RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 7A: Sequences and Series

SYLLABUS INCLUDES

- Concepts of sequence and series for finite and infinite cases
- Sequence as function $y = \mathrm{f}(n)$ where n is a positive integer
- Sequence given by a formula for the nth term
- Relationship between u_n (the nth term) and S_n (the sum to n terms)
- Sequence generated by the relation $u_{n+1} = \mathrm{f}(u_n)$, including the use of a graphing calculator to generate the sequence
- Sum and difference of two series
- Convergence of a series and the sum to infinity.

CONTENT

INTRODUCTION

Sequences and series occur frequently in real life. In this chapter, we will learn more about these concepts, and special sequences such as arithmetic and geometric progressions. You will also get to solve practical problems involving loan repayment, compound interest, etc. You may just find yourself using the knowledge and skills when you take up loans to buy a car or a property some years down the road!

1 Sequences and Series

1.1 Basic Definitions

A **sequence** is a set of numbers arranged in a defined order. Each number in the sequence is called a **term** of the sequence.

A sequence can have finite or infinite number of terms.

A sequence can be defined by a **formula** for the general term $u_n = \mathrm{f}(n)$. Some examples:

	Formula for u_n	n	u_6	Sequence	Finite / Infinite
1.	$u_n = n^2$	$n = 1, 2, 3, ..., 12$	36	1, 4, 9, 16, 25, … , 144	Finite
2.	$u_n = 20 - 5(n-1)$	$n = 2, 3, 4, ...$	-5	$15, 10, 5, 0, -5, ...$	Infinite
3.	$u_n = \frac{1}{2^{n-2}}$	$n = 1, 2, 3, ...$	$\frac{1}{16}$	$2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ ...$	Infinite
4.	$u_n = (-1)^n n$	$n = 0, 1, 2, 3, ..., 50$	6	$0, -1, 2, -3, 4, -5, ..., 50$	Finite
5.	$u_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	$n \in \mathbb{Z}^+$	8	**Fibonacci Sequence** 1, 1, 2, 3, 5, …	Infinite

Example 1 [9740/2009/01/Q1]

(i) The first three terms of a sequence are given by $u_1 = 10$, $u_2 = 6$, $u_3 = 5$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. [4]

(ii) Find the set of values on n for which u_n is greater than 100. [2]

Solutions

(i) Let $u_n = an^2 + bn + c$.

$u_1 = a + b + c = 10$
$u_2 = 4a + 2b + c = 6$
$u_3 = 9a + 3b + c = 5$

Using GC, $a = 1.5$, $b = –8.5$, $c = 17$.

$\therefore\ u_n = 1.5n^2 – 8.5n + 17$.

(ii)

$u_n > 100$

$1.5n^2 - 8.5n + 17 > 100$

$1.5n^2 - 8.5n + 17 - 100 > 0$

Let $y = 1.5n^2 – 8.5n + 17 – 100$

From GC, since $n \in \mathbb{Z}^+$, $n > 10.79$.
Hence solution set = $\{n \in \mathbb{Z}^+ : n \geq 11\}$.

OR

$u_n > 100$

$1.5n^2 - 8.5n + 17 > 100$

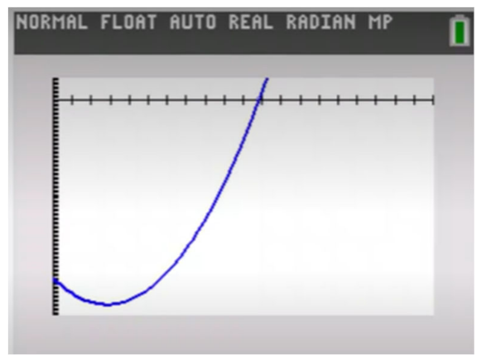
$1.5n^2 - 8.5n + 17 - 100 > 0$

Let $y = 1.5n^2 – 8.5n + 17 – 100$

From G.C.,

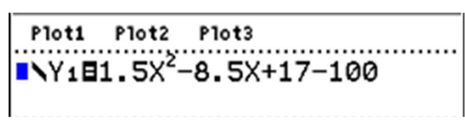
n	y
10	$-18 < 0$
11	$5 > 0$

Hence solution set = $\{n \in \mathbb{Z}^+ : n \geq 11\}$.

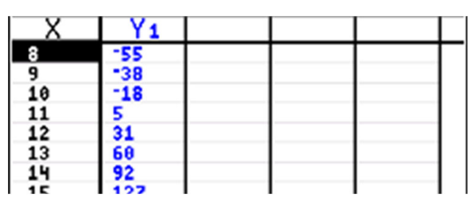


GC Keystrokes

1. Press y= and key in the expression.



2. Press 2nd graph for the table of values. As the question needed the Y_1 value to be positive, scroll down until you reach the <u>first</u> x-value for which Y_1 is positive.



A **series** is the **sum** of the terms of a sequence and is denoted by $S_n = u_1 + u_2 + ... + u_{n-1} + u_n$.

A series can have finite or infinite number of terms.

Note

• $u_n = S_n - S_{n-1}$ for $n \geq 2$
• $u_1 = S_1$

Example 2 [9758/2020/02/Q2c]

The sum of the first n terms of a series is $n^3 - 11n^2 + 4n$, where n is a positive integer.

(i) Find an expression for the nth term of this series, giving your answer in its simplest form.

(ii) The sum of the first m terms of this series, where $m > 3$, is equal to the sum of the first three terms of this series. Find the value of m.

Solutions

(i) Let S_n denote the sum of the first n terms.

For $n \geq 2$, n^{th} term

$$\begin{aligned} &= S_n - S_{n-1} \\ &= \left(n^3 - 11n^2 + 4n\right) - \left(\left(n-1\right)^3 - 11\left(n-1\right)^2 + 4\left(n-1\right)\right) \\ &= \left[n^3 - \left(n-1\right)^3\right] + 11\left[\left(n-1\right)^2 - n^2\right] + 4\left[n - \left(n-1\right)\right] \\ &= \left[n^3 - \left(n^3 - 3n^2 + 3n - 1\right)\right] + 11\left[\left(n^2 - 2n + 1\right) - n^2\right] + 4 \\ &= \left(3n^2 - 3n + 1\right) - 22n + 11 + 4 \\ &= 3n^2 - 25n + 16 \end{aligned}$$

1st term $= \mathrm{S}_1 = 1^3 - 11\left(1\right)^2 + 4 = -6$ and it follows the form $3n^2 - 25n + 16$ when $n = 1$. Thus nth term $= 3n^2 - 25n + 16$ for $n \geq 1$.

(ii)

$S_m = S_3$

$m^3 - 11m^2 + 4m = 3^3 - 11\left(3\right)^2 + 4\left(3\right) = -60$

$m^3 - 11m^2 + 4m + 60 = 0$

$m = -2, 3$ or 10

$\therefore m = 10$ since $m > 3$

In the next section, we will look at convergence of a sequence and a series.

1.2 Convergence of a Sequence

A sequence u_n is said to be convergent if it approaches a ***unique*** value as *n* approaches infinity. For example, from the previous section, given $u_n = \frac{1}{2^{n-2}}$, where $n = 1, 2, 3, \ldots$, it creates the sequence as follows: $2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$.

As $n \to \infty$, we observed that $\frac{1}{2^{n-2}} \to 0$, hence we can conclude that $u_n \to 0$ and this sequence converges.

1.3 Convergence of a Series

If $S_n = u_1 + u_2 + \ldots + u_{n-1} + u_n$ approaches a ***unique*** value as *n* approaches infinity, we say that the series **converges** and that the sum to infinity S_∞ exists. Otherwise, the series is said to **diverge** or that the sum to infinity S_∞ does not exist.

For example, consider the series $S_n = 2 + 1 + \cdots + \frac{1}{2^{n-2}}$.

It can be proven that $S_n = 4\left(1 - \frac{1}{2^n}\right)$ (try proving this on your own after you have studied geometric series).

We see that as $n \to \infty$, $\frac{1}{2^n} \to 0$ and $S_n \to 4$. So, we say that the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ **converges** to 4 and that $S_\infty = 4$. Note that from Section 1.2, the corresponding sequence $2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$ converges to 0.

Here are some examples of a **divergent** series.

(a) If $u_n = \frac{1}{n}$, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ converges to 0 since as $n \to \infty$, $\frac{1}{n} \to 0$.

However, the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges since

$$\begin{aligned} 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \end{aligned}$$

and $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \to \infty$ as $n \to \infty$

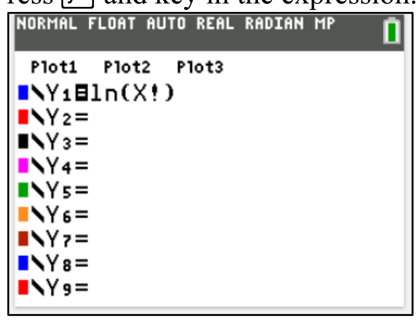
(b) If $u_n = (-1)^n$, then $S_n = -1 + 1 - 1 + 1 - \cdots + (-1)^n$ is divergent.

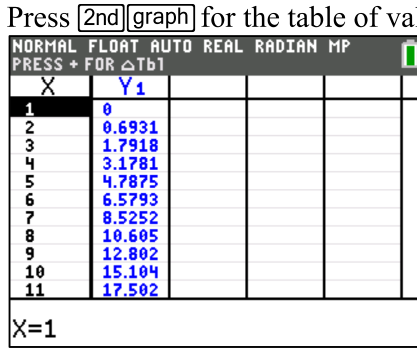
$n=1,\ S_1=-1$
$n=2,\ S_2=-1+1=0$
$n=3,\ S_3=-1+1-1=-1$
$n=4,\ S_4=-1+1-1+1=0$

As a result, we can conclude that $S_n=-1$ if n is odd and $S_n=0$ if n is even. Hence, the series S_n does not approach a unique value and is divergent. Note that the sequence $-1,1,-1,1,\cdots$ is also divergent since $u_n=\left(-1\right)^n$ does not approach a unique value as $n\to\infty$.

(c) If $u_n=\ln n$, then both the sequence $\ln 1, \ln 2, \ln 3, \cdots$ and the corresponding series $S_n=\ln 1+\ln 2+\ln 3+\cdots$ is divergent since
$S_n=\ln 1+\ln 2+\ln 3+\cdots+\ln n=\ln(1\times 2\times 3\times\ldots\times n)=\ln\left(n!\right)$
and $\ln n\to\infty$ and $\ln n!\to\infty$ as $n\to\infty$.

We can verify the result by using a GC, as shown below.

<table>
<tr><th></th><th>GC Keystrokes</th></tr>
<tr><td>From GC, we see that as n increases, the value of S_n increases and does not approach a unique value. Hence, we can conclude that the series is divergent.</td><td>1. Press [y=] and key in the expression.


2. Press [2nd][graph] for the table of values.


Note: S_n is strictly increasing without bound as n increases.</td></tr>
</table>

1.4 Sequence generated by a Recurrence Relation

A sequence can be generated by a **recurrence relation** of the form $u_{n+1} = \mathrm{f}(u_n)$, where $n \in \mathbb{Z}^+$ and the $(n+1)^{\text{th}}$ term, u_{n+1}, is linked to its previous term u_n, by a formula. If the initial condition (i.e. the value of u_1) is given, then we can determine $u_2, u_3, ...$ recursively by $u_2 = \mathrm{f}(u_1)$, $u_3 = \mathrm{f}(u_2)$ and so on.

For example, the sequence 5, 8, 11, 14, … can be defined by a recurrence relation of the form $u_n = u_{n-1} + 3$, with $u_1 = 5$.

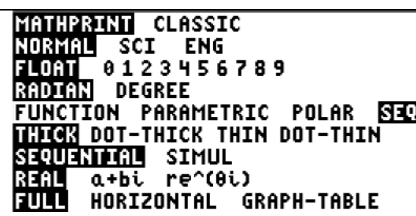
Example 3

A sequence u_1, u_2, u_3, ... is defined by $u_n = 3u_{n-1} + 2$, for $n = 2, 3, 4, ...$.
Given that $u_1 = 3$, find the values of u_4 and u_6.

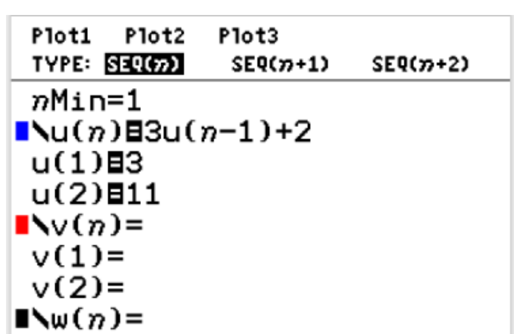
Solution:

$u_1 = 3$	$u_2 = 3u_1 + 2 = 3(3) + 2 = 11$
$u_3 = 3u_2 + 2 = 3(11) + 2 = 35$	$u_4 = 3u_3 + 2 = 3(35) + 2 = 107$
$u_5 = 3u_4 + 2 = 3(107) + 2 = 323$	$u_6 = 3u_5 + 2 = 3(323) + 2 = 971$

Using the GC to obtain terms in a sequence generated by a recurrence relation

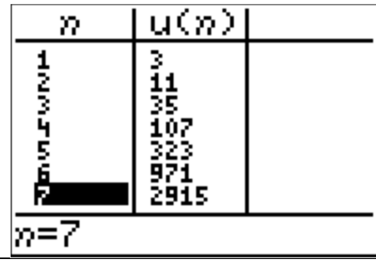
<table>
<tr>
<td>1. Press mode and then scroll down to the fifth line and change to SEQ mode.</td>
<td>
</td>
</tr>
<tr>
<td>2. Press y=

a. Key in the starting value of n: $n\text{Min} = 1$
b. Key in the recurrence relation
$u(n) = 3u(n-1) + 2$.
(to obtain the letter u, press 2nd 7;
to obtain the letter n, press X,T,θ,n.)

c. Key in the first two terms:
$u(1) = 3$
$u(2) = 11$</td>
<td>
</td>
</tr>
</table>

3. Press 2nd graph to check the values of u_4 and u_6 in the table of values.

From the GC, $u_4 = 107$, $u_6 = 971$



Example 4

A sequence of real numbers $u_1, u_2, u_3, \ldots$ satisfies $u_1 = 1$ and the recurrence relation $u_{n+1} = \sqrt{u_n + 3}$ for $n = 1, 2, 3, \ldots$

(i) Use a graphing calculator to determine the behaviour of the sequence.

(ii) Given that as $n \to \infty$, $u_n \to l$, find the exact value of l using an algebraic method.

Solution

1. Change to **SEQ** mode.

2. Press y=

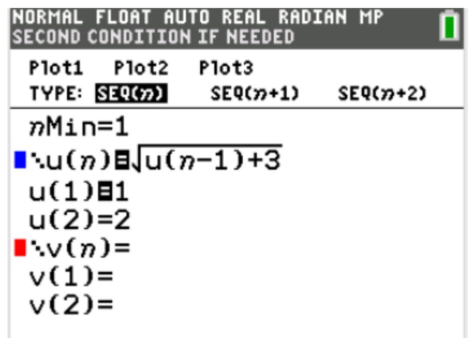
3a. Key in the starting value of n : $n\text{Min} = 1$

3b. $u_{n+1} = \sqrt{u_n + 3}$

Replacing n by $n-1$, $u_n = \sqrt{u_{n-1} + 3}$.

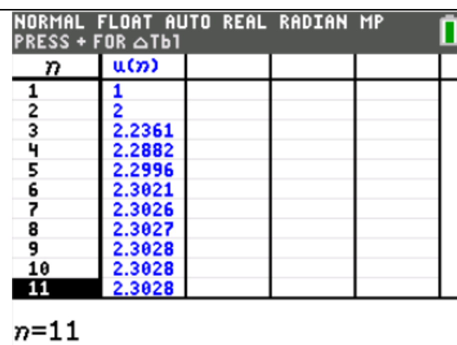
Key in the recurrence relation $u(n) = \sqrt{u(n-1) + 3}$.

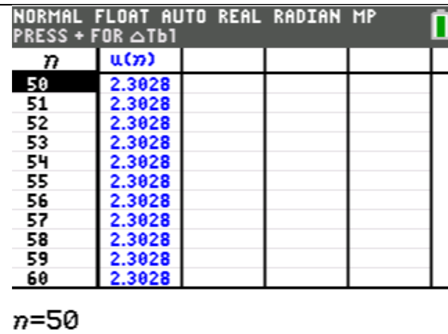
4. Key in the first two terms:
$u(1) = 1$
$u(2) = 2$



5. Press 2nd graph and scroll down the table to check the behaviour of the sequence.

We note that the terms of the sequence are increasing as n increases, but it approaches a unique value as n approaches infinity.





NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

n	u(n)
50	2.3028
51	2.3028
52	2.3028
53	2.3028
54	2.3028
55	2.3028
56	2.3028
57	2.3028
58	2.3028
59	2.3028
60	2.3028

n=50

(i) From the GC, the sequence is strictly <u>increasing</u> and <u>converges</u> to 2.3028 (to 5 s.f that is, 2.30 (to 3 s.f.).

(ii) As $n \to \infty$, $u_n \to l$, $u_{n+1} \to l$.

From $u_{n+1} = \sqrt{u_n + 3}$, as $n \to \infty$, we have

$$l = \sqrt{l+3}$$

$$l^2 = l + 3$$

$$l^2 - l - 3 = 0$$

$$l = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2} = \frac{1 \pm \sqrt{13}}{2}$$

Since $u_n > 0$, $l = \dfrac{1+\sqrt{13}}{2}$.

2 Representing a Series by Σ (Sigma) Notation

Using the sigma notation, the series $u_1 + u_2 + \cdots + u_n$ can be denoted by $\sum_{r=1}^{n} u_r$, i.e. $\sum_{r=1}^{n} u_r = u_1 + u_2 + \cdots + u_n$.

In general,

$$\sum_{r=k}^{m} u_r = u_k + u_{k+1} + \cdots + u_{m-1} + u_m,$$

m ← **Ending** value of r
u_r ← **General Term**
$r=k$ ← **Starting** value of r

where $k, m \in \mathbb{Z}_0^+$ and $k \le m$.
Note that r is known as the index of summation.

Similarly, the infinite series $u_1 + u_2 + u_3 + \cdots$ can be denoted by $\sum_{r=1}^{\infty} u_r$.

If we can observe the pattern in a series and able to see the general term, we can write the series in Σ (sigma) notation. For example, we can see that the general term of the series

$$2 + 4 + 6 + \cdots + 2(n-1) + 2n$$

is $2r$. We can write the series as $\sum_{r=1}^{n} (2r)$.

Example 5
For each of the following series, write down the rth term and express the series using the Σ (sigma) notation.

	Series	***r*th term**	**In Σ notation**
(a)	$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$	$\frac{1}{3^r}$	$\sum_{r=1}^{\infty} \frac{1}{3^r}$
(b)	$5^2 + 7^2 + 9^2 + \cdots + 39^2$	$(2r+3)^2$	$\sum_{r=1}^{18} (2r+3)^2$
(c)	$(1\times 2) + (2\times 3) + (3\times 4) + \cdots + (n-1)(n)$	$r(r+1)$	$\sum_{r=1}^{n-1} r(r+1)$
(d)	$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{98}$	$\frac{(-1)^{r+1}}{2r}$	$\sum_{r=1}^{49} \frac{(-1)^{r+1}}{2r}$

Example 6

For the following series given in summation notation, write down the terms of the series.

Also, evaluate the sum of the series.

(a) $\sum_{r=1}^{12} r^4$ **(b)** $\sum_{k=2}^{8} \frac{\ln k}{k+1}$ **(c)** $\sum_{r=0}^{4} r!$

Solution

(a) $\sum_{r=1}^{12} r^4 = 1^4 + 2^4 + 3^4 + \cdots + 11^4 + 12^4$

$= 60710$

We can use a graphing calculator to evaluate the sum of the series. (See Use of GC on the next page)

(b) $\sum_{k=2}^{8} \frac{\ln k}{k+1} = \frac{\ln 2}{3} + \frac{\ln 3}{4} + \frac{\ln 4}{5} + \cdots + \frac{\ln 7}{8} + \frac{\ln 8}{9}$

$= 1.78.$ (3 s.f.)

(c) $\sum_{r=0}^{4} r! = 0! + 1! + 2! + 3! + 4!$

Recall : $r! = 1 \times 2 \times 3 \times \cdots \times (r-1) \times r$

$= 1 + 1 + 1 \times 2 + 1 \times 2 \times 3 + 1 \times 2 \times 3 \times 4$

$= 1 + 1 + 2 + 6 + 24$

$= 34$

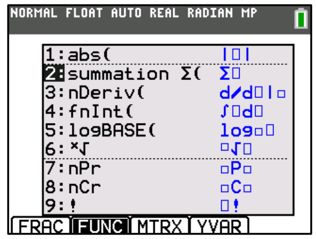
Use of Graphing Calculator to evaluate sums

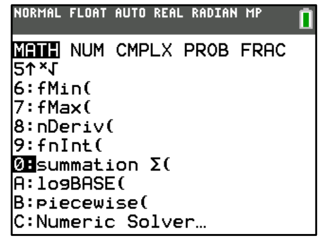
Use the graphing calculator to evaluate the sum in **Example 6(a)**.

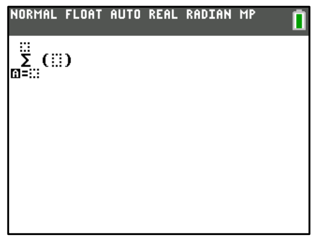
<table>
<tr>
<td>1)
Press either
a) [alpha] [window] [2],

or

b) [math] [0]

to obtain the summation sign</td>
<td>

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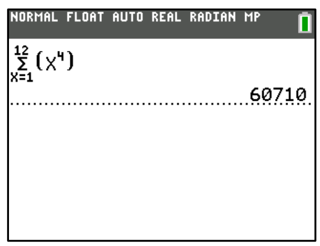
<tr>
<td>2) Key in the parameter, starting value of the parameter, the ending value of the parameter, and the general term.</td>
<td>
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Task: Use the graphing calculator to evaluate the sums in **Example 6(b)** and **6(c)**.

2.1 Properties of the $\sum$ (Sigma) Notation

If u_r and v_r are expressions in terms of r, a and b are constants and k and m are non-negative integers such that $k \le m$, then we have the following results:

(1) $\sum_{r=k}^{m} u_r$ has $(m-k+1)$ terms.	**(2)** For $2 \le k \le m$, $\sum_{r=k}^{m} u_r = \left(\sum_{r=1}^{m} u_r\right) - \left(\sum_{r=1}^{k-1} u_r\right)$.
(3) $\sum_{r=k}^{m} (au_r) = a\sum_{r=k}^{m} u_r$.	**(4)** $\sum_{r=k}^{m} (au_r \pm bv_r) = a\left(\sum_{r=k}^{m} u_r\right) \pm b\left(\sum_{r=k}^{m} v_r\right)$.

Note that $\sum_{r=k}^{m} (u_r v_r) \ne \left(\sum_{r=k}^{m} u_r\right)\left(\sum_{r=k}^{m} v_r\right)$.

2.2 Summation using Standard Results

Let a be a constant and n be a positive integer.

(1) $\sum_{r=1}^{n} a = na$.

(2) $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

(3) $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

(4) $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left(\frac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$.

Remarks:

a. The result in **(1)** can be generalized:

$$\sum_{r=k}^{m} a = \left(number\ of\ terms\right) \times a = \left(m-k+1\right)a,$$

where k and m are integers such that $k \le m$.

b. The result in **(2)** can be derived using the sum of the first n terms of an AP *(Section 3)*

c. The results in **(3)** and **(4)** are not found in the Formula List but these two results will be given when required.

d. The results in **(2)**, **(3)** and **(4)** hold if and only if r starts from 0 or 1.

Note that : $\sum_{r=0}^{n} r = \sum_{r=1}^{n} r$, $\sum_{r=0}^{n} r^2 = \sum_{r=1}^{n} r^2$, $\sum_{r=0}^{n} r^3 = \sum_{r=1}^{n} r^3$ but $\sum_{r=0}^{n} a \ne \sum_{r=1}^{n} a$

Example 7

Evaluate the following sums without using the sigma notation function in a graphing calculator.

(a) $\sum_{r=0}^{10}(2r+3)$ **(b)** $\sum_{r=20}^{40} r^3$

Solution

(a) $\sum_{r=0}^{10}(2r+3) = 2\sum_{r=0}^{10} r + \sum_{r=0}^{10} 3$

$= 2\sum_{r=1}^{10} r + (10-0+1)(3)$

$= 2\left[\frac{1}{2}(10)(10+1)\right] + 33$

$= 110 + 33 = 143$

(b) $\sum_{r=20}^{40} r^3 = \sum_{r=1}^{40} r^3 - \sum_{r=1}^{19} r^3$

$= \frac{1}{4}(40)^2(41)^2 - \frac{1}{4}(19)^2(20)^2$

$= 636300$

Note : If question does not restrict the use of GC, you should use the GC to evaluate the sums in (a) and (b).

Example 8

Find the sum of the first n terms of the series $1 \times 2 + 3 \times 4 + 5 \times 6 + 7 \times 8 + \ldots$.

Solution

Required sum $= \sum_{r=1}^{n}(2r-1)(2r)$

$= \sum_{r=1}^{n}(4r^2 - 2r)$

$= 4\sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r$

$= 4\left(\frac{n}{6}(n+1)(2n+1)\right) - 2\left(\frac{n}{2}(n+1)\right)$

$= \frac{2n}{3}(n+1)(2n+1) - n(n+1)$

$= \frac{n(n+1)}{3}\left[2(2n+1) - 3\right] = \frac{1}{3}n(n+1)(4n-1)$

We revisit verifying the convergence of a series using a GC, now that we have discussed the sigma notation.

How do we determine if a given series $\sum_{r=1}^{n}\frac{1}{r^3}$ is convergent using GC?

1. Press $\boxed{\text{y=}}$

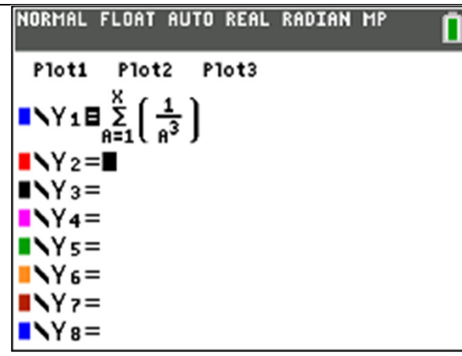
To obtain sigma notation, press either

a) $\boxed{\text{alpha}}\boxed{\text{window}}\boxed{2}$, or

b) $\boxed{\text{math}}\boxed{0}$

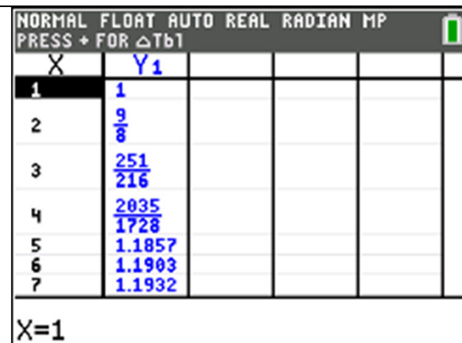
To obtain A, press $\boxed{\text{alpha}}\boxed{\text{math}}$

Note: The index *r* is represented by A while the ending value of *r* is represented by X, since we want to observe the values of the series when *n* tends to a large number.



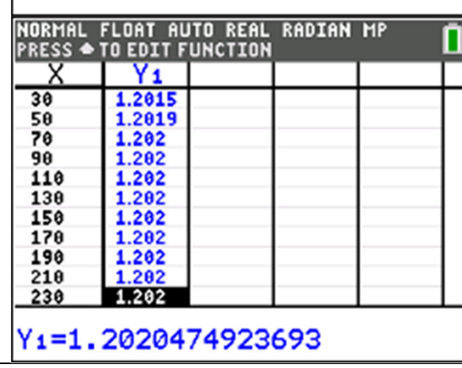
2. Press $\boxed{\text{2nd}}\boxed{\text{graph}}$ to check the values as X increases. Note that X is the number of terms of the series.

We note that $S_n=\sum_{r=1}^{n}\frac{1}{r^3}$ is increasing as n increases.

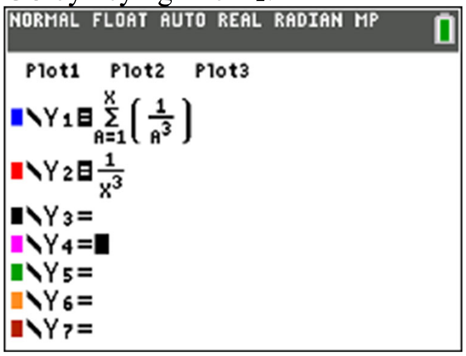


You can press $\boxed{\text{2nd}}\boxed{\text{window}}$ to adjust the table settings to show higher intervals. For example, to show the values of X for every interval of 20 while starting at the value of 10.

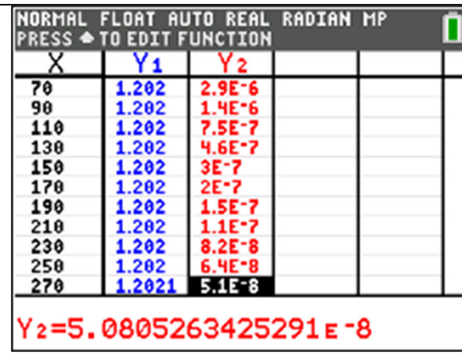
We observed from the table of values that the series $\sum_{r=1}^{n}\frac{1}{r^3}$ is convergent. In fact, as $n\to\infty$, the series converges to 1.202 (3.d.p)



Note that we can also check if the sequence $u_1, u_2, \cdots$ where $u_r = \dfrac{1}{r^3}$ is convergent using the GC by keying into Y2:



From the table, we can see that the sequence $u_1, u_2, \cdots, u_n$ converge since $u_n \to 0$ as $n \to \infty$.



SUMMARY