FINAL EXAMINATION 2016

YEAR 5 IB DIPLOMA PROGRAMME

PAPER 1 MATHEMATICS HIGHER LEVEL – SOLUTIONS

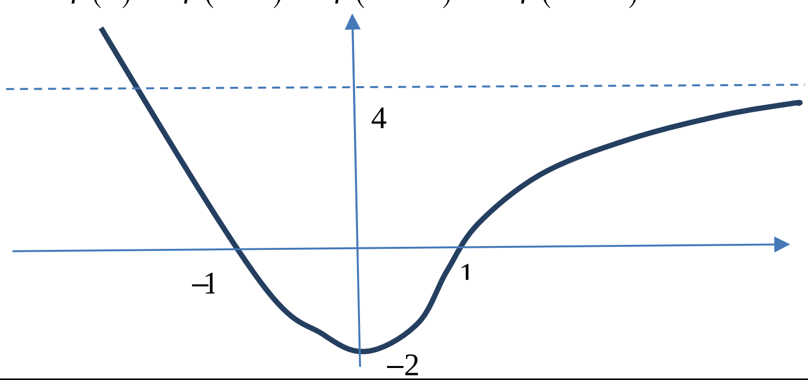
Qn	Solution
1.	
	General Term for $\left(1-\sqrt{x}\right)^{10}$: $\binom{10}{r}1^{10-r}\left(-\sqrt{x}\right)^{r}=\binom{10}{r}\left(-1\right)^{r}x^{r/2}$ $\left(x^{-1}-1\right)\left(1-\sqrt{x}\right)^{10}$ r = 2: $\binom{10}{2}\left(-1\right)^{2}x^{2/2}=\frac{10.9}{2.1}x=45x^{1}$ r = 0: $\binom{10}{0}\left(-1\right)^{0}x^{0/2}=1$ $\left(\frac{1}{x}-1\right)\left(1+45x+..\right)=45-1=44$
	Comment: Mostly well done although a small number do not know how to evaluate 10C2 and a handful do not remember how to do Binomial Expansion.
2.	
(i)	$18+27+...+99$ $=\frac{10}{2}\left(18+99\right)$ $=5\left(117\right)$ $=585$
	Comment: A number of students did by pure arithmetic calculation and some missed out on 99 as the last term.
(ii)	$\sum_{r=1}^{10}9r+9$ (divisible by 9) Total sum – sum of all divisible by 9 = $\sum_{r=1}^{90}\left(r+9\right)-\sum_{r=1}^{10}\left(9r+9\right)$
	Comment: Well done.

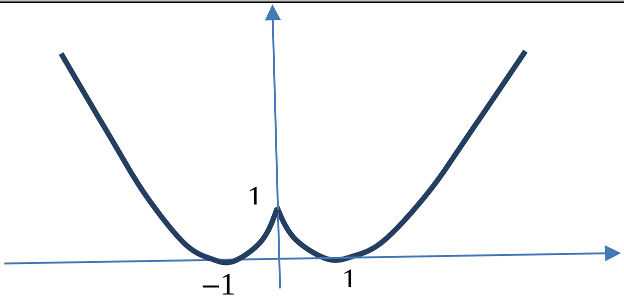
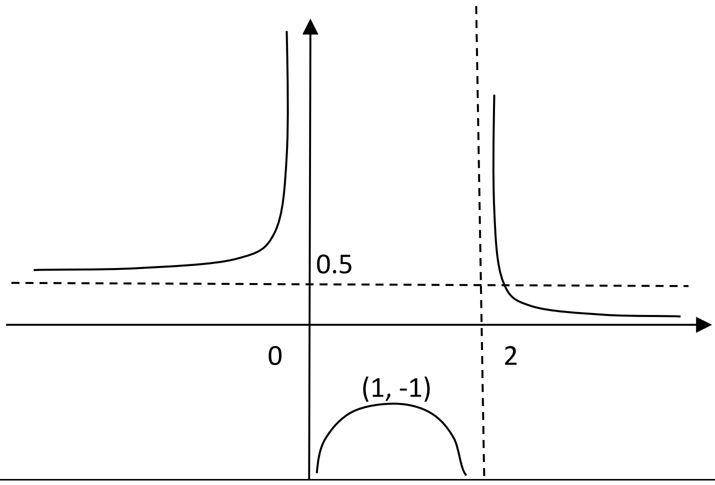
Qn	Solution
3.	
	$\textit{Area}\ \text{A} : \text{B}$ $\frac{1}{2}R^2\theta - \frac{1}{2}r^2\theta \ : \ \frac{1}{2}r^2\theta$ $R^2 - r^2 \quad : \ r^2$ $\frac{R^2 - r^2}{r^2} = 3$ $R^2 = 4r^2$ $R = 2r \qquad (R > 0)$
	Comment: Very well done except some forgot the formula for area of sector.
4.	
(i)	$S_\infty = \frac{2^{-x}}{1-2^{-x}} \times \frac{2^x}{2^x}$ $= \frac{1}{2^x - 1}$ $= \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$ $= \frac{\sqrt{2}+1}{2-1}$ $= \sqrt{2}+1$
	Comment: Some did not manipulate answer to the required form.
(ii)	$2^{-x} \times 2^{-2x} \times 2^{-3x} \times .... \times 2^{-11x}$ $= 2^{-x(1+2+3+..+11)}$ $= 2^{-x\left(\frac{11}{2}(11+1)\right)}$ $= 2^{-0.5(66)}$ $= \frac{1}{2^{33}}$
	Comment: A number forgot to substitute the value of x in their final answer.

Qn	Solution
5.	
	Let P_n be statement $n! > 2^{n-1}$ $n = 3, \quad LHS = 3! = 6$ $\qquad RHS = 2^2 = 4 < LHS$ $\therefore P_3$ is true. Assume P_k is true, $k! > 2^{k-1}$ To show that P_{k+1} is true: $(k+1)! > 2^k$ $k! > 2^{k-1}$ $(k+1)! > 2^{k-1}(k+1)$ $(k+1)! > 2^k\left(\frac{k+1}{2}\right)$ $if\ \ k > 2,\ \left(\frac{k+1}{2}\right) > 1,$ $(k+1)! > 2^k\left(\frac{k+1}{2}\right) > 2^k(1)$ $\therefore (k+1)! > 2^k$ Since P_3 and P_{k+1} is true, by Mathematical Induction, P_n is true for all $n > 2,\ n \in \mathbb{Z}^+$
	Comment: Some vague argument on the validity of P(k+1).
6.	
	$\alpha\beta\gamma = -\frac{16}{8} = -2$ $let\ y = -2x$ $\Rightarrow x = -\frac{y}{2}$ $8\left(-\frac{y}{2}\right)^3 - 16\left(-\frac{y}{2}\right)^2 - 8\left(-\frac{y}{2}\right) + 16 = 0$ $8\left(-\frac{y^3}{8}\right) - 16\left(\frac{y^2}{4}\right) + 4y + 16 = 0$ $-y^3 - 4y^2 + 4y + 16 = 0 \quad$ (or equivalent)
	Comment: A number of the students did by finding out the coefficients of the function in relation to the sum and product of roots. A small group of students did by solving for the original roots.

Qn	Solution
7.	
	$arc\sin(k)=\frac{\pi}{4}-\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{4}-\frac{\pi}{6}$ $k=\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$ $=\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$ $=\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\frac{1}{2}$ $=\frac{\sqrt{3}-1}{2\sqrt{2}}$
	Comment: Well done although some mistakenly gave the value of $\arctan\frac{1}{\sqrt{3}}$ as $\frac{\pi}{3}$.
8.	
(i)	$\frac{1}{a+x}-\frac{1}{a-x}\geq 0 \qquad x\neq a,-a$ $\frac{a-x-(a+x)}{(a+x)(a-x)}\geq 0$ $\frac{-2x}{(a+x)(a-x)}\geq 0$ $x(x+a)(x-a)\geq 0$ $-a<x\leq 0$ or $x>a$
	Comment: Badly done with many stopping at $\frac{-2x}{(a+x)(a-x)}\geq 0$. Some continued with incoherent arguments.
(ii)	a = 2, $subst\ x=e^y$ $-2<e^y\leq 0$ (rej) or $e^y>2\Rightarrow y>\ln 2$
	Some have notion that $a=2$ and $x=e^y$ but could not obtain the right solution.

Qn	Solution
9.	
(i)	$\dfrac{f(x+h)-f(x)}{h}=\dfrac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}$ $=\dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}}$ $=\dfrac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}}\times\dfrac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$ $=\dfrac{x-x-h}{h\left(x\sqrt{x+h}+(x+h)\sqrt{x}\right)}$ $\lim\limits_{h\to 0}\dfrac{-1}{x\sqrt{x+h}+(x+h)\sqrt{x}}=\dfrac{-1}{x\sqrt{x}+x\sqrt{x}}=\dfrac{-1}{2x^{\frac{3}{2}}}$ $\left(\text{or } -\dfrac{1}{2}x^{-\frac{3}{2}} \text{ or } \dfrac{1}{-2x\sqrt{x}}\right)$
	Comment: Several students did not do the key step of multiplying by the conjugate of the surds in order to arrive at the correct answer.
(ii)	$2^{xy}=e^x\cos x$ $xy\ln 2=\ln\left(e^x\cos x\right)$ $xy\ln 2=x+\ln(\cos x)$ $Diff$: $y\ln 2+x\ln 2\dfrac{dy}{dx}=1-\dfrac{\sin x}{\cos x}$ $x\ln 2\dfrac{dy}{dx}=1-\tan x-y\ln 2$ $\dfrac{dy}{dx}=\dfrac{1-\tan x-y\ln 2}{x\ln 2}$ $\left(or\ \dfrac{-\tan x}{x\ln 2}-\dfrac{\ln\cos x}{x^2\ln 2}\quad or\ \dfrac{1-\tan x}{x\ln 2}-\dfrac{y}{x}\right)$ **Alternatively:** $2^{xy}=e^x\cos x$ $Diff$: $2^{xy}\ln 2\left(y+x\dfrac{dy}{dx}\right)=e^x\cos x-e^x\sin x$ $2^{xy}x\ln 2\dfrac{dy}{dx}=e^x(\cos x-\sin x)-2^{xy}y\ln 2$ $\dfrac{dy}{dx}=\dfrac{e^x(\cos x-\sin x)-2^{xy}y\ln 2}{2^{xy}x\ln 2}=\dfrac{e^x(\cos x-\sin x)}{2^{xy}x\ln 2}-\dfrac{y}{x}$ or $\dfrac{1}{x\ln 2}-\dfrac{y}{x}-\dfrac{e^x\sin x}{2^{xy}x\ln 2}$
	Comment: Badly done. There were mistakes in implicit differentiation and algebraic manipulation. Most common mistake is with differentiating 2^{xy} . Some wrote **wrongly** that $2^{xy}=2^x.2^y$

Qn	Solution
10.	
(a)	$y=2x-x^2$ $y=-\left[\left(x^2-2x+1\right)-1\right]$ $y=-\left(x-1\right)^2+1$ $\left(x-1\right)^2=1-y$ $x-1=\pm\sqrt{1-y}$ $x=1\pm\sqrt{1-y}$ $f^{-1}(x)=1-\sqrt{1-x}\qquad (\because x<1)$ $D_{f^{-1}}=R_f=]-\infty,1[$
	Comment: Several students interpreted the question wrongly into testing for whether the inverse function exist and did the horizontal line test. This is not required. Common mistake in forgetting the $\pm$ and in not choosing the correct rule for the function.
(b) (i)	$R_h\subseteq D_g$ $]0,\infty[\ \subseteq\ \mathbb{R}\setminus\{0\}\qquad \therefore gh$ exists.
	Comment: need to state clearly both the range of h and the domain of g in correct notations.
(b) (ii)	$gh(x)=g\left(x(x-2)\right)=\dfrac{1}{x(x-2)}$ $D_{gh}=D_h=]-\infty,0[$
(b) (iii)	$g^2(x)=x\qquad$ (for even numbers) $g\left(g^{2016}(x)\right)=g(x)$ $g\left(h(x)\right)=\dfrac{1}{h(x)}$
	Comment: the question requires the answer in terms of h(x).
(c)	$f(x)\to f(x+1)\to f(-x+1)\to 2f(-x+1)$ 
	Comment: Badly done. Usually the wrong sequence of transformations or wrong transformations given.

Qn	Solution
(d)	
	Comment: some mistakes with the shape and symmetry of the curve.
(e)	
	Comment: badly done. Especially the LHS has a horizontal asymptote to y = 0.5.
11.	
(a)	$\cos\theta = \dfrac{\begin{pmatrix}1\\-1\\4\end{pmatrix}.\begin{pmatrix}1\\-1\\1\end{pmatrix}}{\sqrt{18}\sqrt{3}}$ $= \dfrac{1+1+4}{3\sqrt{6}}$ $= \dfrac{2}{\sqrt{6}} \qquad \left(or \sqrt{\dfrac{2}{3}} \quad or \ \dfrac{\sqrt{6}}{3} \text{ or equivalent}\right)$
	Comment: some wrongly took the angle to be between vectors OA and OB.
(b)	$\underset{\sim}{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB}$ $L_1 : \underset{\sim}{r} = \begin{pmatrix}1\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}, \lambda \in \mathbb{R} \qquad \left(or \begin{pmatrix}2\\-2\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)$
	Comment: well done.

Qn	Solution
(c)	$L_2 : \underset{\sim}{r} = \begin{pmatrix}2\\4\\7\end{pmatrix} + \mu\begin{pmatrix}2\\1\\3\end{pmatrix}, \quad \lambda \in \mathbb{R}$ $\begin{pmatrix}1\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\4\\7\end{pmatrix} + \mu\begin{pmatrix}2\\1\\3\end{pmatrix}$ $\lambda - 2\mu = 1, \quad -\lambda - \mu = 5, \quad \lambda - 3\mu = 3$ $\Rightarrow \mu = -2, \quad \lambda = -3$ $\overrightarrow{OP} = \begin{pmatrix}1\\-1\\4\end{pmatrix} - 3\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-2\\2\\1\end{pmatrix}$ or $(-2,2,1)$
	Comment: Generally well done.
(d)	$\overrightarrow{OF} = \begin{pmatrix}2\\4\\7\end{pmatrix} + \mu\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}2+2\mu\\4+\mu\\7+3\mu\end{pmatrix}$ $\overrightarrow{CF} . \underset{\sim}{d} = 0$ $\left(\begin{pmatrix}2+2\mu-2\\4+\mu-(-10)\\7+3\mu-7\end{pmatrix}\right) . \begin{pmatrix}2\\1\\3\end{pmatrix} = 0$ $2(2\mu) + (\mu+14) + 3(3\mu) = 0$ $14 + 14\mu = 0$ $\mu = -1$ $\begin{pmatrix}2-2\\4-1\\7-3\end{pmatrix} = \begin{pmatrix}0\\3\\4\end{pmatrix}$
	Comment: Generally well done.
(e)	$\overrightarrow{OF} = \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OC}'\right)$ $\overrightarrow{OC}' = 2\overrightarrow{OF} - \overrightarrow{OC}$ $= 2\begin{pmatrix}0\\3\\4\end{pmatrix} - \begin{pmatrix}2\\-10\\7\end{pmatrix} = \begin{pmatrix}-2\\16\\1\end{pmatrix}$
	Comment: Generally well done.

Qn	Solution
(f) (i)	Area of first parallelogram $= \left\|\left(\underset{\sim}{u}-\underset{\sim}{v}\right)\times\underset{\sim}{v}\right\| = \left\|\underset{\sim}{u}\times\underset{\sim}{v}-\underset{\sim}{v}\times\underset{\sim}{v}\right\| = \left\|\underset{\sim}{u}\times\underset{\sim}{v}\right\|.$ Area of second parallelogram $= \left\|\left(\underset{\sim}{u}+\underset{\sim}{v}\right)\times\underset{\sim}{v}\right\| = \left\|\underset{\sim}{u}\times\underset{\sim}{v}+\underset{\sim}{v}\times\underset{\sim}{v}\right\| = \left\|\underset{\sim}{u}\times\underset{\sim}{v}\right\|.$ Hence the areas of the parallelograms are the same. **Alternatively: (or any equivalent graphical approach)** Observe that both parallelogram have the same base $\left\|\underset{\sim}{v}\right\|$ and that the heights for both parallelograms are the same. i.e. $\left\|\underset{\sim}{u}\right\|\sin\theta.$ Hence the areas of the parallelograms are the same.
	Comment: Very badly done. Since a proof is required, the steps need to be very clear. If a diagram is attempted, the vectors must be clearly labelled for the marker to follow.
(f) (ii)	Area $= \left\|\underset{\sim}{u}\right\|\left\|\underset{\sim}{v}\right\|\sin\theta = (1)(1)\sin\theta = \sin\theta.$
	Comment: Very badly done. Many did not use the correct cross products for finding areas of parallelogram. They also did not realize the magnitude of unit vector is 1.
12.	
(a) (i)	$\arg(y+ix) = \arg\left(i(x-iy)\right), \quad \cot(\alpha) = \tan\left(\frac{\pi}{2}-\alpha\right)$ $\arg(i)+\arg(x-iy) = \frac{\pi}{2}-\alpha$
	Comment: Very badly done.
(b)	$\left(\frac{w}{2i}\right)^3 = 1$ $w^3 = -8i$ $w^3 = 8e^{i\left(\frac{-\pi}{2}\right)}$ $w^3 = 2e^{i\left(\frac{-\pi}{2}+2k\pi\right)}$ $w = 2e^{i\left(\frac{2k\pi}{3}-\frac{\pi}{6}\right)}, k = -1, 0, 1$ $w = 2e^{i\left(\frac{-5\pi}{6}\right)}, 2e^{i\left(\frac{-\pi}{6}\right)}, 2e^{i\left(\frac{\pi}{2}\right)}$
	Comment: Badly done. Several could not find the correct argument or the correct modulus. Final answer needs to be in Euler's form.
(c) (i)	$z^n = (\cos\theta + i\sin\theta)^n$ $z^n = \cos n\theta + i\sin n\theta$ $z^{-n} = \cos n\theta - i\sin n\theta$ $z^n + z^{-n} = 2\cos n\theta$
	Comment: Generally good.

Qn	Solution
(c) (ii)	$z^2-5z-5z^{-1}+z^{-2}+6$ $=\left(z^2+z^{-2}\right)-5\left(z+z^{-1}\right)+6$ $=\left(2\cos 2\theta\right)-5\left(2\cos\theta\right)+6$ $=2\cos 2\theta-10\cos\theta+6$ $=2\left(\cos 2\theta-5\cos\theta+3\right)$
	Comment: Generally well done.
(c) (iii)	$\cos 2\theta-5\cos\theta+3=0$ $\left(2\cos^2\theta-1\right)-5\cos\theta+3=0$ $2\cos^2\theta-5\cos\theta+2=0$ $\left(\cos\theta-2\right)\left(2\cos\theta-1\right)=0$ $\cos\theta=2\left(rej\right)\ ,\cos\theta=\frac{1}{2}$ $\sin\theta=\pm\frac{\sqrt{3}}{2}$ $z=\frac{1}{2}+i\frac{\sqrt{3}}{2},\frac{1}{2}-i\frac{\sqrt{3}}{2}$
	Comment: Most can find the values of $\cos\theta$ but did not find $\sin\theta$.