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INTRODUCTION TO COMPLEX NUMBERS

SYLLABUS

Complex numbers expressed in Cartesian form and Argand diagram

- Extension of the number system from real numbers to complex numbers
- Complex roots of quadratic equations
- Modulus, argument and conjugate of a complex number
- Four operations of complex numbers
- Equality of complex numbers
- Conjugate roots of a polynomial equation with real coefficients
- Representation of complex numbers in the Argand diagram
- Geometrical effects of conjugation, negation, addition, subtraction, and multiplication by i

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LECTURE 1

Lesson Outline

- Introduction: imaginary number, Cartesian form of complex number, equality
- Algebraic operations on complex numbers
- Complex conjugate

1 INTRODUCTION

The equation $x^2 + 1 = 0$ does not have a solution in the real number system. An **imaginary number** $i = \sqrt{-1}$ was introduced in the 16th century to solve equations such as $x^2 + 1 = 0$. With this introduction, the equation $x^2 + 1 = 0$ has two roots, i.e. $x = i$ or $x = -i$.

Note that $i = \sqrt{-1}$,
so $i^2 = -1$, thus
 $i^2 + 1 = 0$.

Note

$i = \sqrt{-1}$	$i^5 = i$
$i^2 = -1$	$i^6 = -1$
$i^3 = i^2 \cdot i = (-1) \cdot i = -i$	$i^7 = -i$
$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$	$i^8 = 1$

In general, we have

$$i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i, i^{4n+4} = 1.$$

Definition

The **Cartesian form** of a **complex number** z is given by

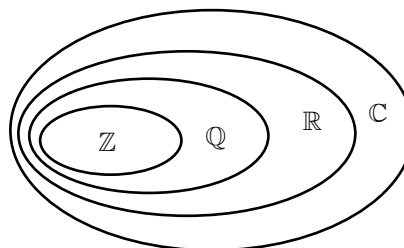
$$z = x + iy, \text{ where } x, y \in \mathbb{R} \text{ and } i = \sqrt{-1}.$$

x is called the **real part** of z , denoted by $\text{Re}(z)$.

y is called the **imaginary part** of z , denoted by $\text{Im}(z)$, which is real.

The set of complex numbers is denoted by $\mathbb{C}$.

The set of all real numbers, $\mathbb{R}$, is a proper subset of the set of all complex numbers $\mathbb{C}$, i.e. $\mathbb{R} \subset \mathbb{C}$. In fact, $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, as represented in the Venn diagram.



Note

For $z = x + iy$,

- $\operatorname{Im}(z) = y$, not iy .
- If $y = 0$, then $z = x$ is a real number.
- If $x = 0$, then $z = iy$ is a purely imaginary number.

Equality of complex numbers

Let $a, b, c, d \in \mathbb{R}$. Let $z_1 = a + ib$ and $z_2 = c + id$.

Two complex numbers z_1 and z_2 are equal **if and only if** their real and imaginary parts are equal.

$$z_1 = z_2 \Leftrightarrow a + ib = c + id \Leftrightarrow a = c \text{ and } b = d$$

Note

We do not define an ordering on complex numbers. In other words, given any two complex z_1 and z_2 , the expressions $z_1 < z_2$ or $z_1 > z_2$ do not mean anything unless z_1 and z_2 happen to be real numbers.

Example 1

Find the real numbers p, q, x and y in the following equations.

(a) $2p + iq = 4 - 7i$

(b) $x + y + i(x - y) = 6 + 2i$

Solution

(a)

(b)

■

2 ALGEBRAIC OPERATIONS ON COMPLEX NUMBERS

Let $z_1 = a + ib$ and $z_2 = c + id$ where $a, b, c, d \in \mathbb{R}$.

Addition

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$$

Subtraction

$$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d)$$

Multiplication

$$kz_1 = k(a + ib) = ka + ikb, k \in \mathbb{R}$$

$$\begin{aligned} z_1 z_2 &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2 bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

Division

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + ib}{c + id} \\ &= \frac{a + ib}{c + id} \times \frac{c - id}{c - id} \\ &= \frac{(ac - iad + ibc - i^2 bd)}{(c^2 - icd + idc - i^2 d^2)} \\ &= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} \\ &= \frac{(ac + bd)}{c^2 + d^2} + i \frac{(bc - ad)}{c^2 + d^2} \end{aligned}$$

The aim is to turn the denominator into a real number.

Example 2

Express $\frac{5 + 2i}{3 + 4i}$ in Cartesian form $x + iy$.

Solution

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To carry out algebraic operations on complex numbers using the GC, we need to press the **MATH** button and select the complex numbers menu **CMPLX**.

Example 3

Simplify the following:

(a) $(4 + 2i) + (3 - i)$

(b) $(4 + 2i)(3 - i)$

(c) $(4 + 2i)^3$

(d) $\frac{4 + 2i}{3 - i}$

Solution

Since the same complex numbers, $4 + 2i$ and $3 - i$ are used, store them into the variables **A** and **B**. Alternatively, they can be typed in with brackets.

The imaginary number i is found above the decimal point, i.e. **[2nd][.]**

The variables **A** and **B** are accessed using **[alpha]**. For **A**, key **[alpha]** **[math]**, for **B** **[alpha]****[apps]**.

1) Key in $4 + 2i$ **[sto>]** **A** **[enter]**

2) Key in $3 - i$ **[sto>]** **B** **[enter]**

Then, perform the computations accordingly.



Extra Practice

Annex: Practice Question 1

Example 4

Two complex numbers w and z are such that $w + z = 6 + 2i$ and

$$w - 3z = \frac{20}{2 - i}.$$

Find w and z , giving your answer in the form $x + iy$.

Solution

■

Extra Practice

Annex: Practice Question 2

3 COMPLEX CONJUGATE

If $z = x + iy$, then the complex number $x - iy$ is called the **complex conjugate** of z .

We denote the complex conjugate of z by z^* (or $\bar{z}$).

Note that $x + iy$ and $x - iy$ are conjugates of each other. For example, $z = 1 + 2i$ and $z^* = 1 - 2i$ are conjugates of each other.

Properties of complex conjugates

Property	Proof	Example
	Let $z = x + iy$, $z^* = x - iy$	$z = 1 + 2i$, $z^* = 1 - 2i$
1. $(z^*)^* = z$	$\begin{aligned}(z^*)^* &= (x - iy)^* \\ &= x + iy \\ &= z\end{aligned}$	$\begin{aligned}(z^*)^* &= (1 - 2i)^* \\ &= 1 + 2i \\ &= z\end{aligned}$
2. $(kz)^* = kz^*$, $k \in \mathbb{R}$	$\begin{aligned}(kz)^* &= [k(x + iy)]^* \\ &= (kx + iky)^* \\ &= kx - iky \\ &= k(x - iy) \\ &= kz^*\end{aligned}$	$\begin{aligned}(3z)^* &= [3(1 + 2i)]^* \\ &= (3 + 6i)^* \\ &= 3 - 6i \\ &= 3(1 - 2i) \\ &= 3z^*\end{aligned}$
3. $z + z^* = 2 \operatorname{Re}(z)$	$\begin{aligned}z + z^* &= (x + iy) + (x - iy) \\ &= 2x \\ &= 2 \operatorname{Re}(z)\end{aligned}$	$\begin{aligned}z + z^* &= (1 + 2i) + (1 - 2i) \\ &= 2 \\ &= 2 \operatorname{Re}(z)\end{aligned}$
4. $z - z^* = 2i \operatorname{Im}(z)$	$\begin{aligned}z - z^* &= (x + iy) - (x - iy) \\ &= 2iy \\ &= 2i \operatorname{Im}(z)\end{aligned}$	$\begin{aligned}z - z^* &= (1 + 2i) - (1 - 2i) \\ &= 4i \\ &= 2i \operatorname{Im}(z)\end{aligned}$
5. $zz^* = x^2 + y^2$	$\begin{aligned}zz^* &= (x + iy)(x - iy) \\ &= x^2 - (iy)^2 \\ &= x^2 + y^2\end{aligned}$	$\begin{aligned}zz^* &= (1 + 2i)(1 - 2i) \\ &= 1 + 2^2 \\ &= 5\end{aligned}$

Property	Proof	Example
	Let $z = x + iy$, $w = a + ib$	$z = 1 + 2i$, $w = 3 + 4i$
6. $(z \pm w)^* = z^* \pm w^*$	$\begin{aligned} (z \pm w)^* &= [(x + iy) \pm (a + ib)]^* \\ &= [(x \pm a) + i(y \pm b)]^* \\ &= (x \pm a) - i(y \pm b) \\ &= (x - iy) \pm (a - ib) \\ &= z^* \pm w^* \end{aligned}$	$\begin{aligned} (z + w)^* &= [(1 + 2i) + (3 + 4i)]^* \\ &= (4 + 6i)^* \\ &= 4 - 6i \\ z^* + w^* &= (1 - 2i) + (3 - 4i) \\ &= 4 - 6i \end{aligned}$
7. $(z \cdot w)^* = (z^*) \cdot (w^*)$ In general, $(z^n)^* = (z^*)^n$, where $n \in \mathbb{Z}^+$	$\begin{aligned} (z \cdot w)^* &= [(x + iy) \cdot (a + ib)]^* \\ &= [(xa + i^2 yb) + i(ya + xb)]^* \\ &= (xa + i^2 yb) - i(ya + xb) \\ &= (x - iy) \cdot (a - ib) \\ &= (z^*) \cdot (w^*) \end{aligned}$	$\begin{aligned} (z \cdot w)^* &= [(1 + 2i)(3 + 4i)]^* \\ &= (-5 + 10i)^* \\ &= -5 - 10i \\ (z^*) \cdot (w^*) &= (1 - 2i)(3 - 4i) \\ &= -5 - 10i \end{aligned}$
8. (a) $\left(\frac{1}{w}\right)^* = \frac{1}{w^*}$ (b) $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$	(a) See https://bit.ly/conjprop8a (b) $\begin{aligned} \left(\frac{z}{w}\right)^* &= \left(z \cdot \frac{1}{w}\right)^* \\ &= (z^*) \cdot \left(\frac{1}{w}\right)^* \quad (\text{property 7}) \\ &= (z^*) \cdot \left(\frac{1}{w^*}\right) \quad (\text{by (a)}) \\ &= \frac{z^*}{w^*} \end{aligned}$	(b) $\begin{aligned} \left(\frac{z}{w}\right)^* &= \left(\frac{1 + 2i}{3 + 4i}\right)^* \\ &= (0.44 + 0.08i)^* \\ &= 0.44 - 0.08i \\ \frac{z^*}{w^*} &= \frac{1 - 2i}{3 - 4i} \\ &= 0.44 - 0.08i \end{aligned}$
9. $z = z^* \Leftrightarrow z$ is real	$\begin{aligned} z = z^* &\Leftrightarrow x + iy = x - iy \\ &\Leftrightarrow y = 0 \\ &\Leftrightarrow z \text{ is real} \end{aligned}$	$\begin{aligned} 0 &= 0^*, \\ 2 &= 2^*, \\ -2 &= (-2)^* \end{aligned}$

Example 5

Given that $az^2 + bz + c = 0$, where a , b and c are real numbers, prove that

$$a(z^*)^2 + b(z^*) + c = 0.$$

Solution

If z is a root of the
polynomial
equation, is z^* also
a root?

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LECTURE 2

Lesson Outline

- Complex roots of a polynomial equation where all coefficients are real
 - Complex roots of a polynomial equation where some coefficients might be complex
 - The Argand diagram as a pictorial representation of complex numbers
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4 COMPLEX ROOTS OF A POLYNOMIAL EQUATION

Generalisation of Example 5

If $z = a + ib$ is a root of the equation $P(z) = 0$ where,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

is a polynomial of degree n with **real coefficients**, then $z^* = a - ib$ is also a root of the equation $P(z) = 0$.

With the introduction of complex numbers, we have one of the most important theorems in algebra.

Fundamental Theorem of Algebra

Any polynomial equation $P(z) = 0$, where $P(z)$ is a **polynomial** of degree n , has exactly n roots, real or complex.

Example 6

Find the exact roots of the equation $x^2 + x + 1 = 0$.

Solution

Alternatively,

$$x^2 + x + 1 = 0$$

$$\left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$$

$$x + \frac{1}{2} = \pm \sqrt{-\frac{3}{4}} = \pm \frac{\sqrt{3}}{2} i$$

$$x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

■

Example 7

Given that $1 + \sqrt{3}i$ is a root of the equation $2z^3 + az^2 + bz + 4 = 0$, find the values of the real numbers a and b . Hence solve the equation $2z^3 + az^2 + bz + 4 = 0$. [9740 N2008/I/8 (modified)]

Solution

Since $1 + \sqrt{3}i$ is a root, $1 - \sqrt{3}i$ is also a root.

$$\begin{aligned} (z - (1 + \sqrt{3}i))(z - (1 - \sqrt{3}i)) &= ((z - 1) + \sqrt{3}i)((z - 1) - \sqrt{3}i) \\ &= (z - 1)^2 + 3 = z^2 - 2z + 4 \end{aligned}$$

$$2z^3 + az^2 + bz + 4 = (z^2 - 2z + 4)(2z + a) = 0$$

Comparing coefficients,

$$z^0: 4 = 4k \Rightarrow k = 1$$

$$z^2: a = k - 4$$

$$z: b = -2k + 8$$

Hence the roots of the equation are

Alternatively,

$$\text{If } z = 1 + \sqrt{3}i,$$

$$\text{then } z^2 = 1 + 2(1)(\sqrt{3}i) + 3i^2 = -2 + 2\sqrt{3}i$$

$$\text{and } z^3 = (-2 + 2\sqrt{3}i)(1 + \sqrt{3}i) = -2 + 2\sqrt{3}i - 2\sqrt{3}i - 2(3) = -8.$$

Since $1 + \sqrt{3}i$ is a root of $2z^3 + az^2 + bz + 4 = 0$, substitute $z = 1 + \sqrt{3}i$ into equation:

$$2(-8) + a(-2 + 2\sqrt{3}i) + b(1 + \sqrt{3}i) + 4 = 0 \Rightarrow (b - 2a) + (2a + b)\sqrt{3}i = 12$$

Comparing real and imaginary parts,

$$b - 2a = 12, \quad 2a + b = 0$$

Solving using the GC, $a = -3$ and $b = 6$.

Since $2z^3 - 3z^2 + 6z + 4 = 0$ only has real coefficients,

$1 + \sqrt{3}i$ is a root $\Rightarrow 1 - \sqrt{3}i$ is also a root.

Let the third root be α , where α is a real number. Then

$$2z^3 - 3z^2 + 6z + 4 \equiv 2(z - (1 + \sqrt{3}i))(z - (1 - \sqrt{3}i))(z - \alpha)$$

Comparing the constant,

$$4 = -2\alpha(1 + \sqrt{3}i)(1 - \sqrt{3}i) = -2\alpha(1 + 3) = -8\alpha \Rightarrow \alpha = -\frac{1}{2}$$

Hence the roots of the equation are $1 + \sqrt{3}i$, $1 - \sqrt{3}i$ and $-\frac{1}{2}$. ■

If question allows use of GC, we can use it to calculate z^2 and z^3 .

The GC can be used to check if the roots you obtained are the correct roots for the equation, regardless of whether the question allows you to use the GC for the intermediate workings.

1) Store one of the roots, e.g. Key in $1 - \sqrt{3}i$ **[sto>] [X,T,θ,n]**.

2) Enter the expression $2X^3 - 3X^2 + 6X + 4$, and press **[enter]**.

If X is indeed a root of the equation, then you should obtain 0 as the result.

If a polynomial only has real coefficients, then any complex roots of the polynomial equation exist as conjugate pairs. If a polynomial has **non-real coefficients**, then its roots may not be complex conjugates of each other.

Example 8

Find the roots of the equation $z^2 + \sqrt{3}z + i = 0$.

Solution

Using the quadratic formula,

$$z = \frac{-\sqrt{3} \pm \sqrt{3 - 4i}}{2}$$

We can use the GC to evaluate:

$$\sqrt{3 - 4i} = 2 - i.$$

(See box after Example 9)

If GC is not allowed, then refer to example 9.



Example 9**Do not use a calculator in answering this question.**

Solve the equation $w^2 = 3 - 4i$, giving your answers in the form $a + ib$, showing your working.

Hence, by expressing $6 - 8i$ in the form $k(3 - 4i)$, where k is a constant, find the roots of the equation $z^2 = 6 - 8i$.

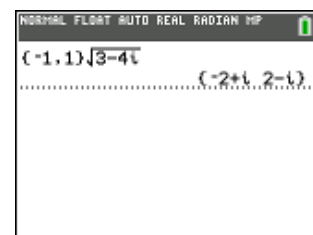
[IJC Prelim 06 P1 Qn12 (modified)]

Solution

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If the use of GC is allowed, we can use it to obtain both complex roots of the equation $w^2 = 3 - 4i$.

On the Home Screen of the GC, key in $\{-1, 1\}\sqrt{3 - 4i}$. This gives a set of two numbers, i.e. $\{-\sqrt{3 - 4i}, \sqrt{3 - 4i}\}$. Press **[enter]**.

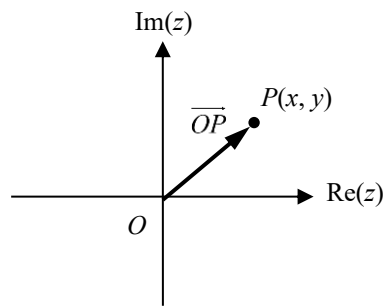
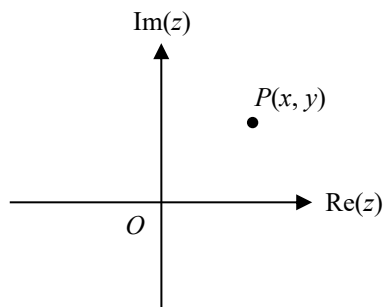
**Extra Practice**

Annex: Practice Question 3

5 THE ARGAND DIAGRAM

The complex number $x + iy$ can be represented geometrically on an **Argand diagram** by the coordinates (x, y) on a Cartesian plane (where x, y are real numbers) as

- (i) a **point** $P(x, y)$ or as
- (ii) a **position vector** $\overrightarrow{OP} = xi + yj$



This geometrical representation is called an **Argand Diagram**.

Notes

- (i) Points on the horizontal axis (known as the **real axis**) correspond to real numbers.
- (ii) Points on the vertical axis (known as the **imaginary axis**) correspond to purely imaginary numbers.

LECTURE 3

Lesson Outline

- Modulus and argument of a complex number
 - Geometrical effects of conjugation, negation, addition, subtraction and multiplication by i
-

5.1 Modulus and Argument of a Complex Number

Let $P(x, y)$ be the point representing the complex number $z = x + iy$ in the Argand diagram, where $x, y \in \mathbb{R}$.

Definitions

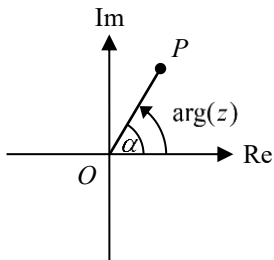
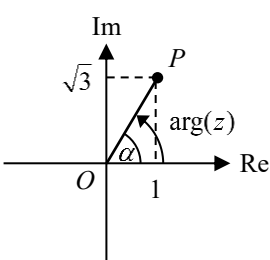
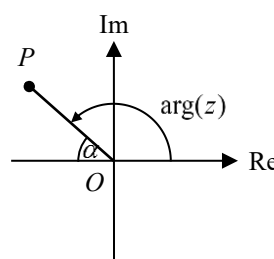
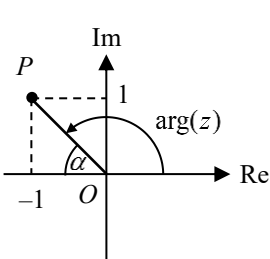
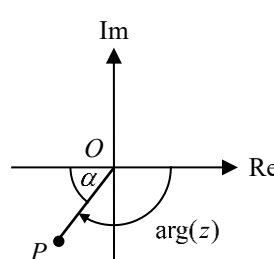
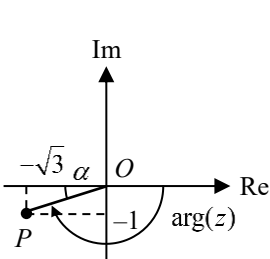
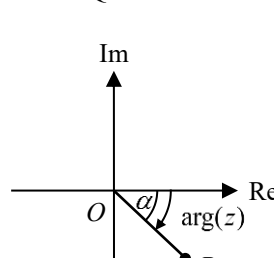
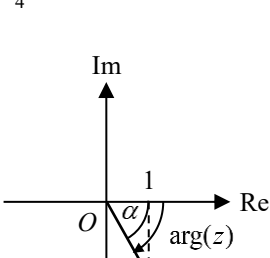
Modulus of z is the distance between P and the origin and is denoted by $|z|$, i.e.

$$|z| = \sqrt{x^2 + y^2}.$$

Argument of z (denoted by $\arg z$) is the angle measured in radians from the positive real axis to the line segment joining P and the origin.

Let $\alpha = \tan^{-1} \left| \frac{y}{x} \right|$ denote the basic angle of the line segment joining P and the origin. Since α is an acute angle, we need to take the modulus of the ratio $\frac{y}{x}$.

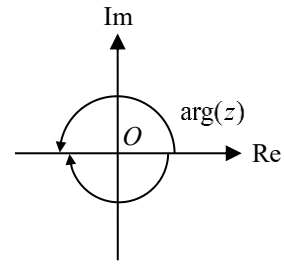
Calculating $\arg z$ depends on which quadrant the complex number is in.

Argand diagram	Example
First Quadrant  $\arg z = \alpha$	$z_1 = 1 + \sqrt{3}i$  Basic angle $\alpha = \tan^{-1} \left \frac{\sqrt{3}}{1} \right = \frac{\pi}{3}$ $\arg(z_1) = \frac{\pi}{3}$
Second Quadrant  $\arg z = \pi - \alpha$	$z_2 = -1 + i$  Basic angle $\alpha = \tan^{-1} \left \frac{1}{-1} \right = \frac{\pi}{4}$ $\arg(z_2)$ $= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$
Third Quadrant  $\arg z = -(\pi - \alpha) = \alpha - \pi$	$z_3 = -\sqrt{3} - i$  Basic angle $\alpha = \tan^{-1} \left \frac{-1}{-\sqrt{3}} \right = \frac{\pi}{6}$ $\arg(z_3)$ $= -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$
Fourth Quadrant  $\arg z = -\alpha$	$z_4 = 1 - 2i$  Basic angle $\alpha = \tan^{-1} \left \frac{-2}{1} \right $ $= 1.11 \text{ (3 s.f.)}$ $\arg(z_4) = -1.11$

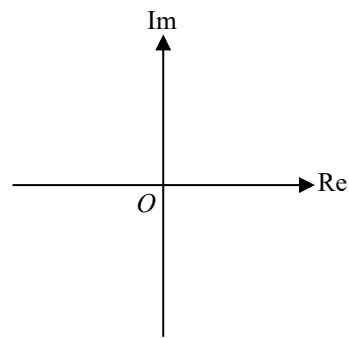
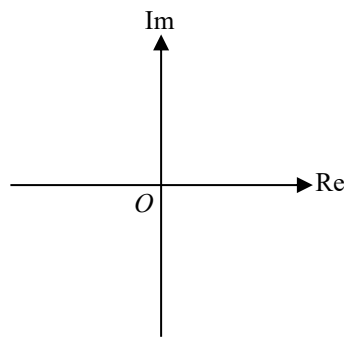
Notes

Let $|z|$ be r and $\arg z$ be θ .

- Arguments are measured from the positive real axis. Arguments are positive when measured in the anti-clockwise direction, and negative when measured in the clockwise direction.
- The range $-\pi < \theta \leq \pi$ is called the **principal range** for arguments. When evaluating arguments, we adjust arguments that fall outside the principal range by adding or subtracting multiple(s) of 2π .
- $\arg 0$ is undefined.
- z is real if and only if $\arg z = 0$ or π .



z is purely imaginary if and only if $\arg z = \pm \frac{\pi}{2}$.



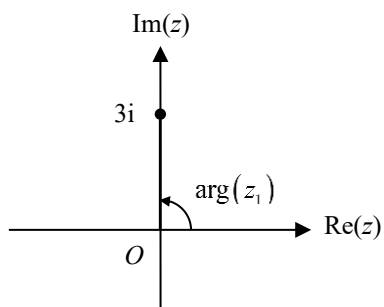
- $zz^* = x^2 + y^2 = \left(\sqrt{x^2 + y^2}\right)^2 = |z|^2 = r^2$

Example 10

Find, in exact form, the modulus and argument of $3i$, $-\sqrt{3}+i$, 5 and $-1-i$.

Solution

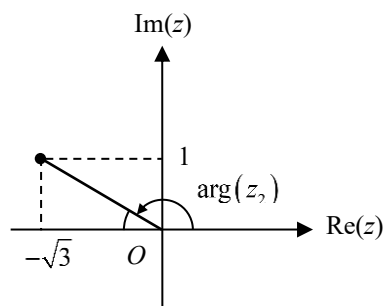
Let $z_1 = 3i$.



$$|z_1| = |0 + 3i| = \sqrt{0^2 + 3^2} = 3 \text{ (or by observation)}$$

$$\arg(z_1) = \frac{\pi}{2} \text{ (since the point lies on the positive imaginary axis)}$$

Let $z_2 = -\sqrt{3} + i$.



$$|z_2| = |-\sqrt{3} + i| = \sqrt{(-\sqrt{3})^2 + 1} = 2$$

$$\text{Basic angle} = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\arg(z_2) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

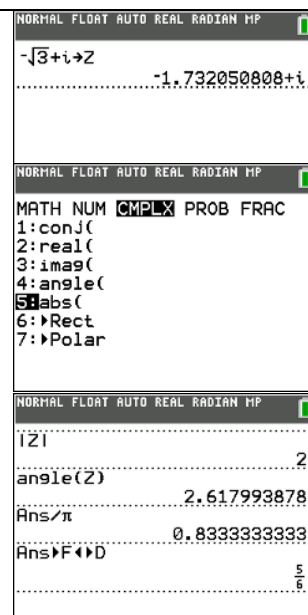
Let $z_3 = 5$.

Let $z_4 = -1 - i$.

Using GC to find modulus and argument of a complex number

Example: $z = -\sqrt{3} + i$

- 1) Key in $-\sqrt{3} + i$ **[sto>] Z [enter]**.
- 2) To find the modulus, select **abs(** in the **MATH CMPLX** Menu.
Press **[math] [>] [>]** to select the **CMPLX** menu.
Press **[5] Z [enter]**. Thus the modulus, $|z| = 2$.
- 3) To find the argument, select **angle(** in the **MATH CMPLX** Menu.
Press **[math] [>] [>]** to select the **CMPLX** menu.
Press **[4] Z [)] [enter]**.
Thus the argument of z , $\arg(z) = 2.617993878$.
- 4) To check whether the angle is a fractional multiple of π , divide the answer by π and convert it to a fraction.
Press **[alpha] [y=] [4] [enter]** to convert it to a fraction.
Thus $\arg(z) = \frac{5\pi}{6}$.



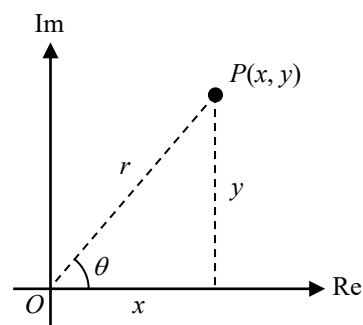
Finding Cartesian form of complex number

If you are given the modulus and argument of a complex number, you can find the real and imaginary parts of the complex number and hence write down the Cartesian form of the complex number.

Given that the complex number $z = x + iy$ has modulus r and argument θ , then

$$\operatorname{Re}(z) = x = r \cos \theta,$$

$$\operatorname{Im}(z) = y = r \sin \theta.$$



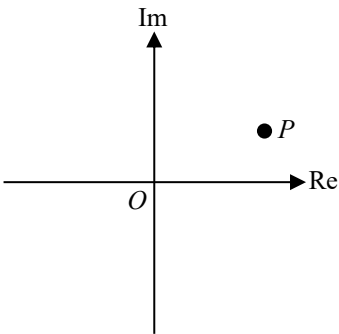
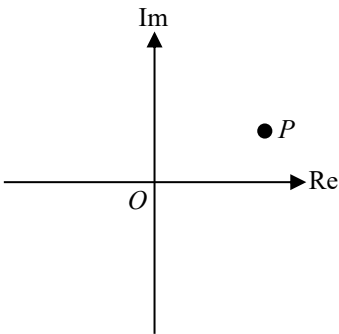
Extra Practice

Annex: Practice Question 4

5.2 Geometrical Effects of Algebraic Operations on Complex Numbers

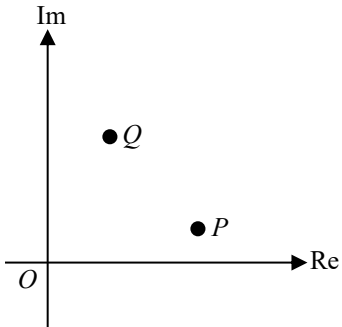
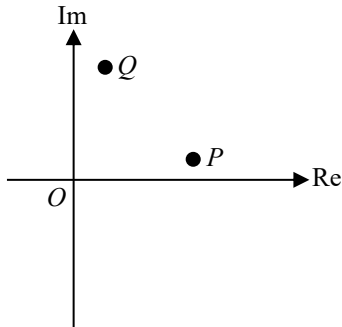
Let the points P and Q represent the complex numbers $z = a + ib$ and $w = c + id$ respectively on the Argand diagram. When we perform simple algebraic operations on complex numbers, there are corresponding geometrical effects on the points representing these complex numbers on the Argand diagram. This allows us to visualise relationships between complex numbers in a different way.

Conjugation and Negation

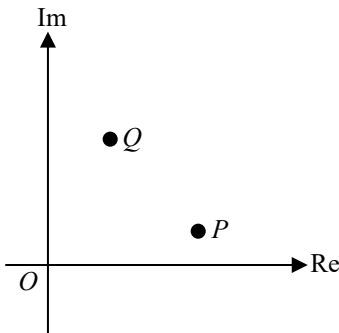
Conjugation	Negation
	
$z^* = \underline{\hspace{2cm}}$ is represented by the point P' with coordinates $\underline{\hspace{2cm}}$. The point P' is a $\underline{\hspace{2cm}}$. $\underline{\hspace{2cm}}$.	$-z = \underline{\hspace{2cm}}$ is represented by the point P'' with coordinates $\underline{\hspace{2cm}}$. The point P'' is a $\underline{\hspace{2cm}}$. $\underline{\hspace{2cm}}$.
Notes • $z^* = z = r$ • $\arg z^* = -\arg z = -\theta$	Notes • $-z = z = r$ • $\arg(-z) = \arg z - \pi = \theta - \pi$

Addition and Subtraction

Geometrically, adding and subtracting 2 complex numbers are similar to adding and subtracting 2 vectors geometrically using a vector triangle or parallelogram.

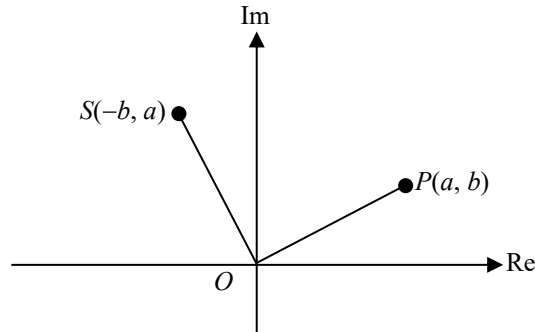
Addition	Subtraction
	
$z + w = (a + c) + i(b + d)$ is represented by the point R , where _____ _____.	$z - w = z + (-w) = (a - c) + i(b - d)$ is represented by the point R' , where _____ _____.

Alternatively, we can say that the complex numbers $z + w$ and $z - w$ are represented by the diagonals of the parallelogram whose sides are represented by z and w .



Multiplication by i

Now consider the complex numbers $z = a + ib$ and $iz = -b + ia$. These complex numbers are represented on an Argand diagram by the points $P(a, b)$ and $S(-b, a)$ respectively.



- By considering the position vectors $\vec{OP}$ and $\vec{OS}$, we note that

$$\vec{OP} \cdot \vec{OS} = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix} = -ab + ab = 0$$
 $\Rightarrow$ The line segments OP and OS are perpendicular.
- $OP = \sqrt{a^2 + b^2}$, and $OS = \sqrt{(-b)^2 + a^2} = \sqrt{a^2 + b^2}$.
 $\Rightarrow$ The line segments OP and OS have the same length.

With these observations, we can conclude that the geometrical effect of multiplying a complex number z by i is a _____ of the position vector representing z _____ about the origin by $\frac{\pi}{2}$ radians.

Example 11

The points A , B and C represent the complex numbers $-3+i$, $-2+4i$ and $p+iq$ respectively.

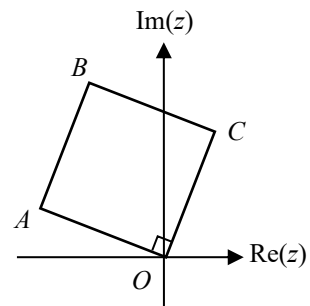
- (a) Find p, q such that $OABC$ is a parallelogram.
 (b) For the values of p, q found in (a), show that $OABC$ is a square.

Solution

- (a) Since $OABC$ is a parallelogram,

- (b) We note that

This implies that the line segment OA is a rotation of the line segment OC anticlockwise about the origin by $\frac{\pi}{2}$ radians.



Since the two adjacent sides OA and OC are perpendicular and have the same length, we conclude that $OABC$ is a square. (shown) ■

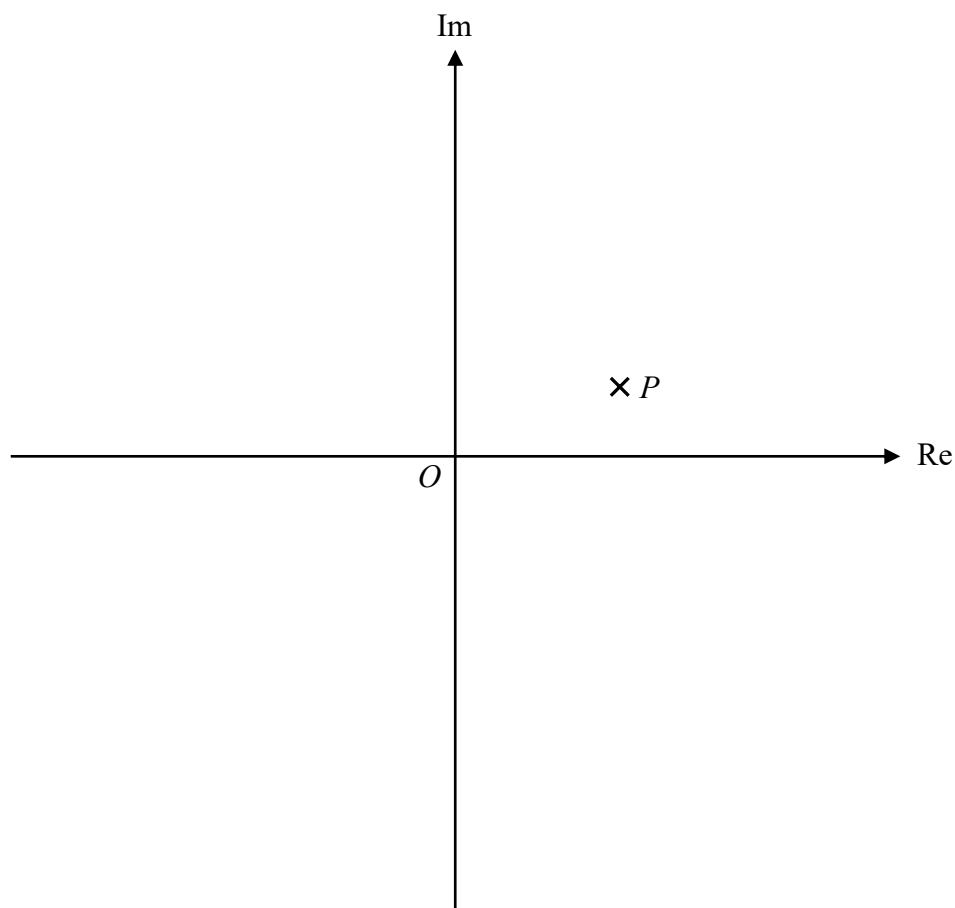
Example 12

The point P on the Argand diagram below represents the complex number z . On the same Argand diagram, plot the points representing the following complex numbers.

- (a) Point A representing the complex number iz
- (b) Point B representing the complex number $z + iz$
- (c) Point C representing the complex number z^*
- (d) Point D representing the complex number $z + z^*$
- (e) Point E representing the complex number $z + 2$
- (f) Point F representing the complex number $2z$

Given that $\arg z = \theta$, write down the arguments of iz , $z + iz$, z^* and $z + z^*$.

Solution



Complex number	iz	$z + iz$	z^*	$z + z^*$
Argument				

■

ANNEX**Practice Questions on Complex Numbers**

1 Simplify the following without using a calculator. (* Use the GC to check your answer)

(i) $i(1-i)^2$

(ii) $(1+2i)(i^2-1)$

(iii) $\frac{1}{i}$

(iv) $\frac{1}{1-i}$

(v) $\frac{1-i}{i^2-2}$

(vi) $\frac{2+i}{i^3-2}$

(vii) $\frac{2}{i^{12}}$

2 Without the use of a GC, solve the following simultaneous equations:

(i) $w+2z=2+5i$ and $iz-w=-1-2i$

(ii) $w-2z=3i$ and $z-w^*=i^2$

3 Without the use of a GC, solve the following equations:

(i) $z^2=3+4i$

(ii) $x^2-2x+5=0$

(iii) $2z^2-(4+i)z+3+i=0$

(iv) $z^3-z^2+z-1=0$

4 Find the modulus and argument of the following complex numbers in exact form.

(i) 1

(ii) i

(iii) -1

(iv) $-i$

(v) $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(vi) $-3+\sqrt{3}i$

(vii) $1+i$

(viii) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

(ix) $2\sqrt{3}-6i$

Answers

1 (i) $2i$ (ii) $-2-4i$ (iii) $-i$ (iv) $\frac{1}{2} + \frac{1}{2}i$ (v) $-\frac{1}{3} + \frac{1}{3}i$ (vi) -1 (vii) 2

2 (i) $z=1+i, w=3i$ (ii) $z=1-i, w=2+i$

3 (i) $2+i$ or $-2-i$ (ii) $1+2i$ or $1-2i$ (iii) $1+i$ or $1-\frac{1}{2}i$ (iv) 1 or i or $-i$

4 (i) $1, 0$ (ii) $1, \frac{1}{2}\pi$ (iii) $1, \pi$ (iv) $1, -\frac{1}{2}\pi$ (v) $1, \frac{1}{6}\pi$ (vi) $2\sqrt{3}, \frac{5}{6}\pi$ (vii) $\sqrt{2}, \frac{1}{4}\pi$

(viii) $1, -\frac{3}{4}\pi$ (ix) $4\sqrt{3}, -\frac{1}{3}\pi$