6 \end{pmatrix} \times \frac{1}{\sqrt{11}} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \right| = \sqrt{90}$

(b) $\perp$ Distance from Point to Plane

Given: $\Pi: \mathbf{r}: \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1$

Point $P(7, 0, 3)$

M1: Let N be foot of $\perp$ from point P to plane Π .

From (F), $\therefore \overrightarrow{ON} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$

$\therefore \perp \text{ distance} = |\overrightarrow{PN}| = \left| \begin{pmatrix} -2 \\ -2 \\ -4 \end{pmatrix} \right| = 2\sqrt{6}$

M2: $\perp \text{ distance} = |\overrightarrow{AP} \cdot \hat{\mathbf{n}}|$ where $A(1, 0, 0)$ lies on Π

$= \left| \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right| = 2\sqrt{6}$

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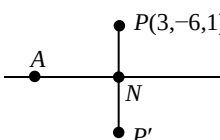
Reflection of Point in (a) Line (b) Plane

(a) Reflection of a point in Line

Given:

$L: \mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

Point $P(3, -6, 1)$



Let N be foot of $\perp$ from point P to line L .

From (F), $\overrightarrow{ON} = \begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix}$

M1: Since $\overrightarrow{ON} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OP'})$,

$\begin{pmatrix} 8 \\ -2 \\ -6 \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix} + \overrightarrow{OP'} \right] \Rightarrow \overrightarrow{OP'} = \begin{pmatrix} 13 \\ 2 \\ -13 \end{pmatrix}$

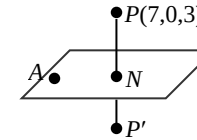
M2: $\overrightarrow{PN} = \overrightarrow{NP'} \Rightarrow \overrightarrow{OP'} = \begin{pmatrix} 13 \\ 2 \\ -13 \end{pmatrix}$

(b) Reflection of a point in Plane

Given:

$\Pi: \mathbf{r}: \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1$

Point $P(7, 0, 3)$



Let N be foot of $\perp$ from point P to plane Π .

From (F), $\overrightarrow{ON} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$

M1: Since $\overrightarrow{ON} = \frac{1}{2}(\overrightarrow{OP} + \overrightarrow{OP'})$,

$\begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} = \frac{1}{2} \left[\begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} + \overrightarrow{OP'} \right] \Rightarrow \overrightarrow{OP'} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$

M2: $\overrightarrow{PN} = \overrightarrow{NP'} \Rightarrow \overrightarrow{OP'} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$