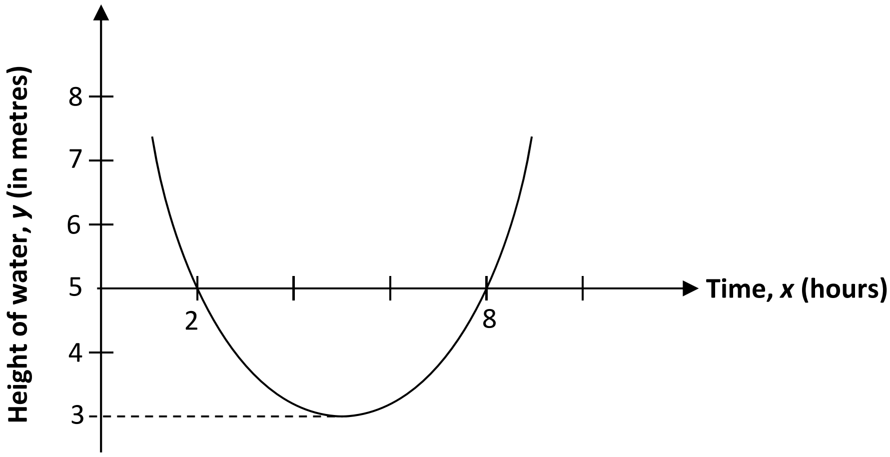
Topic: Coordinate geometry

Question 1 [2008 EOY, Q9] — marks

The graph below shows part of a quadratic curve which represents the water level, *y* metres, of a river over a period of time in hours during the operation of a water dam nearby. At 2 pm and 8 pm, the water level was at the recommended level of 5 metres. Let *x* represents the time in hours since 12pm.



(i) At what time did the water level reaches its minimum? [1]

Solution:

$\dfrac{2+8}{2}=5$; at 5pm

(ii)

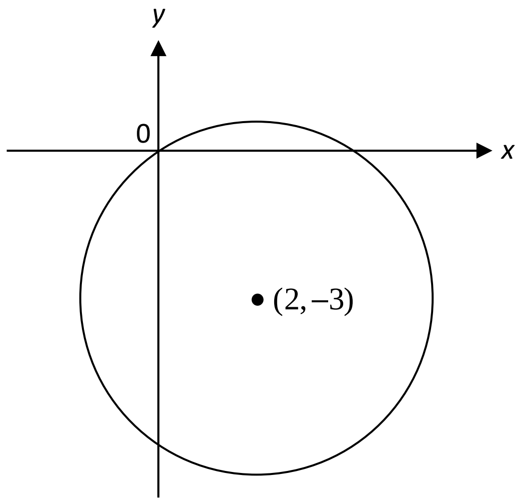
Find the equation of the curve, expressing it in the form $y=a(x-h)^2+k$ where *a*, *h* and *k* are real numbers. [4]

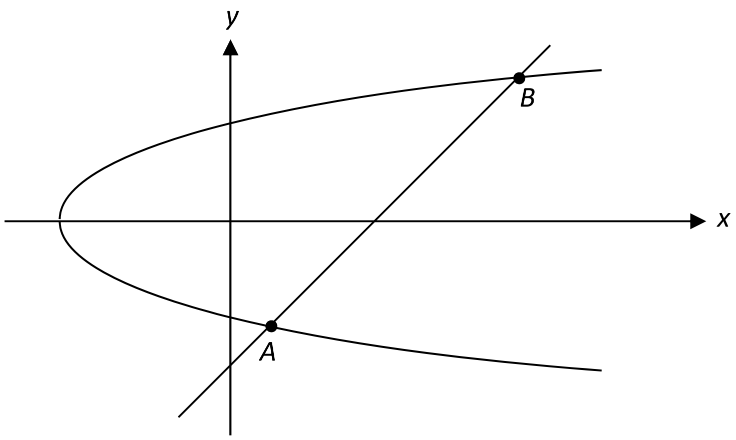
Solution:

(a) The vertex is at (5, 3) so $y=a(x-5)^2+3$

Sub (2, 5) into the equation to get $5=9a+3 \Rightarrow a=\dfrac{2}{9}$

Hence $y=\dfrac{2}{9}(x-5)^2+3$

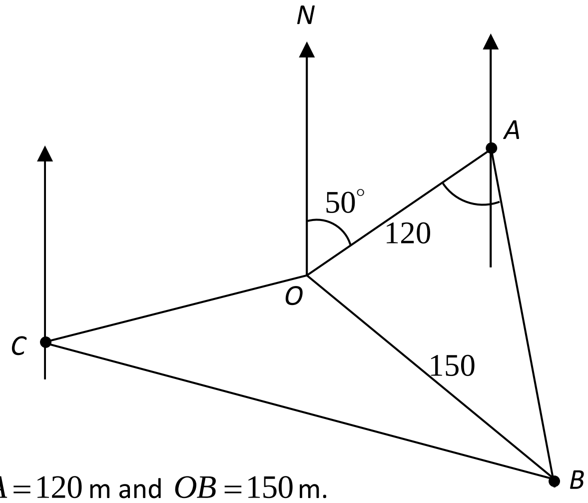
(iii) Hence, or otherwise, determine the height of the water level at 12 pm. **Solution:** Sub $x=0$ the height at 12 pm is $y=\frac{2}{9}(25)+3=8\frac{5}{9}$ metres	[2]
Question 2 [2008 EOY, Q10]	marks
Find the range of values of *k* if the line $2y=x+k$ intersects the curve $y=2x+\frac{6}{x}$ more than once. **Solution:** Equate the two equations to get $\frac{x+k}{2}=2x+\frac{6}{x}\Rightarrow x^2+kx=4x^2+12\Rightarrow 3x^2-kx+12=0$ For more than one root, $(-k)^2-4(3)(12)>0\Rightarrow k^2-144>0$ Hence $(k-12)(k+12)>0\Rightarrow k<-12$ or $k>12$	[5]
Question 3 [2008 EOY, Q12]	marks
The diagram below shows a circle with radius *r* units and its centre is $(2,-3)$.  (i) Given that the circle passes through the origin, find the exact value of *r*. **Solution:** $r=\sqrt{2^2+(-3)^2}=\sqrt{13}$	[2]

(ii) The circle is reflected about the *y*-axis and then moved 5 units downwards. Write down the coordinates of the new centre of the circle. **<u>Solution:</u>** $(-2,-3-5)=(-2,-8)$	[1]
(iii) Find the equation of the new circle in the form $x^2+y^2+2gx+2fy+c=0$ where *g*, *f* and *c* are real constants. **<u>Solution:</u>** $(x+2)^2+(y+8)^2=13$ $\Rightarrow x^2+4x+4+y^2+16y+64=13 \Rightarrow x^2+y^2+4x+16y+55=0$	[3]
Question 4 [2008 EOY, Q15]	marks
The figure below shows the graphs of $2y^2=x+7$ and $y=x-3$.  The two curves meet at *A* and *B*. (i) Show that the coordinates of *A* are $(1,-2)$ and find the coordinates of *B*. **<u>Solution:</u>** $2(x-3)^2=x+7 \Rightarrow 2x^2-12x+18=x+7 \Rightarrow 2x^2-13x+11=0$ Solve to get $(2x-11)(x-1)=0 \Rightarrow x=1$ or $\frac{11}{2}$ Sub $x=1$ to get $y=-2$ hence *A* is $(1,-2)$ (shown) and sub $x=\frac{11}{2}$ to get $y=\frac{5}{2}$ so *B* is $\left(\frac{11}{2},\frac{5}{2}\right)$	[3]

(ii) Find the length of *AB*. **Solution:** $AB = \sqrt{\left(\frac{11}{2}-1\right)^2+\left(\frac{5}{2}+2\right)^2} = \sqrt{\frac{2(81)}{4}} = \frac{9\sqrt{2}}{2}$ or 6.36 units	[2]
(iii) Find the equation of the perpendicular bisector of *AB*. **Solution:** Midpoint $= \frac{1}{2}\left(1+\frac{11}{2}, -2+\frac{5}{2}\right) = \left(\frac{13}{4}, \frac{1}{4}\right)$ Gradient of perpendicular $= -1 \div \frac{\frac{5}{2}+2}{\frac{11}{2}-1} = -1$ hence Perpendicular bisector of *AB* is $y-\frac{1}{4} = -\left(x-\frac{13}{4}\right) \Rightarrow y = \frac{7}{2}-x$	[3]

Question 5 [2008 EOY, Q17]	marks

The diagram below shows the position of three locations *A*, *B* and *C* with respect to a reference point *O*. The bearings of *A*, *B* and *C* from *O* are 050°, 135° and 240° respectively.



It is also given that $OA = 120$ m and $OB = 150$ m.

(i)

Find the bearing of *B* from *A*. [3]

Solution:

$$\measuredangle BOA = 135^\circ - 50^\circ = 85^\circ$$

$$AB = \sqrt{150^2 + 120^2 - 2(150)(120)\cos 85^\circ} = 183.7$$

$$\frac{\sin OAB}{150} = \frac{\sin 85^\circ}{AB} \Rightarrow OAB = 54.4^\circ$$

Hence bearing of *B* from *A* $= 360^\circ - (180^\circ - 50^\circ) - 54.4^\circ = 175.6^\circ$

(ii)

If the bearing of *B* from *C* is 100°, calculate the length of *OC*. [3]

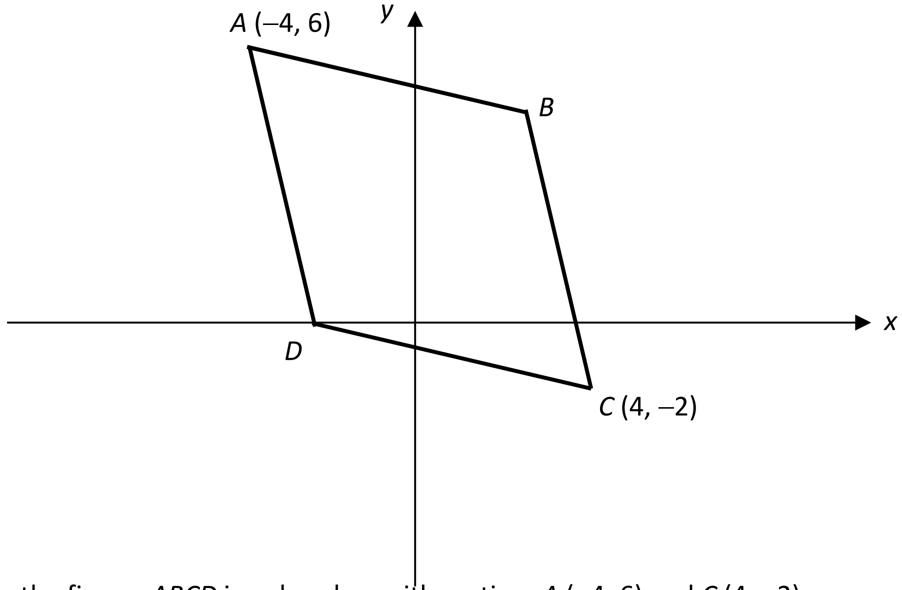
Solution:

$\measuredangle COB = 240^\circ - 135^\circ = 105^\circ$ and $\measuredangle OCB = 100^\circ - (180 - 120)^\circ = 40^\circ$

Hence by sine rule $\dfrac{OC}{\sin 35^\circ} = \dfrac{150}{\sin 40^\circ} \Rightarrow OC = 133.8$ m

(iii)	
A mast stands vertically at *B* such that the angle of elevation of the top of the mast from *O* is 62°. Find the height of the mast. **Solution:** $\tan 62^\circ = \frac{\text{height}}{150} \Rightarrow$ height $= 282.1$ m	[2]
Question 6 [2009 EOY, Q1]	marks
Find the coordinates of the points at which the straight line $y = -x - 5$ intersects the curve $x^2 + (y+2)^2 = 5$. **Solution:** Sub $y = -x - 5$ into $x^2 + (y+2)^2 = 5$: $x^2 + (-x-3)^2 = 5$ $x^2 + x^2 + 6x + 9 = 5$ $2x^2 + 6x + 4 = 0$ $x^2 + 3x + 2 = 0$ $(x+2)(x+1) = 0$ $x = -1$ or -2 $\therefore y = -4$ or -3 The coordinates of the points are (−1, −4) and (−2, −3).	[4]
Question 7 [2009 EOY, Q9]	marks
The equation of a circle *A* is $x^2 + y^2 + 6x - 4y + 4 = 0$. Find the coordinates of the centre of circle *A* and state its radius. **Solution:** $(x+3)^2 - 9 + (y-2)^2 - 4 + 4 = 0$ $(x+3)^2 + (y-2)^2 = 3^2$ A has centre (− 3, 2), radius = 3	[3]

Question 8 [2009 EOY, Q11]	marks
Solutions to this question by accurate drawing will not be accepted.	



In the figure, *ABCD* is a rhombus with vertices *A* (–4, 6) and *C* (4, –2). [3]

Given that the vertex *B* lies on the line $3x+2y=14$, calculate the

(i) equations of *AC* and *BD*,

Solution:

midpoint of *AC* = midpoint of *BD* = (0, 2)

gradient of *AC* = $\dfrac{6-(-2)}{-4-4}=-1$

Equation of *AC*: $y-6=-1(x+4)$

$y=-x+2$

Gradient of *BD* = 1

Equation of *BD*: $y-2=1(x-0)$

$y=x+2$

(ii) coordinates of *B* and *D*, **Solution:** Coordinates of *B*: Solve $y = x + 2$ and $3x + 2y = 14$ simultaneously. $y = x + 2 \rightarrow (1)$ $3x + 2y = 14 \rightarrow (2)$ Solving, $x = 2$, $y = 4$ $B = (2, 4)$ Use midpoint to calculate coordinates of *D*: $\left(\frac{x_1+2}{2}, \frac{y_1+4}{2}\right) = (0, 2)$ $D = (-2, 0)$	[3]
(iii) equation of *AB*, **Solution:** Gradient of *AB* = $\frac{6-4}{-4-2} = -\frac{1}{3}$ Equation of *AB*: $y - 6 = -\frac{1}{3}(x+4)$ $y = -\frac{1}{3}x + \frac{14}{3}$	[2]

(iv) area of the rhombus *ABCD* and hence find the perpendicular distance from *B* to the line *CD*.	[4]

Solution:

$$\text{Area of } ABCD = \frac{1}{2}\begin{vmatrix} -4 & 2 & 4 & -2 & -4 \\ 6 & 4 & -2 & 0 & 6 \end{vmatrix}$$

$$= \frac{1}{2}\left|-16-4+0-12-(0+4+16+12)\right|$$

$$= \frac{1}{2}\left|-64\right|$$

$$= 32 \text{ units}^2.$$

Let perpendicular distance from *B* to line *CD* be *h*.

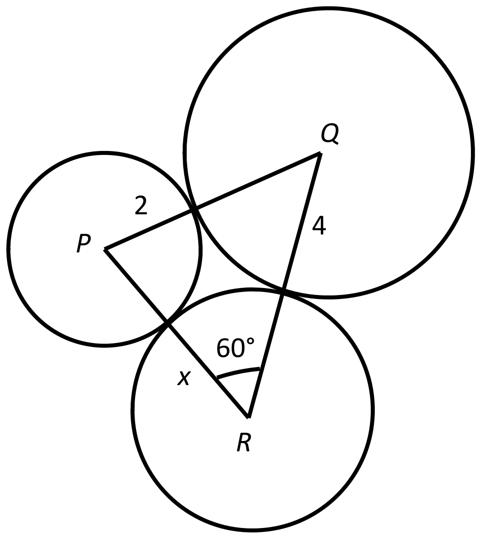
$$AB \times d = 32$$

$$d = \frac{32}{\sqrt{(6-4)^2+(-4-2)^2}}$$

$$= \frac{32}{\sqrt{40}}$$

$$= 1.6\sqrt{10} \text{ or } 5.06 \text{ units}$$

Question 9 [2009 EOY, Q14]	marks
The diagram below shows three circular fields with centres *P*, *Q* and *R* tangential to one another. The radii of the three fields are 2 m, 4 m and *x* m respectively and $\angle PRQ$ = 60°.	



(i)

Form an equation in *x* and show that it reduces to $x^2 + 6x - 24 = 0$. [3]

Solution:

Using cosine rule,

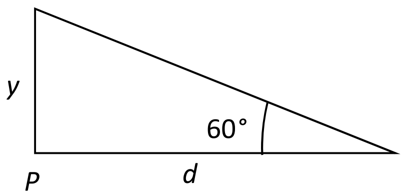
$$PQ^2 = PR^2 + QR^2 - 2(PR)(QR)\cos 60^\circ$$

$$6^2 = (x+4)^2 + (x+2)^2 - 2(x+4)(x+2)\left(\frac{1}{2}\right)$$

$$36 = x^2 + 8x + 16 + x^2 + 4x + 4 - (x^2 + 6x + 8)$$

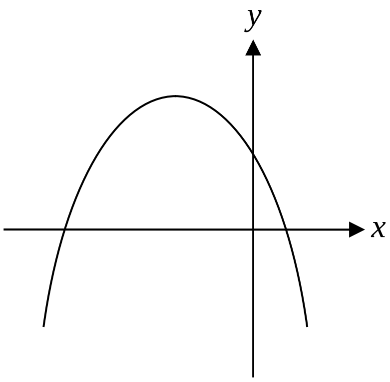
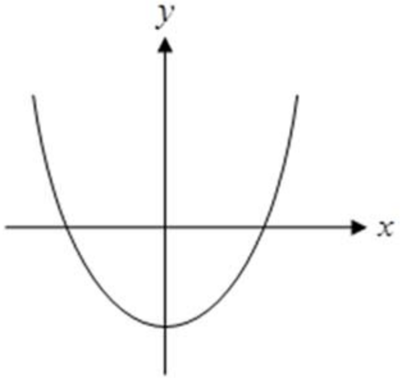
$$36 = x^2 + 6x + 12$$

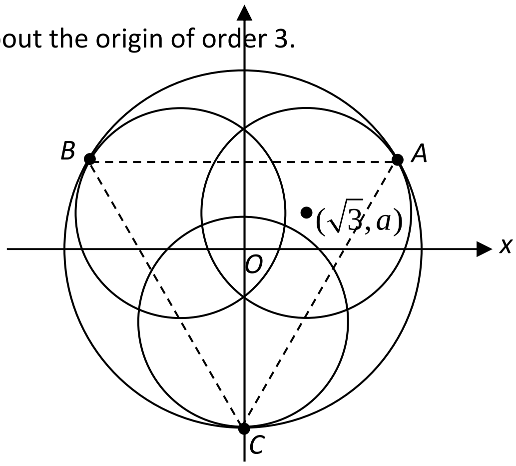
$$x^2 + 6x - 24 = 0$$

(ii) Solve the equation, giving your answer in surd form and find the exact area of triangle *PQR*. **<u>Solution:</u>** $x=\frac{-6\pm\sqrt{6^2-4(1)(-24)}}{2}$ $x=-3+\sqrt{33}$ or $-3-\sqrt{33}$ (rej) (2.74456...) or (−8.74456...) Area = $\frac{1}{2}(PR)(QR)\sin 60^\circ$ $=\frac{1}{2}(-1+\sqrt{33})(1+\sqrt{33})\frac{\sqrt{3}}{2}$ $=\frac{\sqrt{3}}{4}(32)$ $=8\sqrt{3}$	[4]
(ii) A pole of *y* m stands at *P*. A boy walks from *R* to *Q*. Given that the maximum angle of elevation of the top of the pole during the walk is 60°, find *y*. **<u>Solution:</u>** Maximum angle of elevation occurs when distance, *d* from *P* to *QR* is the smallest.  $8\sqrt{3}=\frac{1}{2}(1+\sqrt{33})d$ $d=\frac{16\sqrt{3}}{1+\sqrt{33}}$ $\tan 60^\circ=\frac{y}{d}$ $y=d\times\tan 60^\circ$ $=\frac{16\sqrt{3}}{1+\sqrt{33}}\times\sqrt{3}$ $=\frac{48}{1+\sqrt{33}}$ or 7.12 m	[3]

Question 10 [2010 EOY, Q4]	marks
Find the range of values of *k* such that the curve $y+2x^2=k(x-1)$ meets the line $2y=6kx+2k^2-3$ at two distinct points. **Solution:** Substitute *y*: $2(kx-k-2x^2)=6kx+2k^2-3 \Rightarrow 4x^2+4kx+2k^2-3+2k=0$ For two distinct roots, $(4k)^2-4(4)(2k^2+2k-3)>0 \Rightarrow k^2+2k-3<0$ Hence $(k+3)(k-1)<0 \Rightarrow -3<k<1$	[5]

Question 11 [2010 EOY, Q5]	marks
A quadratic curve is such that it is symmetrical about the line $x=2k$, where $k>1$. Given that it cuts the *x*-axis when $x=2$, and the maximum value of the curve is $k-1$, (i) explain why the other *x*-intercept is $4k-2$, and **Solution:** The roots are equidistant from the line of symmetry, so the other *x*-intercept is at $2k+(2k-2)=4k-2$	[1]
(ii) find, in terms of *x*, *y* and *k*, the equation of the curve **Solution:** Vertex form: $y=a(x-2k)^2+(k-1)$ Sub $(2,0)$ to get $a=\dfrac{1}{4(1-k)}$ [A1], hence $y=\dfrac{(x-2k)^2}{4(1-k)}+(k-1)$ **OR** Product form: $y=p(x-2)(x-4k+2)$ Sub $(2k,k-1)$ to get $p=\dfrac{1}{4(1-k)}$, hence $y=\dfrac{(x-2)(x-4k+2)}{4(1-k)}$	[3]

(cont'd) Given further that the distance between the two x-intercepts is 6 units, find the coordinates of the y-intercept. **Solution:** $4k-2-2=6 \Rightarrow k=\frac{5}{2}$ [B1] hence $y=-\frac{(x-5)^2}{6}+\frac{3}{2}$ (using vertex form) Hence y-intercept $=-\frac{25}{6}+\frac{3}{2}$ $=-\frac{8}{3}$	[3]
Question 12 [2010 EOY, Q6]	marks
On separate axes, sketch the graphs of (i) $y=a(x-h)^2+k$ for $a<0$, $h<0$ and $k>0$ **Solution:**  [G1] – Open downwards [G1] – Vertex in 2nd quadrant	[2]
(ii) $y=k(x-\alpha)(x-\beta)$ for $\alpha\beta<0$, $\|\alpha\|=\|\beta\|$ and $k>0$ **Solution:** [G1] – Open upwards [G1] – Vertex lies on y-axis 	[2]

Question 13 [2010 EOY, Q7]	marks
The diagram below shows three small identical circles inscribed in a big circle centred at the origin. Each small circle has radius *r* units (where $r > 2$) and one of the small circles has its centre at the point $(\sqrt{3}, a)$. The small circles are arranged with a rotational symmetry about the origin of order 3.  (i) By considering the rotational symmetry of the diagram, explain fully why $a = 1$. **Solution:** Due to rotational symmetry of order 3, *OA* (the radius of the bigger circle) is part of the line of symmetry of the equilateral triangle *ABC*, and hence makes an angle of 30° [M1] with the *x*-axis (alternate angle with $\measuredangle OAB$). Hence, $\tan 30° = \frac{a}{\sqrt{3}} \Rightarrow a = \sqrt{3} \times \frac{1}{\sqrt{3}} \Rightarrow a = 1$	[2]
(ii) Write down the equation of the bigger circle, in terms of *x*, *y* and *r*. **Solution:** Radius of big circle $= r + 2$ units [B1] Hence $x^2 + y^2 = (r+2)^2$ or $x^2 + y^2 = r^2 + 4r + 4$	[2]

Given further that $r = 3$, (iii) find the coordinates of *A*, the point of contact between the smaller circle and the bigger circle as shown in the diagram, and **Solution:** When $r = 3$, $OA = 5$ units [B1] hence coordinates of *A* are $(5\cos 30°, 5\sin 30°) = \left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$	[3]
(iv) show that the area of the triangle *ABC* is $\frac{75\sqrt{3}}{4}$ units². **Solution:** Due to symmetry, $AB = \frac{5\sqrt{3}}{2} \times 2$ units $= 5\sqrt{3}$ units Hence area of triangle $= \frac{1}{2} \times 5\sqrt{3} \times 5\sqrt{3} \times \sin 60°$ units^2 $= \frac{75\sqrt{3}}{4}$ units^2	[3]

Question 14 [2010 EOY, Q9]	marks

Three points A, B and C on level ground are such that A is due north of B, $BC = 20$ km, the bearing of C from B is $042°$ and $AC = 16$ km.

(i)

Find the possible distances of A from B. [5]

Solution:

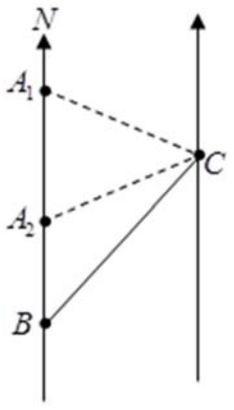
By sine rule, $\dfrac{\sin \measuredangle BAC}{20} = \dfrac{\sin 42°}{16}$ [M1]

Hence $\measuredangle BA_1C = 56.76°$ (2 d.p.) and $\measuredangle BA_2C = 123.24°$ (2 d.p.)

So $\measuredangle BCA_1 = 81.24°$ (2 d.p.) and $\measuredangle BCA_2 = 14.76°$ (2 d.p.)

$$\frac{BA_1}{\sin 81.24°} = \frac{16}{\sin 42°} \Rightarrow BA_1 = 23.6 \text{ km (3 s.f.)}$$

$$\frac{BA_2}{\sin 14.76°} = \frac{16}{\sin 42°} \Rightarrow BA_2 = 6.09 \text{ km (3 s.f.)}$$



(ii)

Taking the shorter distance between A and B, determine the bearing of A from C. [2]

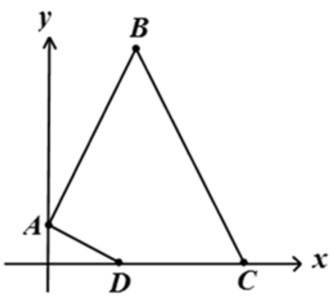
Solution:

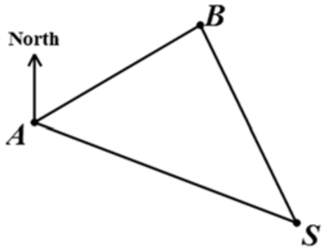
Bearing of A_2 from $C = 180° + 42° + 14.8°$

$= 236.8°$ (1 d.p.)

Question 15 [2011 EOY, Q6]	marks
The equation of a circle is given by $x^2 + y^2 = 6x - 8y$. (i) Find the centre and the radius of the circle. **Solution:** $\left(x^2 - 6x + 9\right) + \left(y^2 + 8y + 16\right) = 25$ $\left(x - 3\right)^2 + \left(y + 4\right)^2 = 25$ $\left(3, -4\right)$ radius 5 .	[4]
(ii) Hence or otherwise, find the greatest possible value of *y*. **Solution:** By sketching the circle, or show $y = -4 + 5$: Greatest *y* is 1	[2]
Question 16 [2011 EOY, Q7]	marks
The variables *x* and *y* are related by the equation $\sqrt{2^y} = ay + \frac{b}{\sqrt{x}}$. A straight line passes through (5, 7) is obtained when $\sqrt{2^y x}$ is plotted against $y\sqrt{x}$. (i) Given that the gradient of this line is 2, calculate the value of *a* and of *b*. **Solution:** $\sqrt{2^y} = ay + \frac{b}{\sqrt{x}}$ $\Rightarrow \sqrt{2^y x} = ay\sqrt{x} + b$ $a = gradient = 2$ A1 $\Rightarrow \left(7\right) = 2\left(5\right) + b \Rightarrow b = -3$	[3]
(ii) Given that this line also passes through (*k*, 13), find the value of *k*. **Solution:** $\Rightarrow \left(13\right) = 2\left(k\right) - 3$ $\Rightarrow k = \frac{16}{2} = 8$	[2]

Question 17 [2011 EOY, Q10a]	marks
A quadratic curve has a minimum point at $(1,\ 5)$ and passes through $(0,\ 7)$. If the curve does not meet the line $2(y-mx)=13$, find the range of values of *m*. **Solution:** Equation of the curve is $y=a(x-1)^2+5$ At $(0,7)$, $7=a(-1)^2+5 \Rightarrow a=2$ $y=2(x-1)^2+5........(1)$ $y=mx+6.5......(2)$ $2(x-1)^2+5=mx+6.5$ $2x^2-x(4+m)+0.5=0$ Since line does not meets the curve, $b^2-4ac<0$ $(4+m)^2-4(2)(0.5)<0$ $m^2+8m+12<0$ $(m+2)(m+6)<0$ $-6<m<-2$	[5]

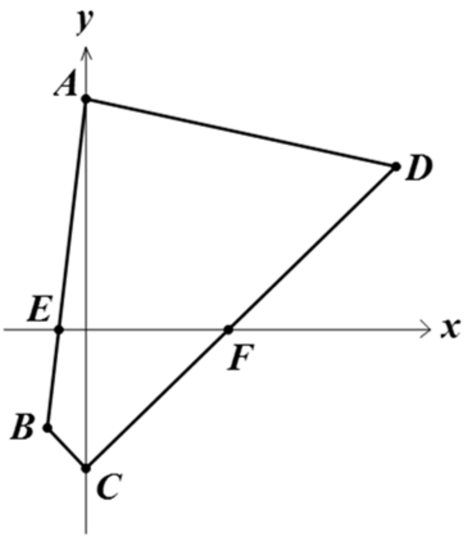
Question 18 [2011 EOY, Q14]	marks
Solutions to this question by accurate drawing will not be accepted. The diagram shows the quadrilateral *ABCD*. The coordinates of *A* and *B* are $(0,1)$ and $(3,7)$ respectively. Points *C* and *D* lie on the *x*-axis and $\angle BAD = 90°$.  (i) Find the equation of the perpendicular bisector of line *AB*. **Solution:** Midpoint of *AB*, $M(1.5,4)$ Gradient of *AB* $=\dfrac{7-1}{3-0}=2$ Equation of perpendicular of *AB* is: $y-4=-0.5(x-1.5)$ $y=-0.5x+4.75$	[3]

(ii) Find the equation of *AD* and hence find the coordinates of *D*. **Solution:** Equation of *AD* is $y=-0.5x+1$ At *D*, $y=0$, $\therefore x=2$ $D(2,0)$	[2]
(iii) If the area of ΔBCD is 14 square units, find the coordinates of *C* **Solution:** $\frac{1}{2}\times CD\times 7=14$ $CD=4$ $\therefore C(2+4,0),\quad i.e.\ C(6,0)$	[1]
(iv) Find all the possible coordinates of *E* for which *A*, *B*, *C* and *E* are vertices of a parallelogram. **Solution:** $E(6-3,0-6),\quad i.e.\ E(3,-6)$ or $E(6+3,0+6),\quad i.e.\ E(9,6)$ $or\quad E(3-6,8-0),\quad i.e.\ E(-3,8)$	[3]
Question 19 [2011 EOY, Q15]	marks
In the diagram *A*, *B* and *S* represent three points in the sea. *B* is 3 km from *A* on a bearing of 060°. The bearings of *S* from *A* and *B* are 110° and 155° respectively.  (i) Calculate the distance *AS*. **Solution:** $\angle BAS=110^{0}-60^{0}=50^{0}$ $\angle ABS=60^{0}+(180^{0}-155^{0})=85^{0}$ $\angle ASB=180^{0}-85^{0}-50^{0}=45^{0}$ $\frac{AS}{\sin 85^{0}}=\frac{3}{\sin 45^{0}}$ $AS=\frac{3siin85^{0}}{\sin 45^{0}}=4.226\approx 4.23km$	[3]

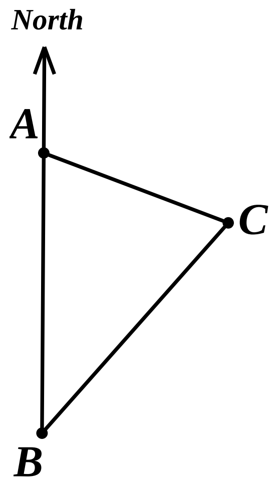
(ii) A ship at *S* sails directly to *A* at a steady speed of 2 kmh^{-1}. After 40 minutes, the ship is at *T,* a point on *AS.* (a) Calculate the distance *BT*. **<u>Solution:</u>** Distance = Speed x Time $=2\times\frac{40}{60}=\frac{4}{3}$ km AT = 4.226 – 1.333 km = 2.893 km….M1 Using cosine rule, $BT^2=3^2+2.893^2-2(3)(2.893)\cos 50^0$ $BT=2.492$ $BT=2.49km$	[3]
(b) Find the bearing of *B* from *T*. **<u>Solution:</u>** $\frac{\sin\angle ABT}{2.892}=\frac{\sin 50^0}{2.492}$ $\angle ABT=\frac{2.892\sin 50^0}{2.492}=62.74^0$ $\angle y=62.74^0-60^0=2.74^0$ *The bearing of B from T is* $360^0-2.74^0=357.26^0\approx 357.3^0$	[3]
(iii) A hot air balloon, *H*, is lowering at a point 600 m vertically above ***B***. Calculate the greatest angle of elevation of ***H*** from an observer on the ship as the ship sails from ***S*** to ***A***. **<u>Solution:</u>** Shortest distance from observer to *B* = $3\sin 50^0$ $\theta=\tan^{-1}\left(\frac{0.6}{3\sin 50^0}\right)$ $\approx 14.6^0$	[2]

Question 20 [2012 EOY, Q5]	marks
A set of points with coordinates (x, y) lies on the graph whose equation is $\frac{x^2+2x}{4}+\left(\frac{y}{2}-1\right)^2=2$. The points are *d* units from a fixed point *P*. Find the coordinates of *P* and the value of *d*. **<u>Solution:</u>** $\frac{x^2+2x}{4}=2-\left(\frac{y}{2}-1\right)^2$ $(x+1)^2+(y-2)^2=9$ $\therefore P(-1,2)$ and $d=3$	[4]

Question 21 [2012 EOY, Q10 c]	marks
(c) Find the coordinates of the points of intersection of the line $10x+5y+4=0$ and the curve $4x^2+y^2=16$. **<u>Solution:</u>** $4x^2+y^2=16 \rightarrow (1)$ $10x+5y=-4 \rightarrow (2)$ From (2): $y=\frac{-4-10x}{5} \rightarrow (3)$ Sub (3) into (1): $4x^2+\left(\frac{-4-10x}{5}\right)^2=16$ $100x^2+(-4-10x)^2=400$ $100x^2+16+80x+100x^2=400$ $200x^2+80x-384=0$ $25x^2+10x-48=0$ $(5x-6)(5x+8)=0$ $x=\frac{6}{5} or -\frac{8}{5}$ $y=-\frac{16}{5} or \frac{12}{5}$ $\therefore$ the points of intersection are $\left(\frac{6}{5},-\frac{16}{5}\right)and\left(-\frac{8}{5},\frac{12}{5}\right)$	[5]

Question 22 [2012 EOY Q11]	marks
Solutions to this question by accurate drawing will not be accepted. The diagram shows a quadrilateral *ABCD* with $A(0,6)$, $B(-1,-3)$, $C(0,-4)$, and $D(9,\ 5)$. *AB* and *CD* intersect the *x*-axis at points $E\left(-\frac{2}{3},0\right)$ and $F\ (4,0)$ respectively.  (i) Show that angle $BAD = 90^\circ$. **Solution:** gradient of $AB = \frac{6-(-3)}{0-(-1)} = 9$ gradient of $AD = \frac{6-5}{0-9} = -\frac{1}{9}$ gradient of $AB \times$ gradient of $AD = 9 \times \left(-\frac{1}{9}\right) = -1$ $\therefore AB \perp AD$	[2]

(ii) Find the equation of the perpendicular bisector of *AD*. **<u>Solution:</u>** midpoint of $AD=\left(\frac{0+9}{2},\frac{6+5}{2}\right)=(4.5,5.5)$ the equation of the perpendicular bisector of line AD: $\frac{y-5.5}{x-4.5}=9 \qquad y-5.5=9x-40.5$ $y=9x-35$	[3]
(iii)Given that *ABCD* is a cyclic quadrilateral, find the exact length of the diameter of the circle passing through *A*, *B*, *C* and *D*. **<u>Solution:</u>** $\because \angle BAD=90^\circ$, BD is the diameter of the circle $(\angle \text{in semi-circle})$ $BD \quad =\sqrt{(-1-9)^2+(-3-5)^2}$ $\quad =\sqrt{164}=2\sqrt{41}$	[2]
(iv)A point *P* is such that *BCDP* is a rectangle, find the coordinates of *P*. **<u>Solution:</u>** Mid-point of $CP=$ Mid-point of BD $\left(\frac{0+x}{2},\frac{-4+y}{2}\right)=\left(\frac{-1+9}{2},\frac{-3+5}{2}\right)$ $(x,y)=(8,6)$	[2]
(v)Find the area of the quadrilateral *AEFD*. **<u>Solution:</u>** Area of $AEFD$ $=\frac{1}{2}\begin{vmatrix} 0 & -\frac{2}{3} & 4 & 9 & 0 \\ 6 & 0 & 0 & 5 & 6 \end{vmatrix}$ $=\frac{1}{2}(20+54+4)=39$	[2]

Question 23 [2012 EOY Q13]	marks
ABC represents a horizontal triangular field such that *A* is due north of *B*, *AB* = 240 m, *AC* = 180m and the bearing of *C* from *A* is 110°.  (i) Find the distance *BC*. **Solution:** **Using Cosine Rule** $BC^2 = 240^2 + 180^2 - 2(240)(180)\cos 70^\circ$ $BC \approx 245.864$ $BC \approx 246\ m (3s.f.)$	[3]
(ii) Find the bearing of *C* from *B*. **Solution:** **Using Sine Rule** $\frac{\sin ABC}{180} = \frac{\sin 70^\circ}{BC}$ $\quad$ $A\hat{B}C = 43.468...$ Bearing of C from B is 043.5°.	[3]

(iii)
A vertical tree, of height 25 m, is planted at *C*. A farmer walks along a footpath from *A* to *B*. Find the maximum angle of elevation of the top of the tree from the farmer during this walk. You may ignore the height of the farmer. [4]

Solution:
Maximum angle of elevation occurs when distance between tree and farmer is the ***shortest.***
Let this shortest distance be *d*.

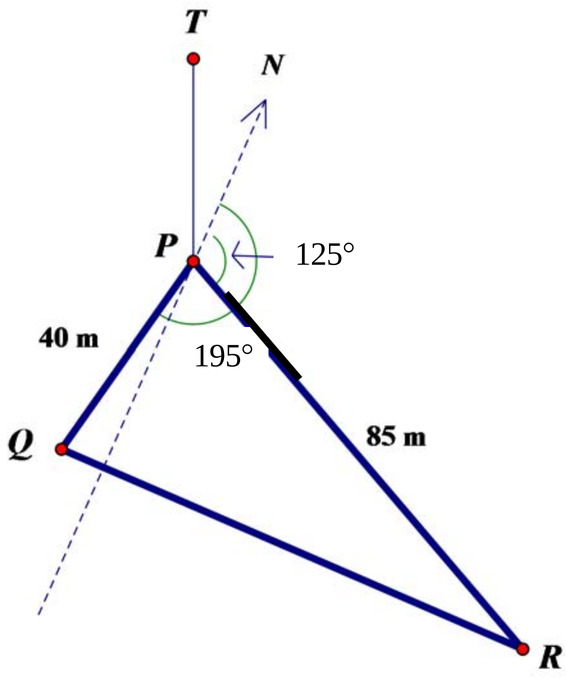
$$d = 180\sin 70^\circ$$

Let the maximum angle of elevation be θ

$$\tan\theta = \frac{25}{180\sin 70^\circ}$$
$$\theta = 8.4075...$$
$$\approx 8.4^\circ$$

Question 24 [2013 EOY, Q12]	marks

12 In the diagram, *PQR* represents a horizontal triangular field such that *PR* = 85 m and *PQ* = 40 m. The bearing of *R* from *P* and the bearing of *Q* from *P* is $125°$ and $195°$ respectively.



(i) Calculate the distance from *Q* to *R*.

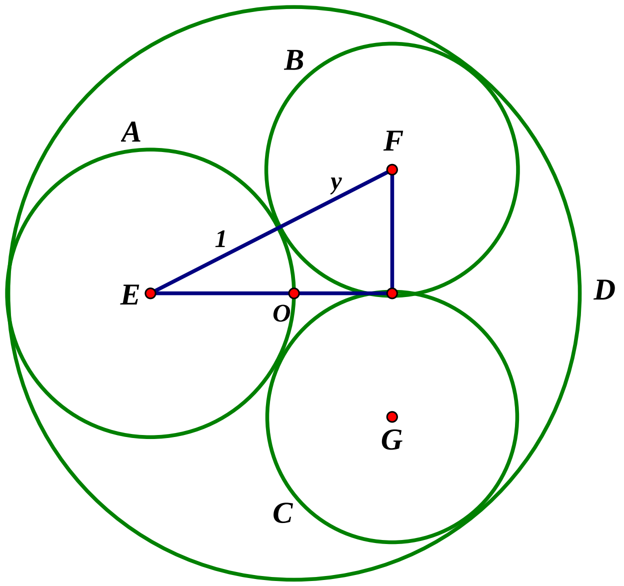
Solution: [2]
In ΔPQR, $\angle QPR = 195^\circ - 125^\circ = 70^\circ$

$$QR = \sqrt{40^2 + 85^2 - 2(40)(85)\cos 70^\circ}$$
$$= \sqrt{1600 + 7225 - 2325.736}$$
$$= \sqrt{6499.24} = 80.61 \simeq 80.6 \text{ m}$$

(ii) Calculate the area of the triangular field *PQR*. **<u>Solution:</u>** Area of the triangular field $\Delta PQR = \frac{1}{2} \times 40 \times 85 \times \sin 70°$ $= 1597.47 \simeq 1600 \text{ m}^2$	[2]
(iii) Calculate the shortest distance from P to the road QR. **<u>Solution:</u>** In ΔPQR, $\frac{40}{\sin \angle PRQ} = \frac{80.61}{\sin 70°}$ $\Rightarrow \quad \angle PRQ = 27.79°$ Hence the shortest distance from P to QR is $d = 85 \sin 27.79° = 39.629 \simeq 39.6 \text{ m}$	[2]
TP represents a building at the corner *P*. Given that the angle of depression of the top of the building *T* to the corner *R* is $20°$, (iv) calculate the height of the building *TP*. **<u>Solution:</u>** In ΔPTR, $\tan 20° = \frac{h}{85}$ where h is the height of the building Hence $h = 85 \tan 20° = 30.93 \simeq 30.9 \text{ m}$	[2]
A man walks along the road from *Q* to *R*, (v) calculate the greatest angle of elevation of the top of the building. **<u>Solution:</u>** The biggest angle of elevation α is $\alpha = \tan^{-1}(\frac{30.93}{39.62}) = 37.97° \simeq 38.0°$	[2]

Question 25 [2013 EOY, Q13]	marks
(i) The equation of a circle is given by $x^2 + y^2 - 10x + 2y + 1 = 0$. Find the centre and the radius of the circle. **<u>Solution:</u>** Centre is (5, -1) and the radius is $r = \sqrt{5^2 + (-1)^2 - 1} = 5$ units	[2]
(ii) The tangent to the circle at a point *A* is parallel to the straight line $x - y + 2 = 0$. Find the equation of the diameter of the circle through *A*. **<u>Solution:</u>** The gradient of the straight line $x - y + 2 = 0$ is *m* = 1 so the gradient of the diameter through *A* is -1. The equation of the diameter passes through *A* is $y - 5 = -1(x+1) \quad \Rightarrow \quad x + y = 4$	[2]
(iii) The circle is then reflected on the line $y = x$. State the equation of the new circle formed. **<u>Solution:</u>** The coordinates of the centre of the circle after the reflection on the line $y = x$ are (-1, 5) Hence the equation of the new circle formed is $(x+1)^2 + (y-5) = 5^2$ $\Rightarrow x^2 + y^2 + 2x - 10y + 1 = 0$	[1]

Question 26 [2013 EOY, Q14]	marks
Circles A, B and C are externally tangent to each other and internally tangent to circle D. Circles B and C are congruent. E, F and G are the centres of circles A, B and C respectively. Circle A has radius 1 and passes through O, the centre of circle D. Both circles B and C have radii y. By considering two right-angled triangles, find the radius y of the circle B.	[6]



Solution:

Let P be the contact point of circle B and circle C.

Let $OP = x$ and join OF, $OF = 2 - y$.

In right-angled triangle ΔEFP,

$$(1+y)^2 = y^2 + (1+x)^2$$

$$\Rightarrow y = x + \frac{x^2}{2} \quad \text{---- (1)}$$

In right-angled triangle ΔFOP,

$$(2-y)^2 = x^2 + y^2$$

$$\Rightarrow y = 1 - \frac{x^2}{4} \quad \text{---- (2)}$$

Solving eqns (1) and (2)

$$x + \frac{x^2}{2} = 1 - \frac{x^2}{4}$$

$$\Rightarrow 3x^2 + 4x - 4 = 0$$

$$(3x-2)(x+2) = 0$$

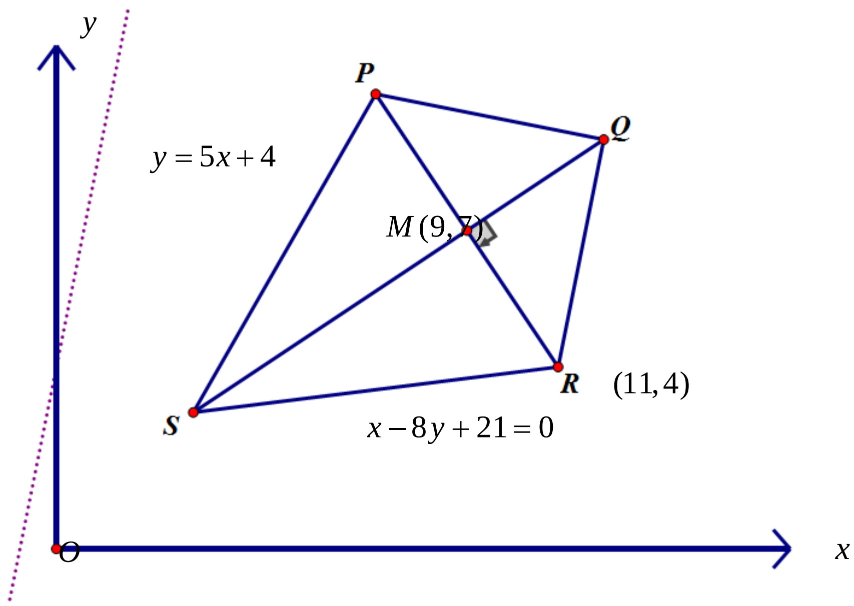
i.e. $x = \frac{2}{3}$ or $x = -2$ (rej)

hence $y = 1 - \frac{(\frac{2}{3})^2}{4} = 1 - \frac{1}{9} = \frac{8}{9}$ (the radius of circle B).

Question 27 [2013 EOY, Q16]	marks
Solutions to this question by accurate drawing will not be accepted	

The diagram shows a quadrilateral *PQRS* in which *QS* is the perpendicular bisector of *PR*. The mid-point of *PR* is *M*.

The coordinates of *R* and *M* are (11, 4) and (9, 7) respectively. *RQ* is parallel to the line $y = 5x + 4$ and the equation of *SR* is $x - 8y + 21 = 0$.



Find

(i) the coordinates of *P*, [2]

Solution:

let *P* = (*s*, *t*), since *M* is the mid-point *PR*

$$\left(\frac{s+11}{2}, \frac{t+4}{2}\right) = (9,7) \quad \Rightarrow \quad s = 7 \text{ and } t = 10$$

$P(7,10)$

(ii)

the equations of *QR* and *QS*, [4]

Solution:

Gradient of the given line is 5 and *RQ*//to the given line

So the equation of *QR* is $y - 4 = 5(x - 11)$ or $y = 5x - 51$

$$\text{Gradient of } RM = \frac{7-4}{9-11} = -\frac{3}{2}$$

$$\text{Gradient of } QS = \frac{2}{3}.$$

So the equation of *QS* is

$$y - 7 = \frac{2}{3}(x - 9) \quad \text{or} \quad 2x - 3y + 3 = 0$$

(iii)

the coordinates of Q and of S, [3]

Solution:

Coordinates of Q : Solve Equation QR : $y = 5x - 51$ and Equation QS : $2x-3y+3=0$

$\Rightarrow \quad x=12$ and $y=9 \quad \therefore \ Q(12,9)$

Coordinates of S : Solve Equation RS : $x-8y+21=0$ and Equation QS :

$2x-3y+3=0$

or $y=\frac{2}{3}x+1$

$\Rightarrow \quad x=3$ and $y=3 \quad \therefore \ S(3,3)$

(iv)

the area of $PQRS$. [2]

Solution:

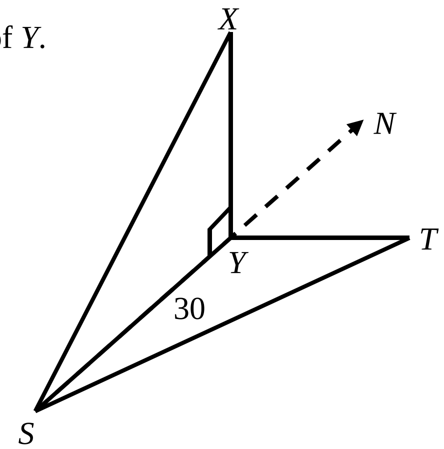
Area of $PQRS = \frac{1}{2}\times\begin{vmatrix} 3 & 11 & 12 & 7 & 3 \\ 3 & 4 & 9 & 10 & 3 \end{vmatrix}$

$=\frac{1}{2}\times(12+99+120+21-33-48-63-30)$

= 39 square unit

Question 28 [2014 EOY, Q14]	marks

1 Three points, *S*, *Y* and *T* are on level ground. *Y* is the foot of a vertical flagpole *XY*, and *S* is 30 m due south of *Y*.



(i)

The angle of elevation of *X* from *S* is $25°$. Calculate the height of the flagpole. [2]

Solution:

Height of the flagpole *XY*

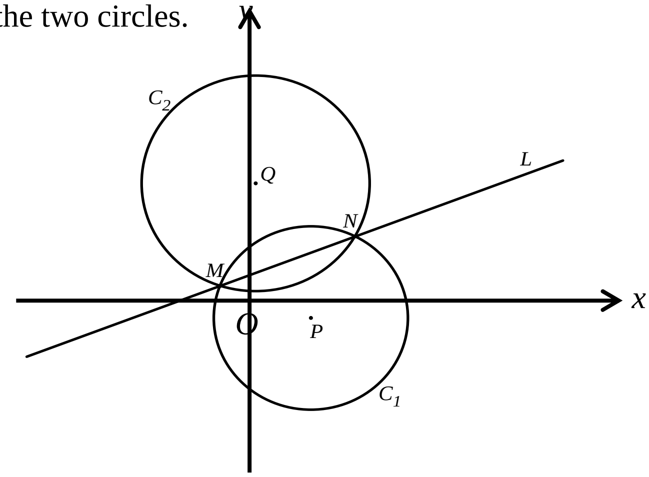
$=30\tan 25°$ [M1]

$\approx 13.989 = 14.0$ m (3 s.f.)

(ii) The bearing of T from S and the bearing of T from Y are $042°$ and $075°$ respectively, calculate the distance ST. **<u>Solution:</u>** The bearing of T from S and the bearing of T from Y are $042°$ and $075°$ respectively. $\Rightarrow \angle YST = 42°, \angle SYT = 180° - 75° = 105°$ $\therefore \angle YTS = 180° - 42° - 105° = 33°$ Alternatively, by stating that $\angle NYT = 75°$, $\angle YTS = 75° - 42° = 33°$ $\frac{ST}{\sin 105°} = \frac{SY}{\sin 33°}$ $\therefore ST = \frac{30}{\sin 33°} \times \sin 105°$ $\approx 53.205 = 53.2$ m (3 s.f.)	[3]
(iii) A man is standing at the midpoint of ST. Calculate the angle of elevation of X from the man. **<u>Solution:</u>** Let the midpoint of S and T be M. $SM = \frac{1}{2} \times ST \approx 26.6025$ Using Cosine Rule, $MY^2 = SY^2 + SM^2 - 2 \times SY \times SM \times \cos \angle YSM$ $\therefore MY = \sqrt{30^2 + 26.6025^2 - 2 \times 30 \times 26.6025 \times \cos 42°}$ ≈ 20.531 Angle of elevation of X from the man $= \tan^{-1} \frac{XY}{MY} = \tan^{-1} \frac{13.989}{20.531}$ $= 34.3°$	[5]

Question 29 [2014 EOY, Q16]	marks

The diagram shows two intersecting circles C_1 and C_2 with centres P and Q respectively. A straight line L passes through M and N, which are the points of intersections of the two circles.



(i)

Given that the equation of the circle C_1 is $x^2 + y^2 - 6x + 4y - 21 = 0$, find the coordinates of P and the radius of C_1. [3]

<u>Solution:</u>

$x^2 + y^2 - 6x + 4y - 21 = 0$

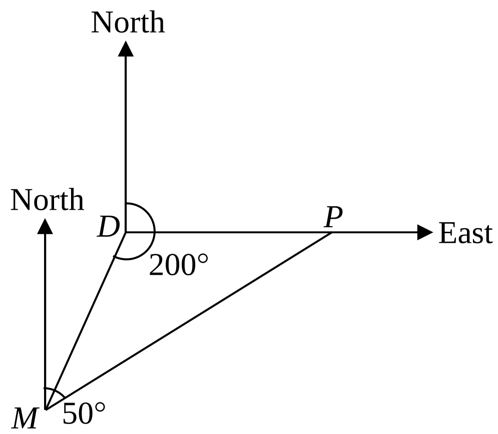
$(x-3)^2 + (y+2)^2 = 34$

Hence, the coordinates of *P* are $(3,-2)$, and the radius of C_1 is $\sqrt{34}$ units.

(ii) It is also given that the equation of the line L is $4y - x = 6$, find the coordinates of M and of N. **Solution:** $4y - x = 6 \quad ...(1)$ $x^2 + y^2 - 6x + 4y - 21 = 0 \quad ...(2)$ From (1), $x = 4y - 6 \quad ...(3)$ Sub (3) into (2): $(4y-6)^2 + y^2 - 6(4y-6) + 4y - 21 = 0$ $17y^2 - 68y + 51 = 0$ $y^2 - 4y + 3 = 0$ $(y-1)(y-3) = 0$ $y = 1$ or 3 $x = -2$ or 6 $\therefore M = (-2,1),\ N = (6,3)$	[4]
(iii) Find the equation of the perpendicular bisector of MN. **Solution:** Equation of $MN: 4y - x = 6$ $\Rightarrow y = \frac{1}{4}x + \frac{3}{2}$ $\therefore$ Gradient of the perpendicular bisector $= \frac{-1}{1/4} = -4$ Midpoint of $MN = \left(\frac{6-2}{2}, \frac{3+1}{2}\right) = (2, 2)$ $\therefore$ Equation of the perpendicular bisector: $y - 2 = -4(x - 2)$ $y = -4x + 10$	[3]

(iv) Given that the radius of circle C_2 is $\frac{1}{2}\sqrt{221}$ units, find the coordinates of Q. **<u>Solution:</u>** Since Q is the centre of C_2, it lies on the perpendicular bisector of chord MN. Hence, the coordinates of Q satisfy the equation $y=-4x+10 \quad ...(1)$ Also, since the radius of $\frac{1}{2}\sqrt{221}$ units, $QM=\frac{1}{2}\sqrt{221}$. $\sqrt{(x+2)^2+(y-1)^2}=\frac{1}{2}\sqrt{221}$ $(x+2)^2+(y-1)^2=\frac{221}{4} \quad ...(2)$ Substitute (1) into (2): $(x+2)^2+(-4x+9)^2=\frac{221}{4}$ $17x^2-68x+\frac{119}{4}=0$ $4x^2-16x+7=0$ $(2x-1)(2x-7)=0$ $x=\frac{1}{2}$ or $\frac{7}{2}$ $y=8$ or -4 Since Q lies above the x-axis, only the pair of solution $\left(\frac{1}{2},8\right)$ is accepted. Hence, coordinates of Q are $\left(\frac{1}{2},8\right)$.	[4]
(v) Calculate the area of the quadrilateral $PMQN$. **<u>Solution:</u>** Area of $PMQN$ $=\frac{1}{2}\begin{vmatrix} 3 & 6 & \frac{1}{2} & -2 & 3 \\ -2 & 3 & 8 & 1 & -2 \end{vmatrix}$ $=42\frac{1}{2}$ units^2	[2]

Question 30 [2015 EOY, Q10]	marks
Two lines l_1 and l_2 are such that they are parallel, and l_2 is the perpendicular bisector of two points $A(-2,4)$ and $B(3,-6)$. (i) Find the equation of l_2. **<u>Solution:</u>** Midpoint of AB $= M\left(\frac{1}{2},-1\right)$ and gradient of AB $=-2$ Hence equation of l_2: $y-(-1)=\frac{1}{2}\left(x-\frac{1}{2}\right)\Rightarrow 4y+5=2x$ or $y=\frac{1}{2}x-\frac{5}{4}$	[3]
(ii) Hence find the equation of l_1 if it passes through the point $C(5,8)$. **<u>Solution:</u>** Equation of l_1: $y-8=\frac{1}{2}(x-5)\Rightarrow 2y=x+11$ or $y=\frac{1}{2}x+\frac{11}{2}$	[2]
(iii) Find the distance between the two lines. **<u>Solution:</u>** Equation of perpendicular to l_1 at $\left(0,\ \frac{11}{2}\right)$: $y=-2x+\frac{11}{2}$ Intersecting with l_2: $-2x+\frac{11}{2}=\frac{1}{2}x-\frac{5}{4}$ $\Rightarrow x=2.7$ and $y=0.1$ Hence distance between the two lines $=\sqrt{2.7^2+(5.5-0.1)^2}$ units	[4]

$= 2.7\sqrt{5}$ or 6.04 (3 sf.) units **<u>OR</u>** Equation of *AB*: $y-4=-2\left(x-(-2)\right) \Rightarrow y=-2x$ Intersecting with l_1 : $\frac{1}{2}x+\frac{11}{2}=-2x$ $\Rightarrow x=-2.2$ and $y=4.4$ Hence distance between the two lines $=\sqrt{(0.5+2.2)^2+(4.4+1)^2}$ units $= 2.7\sqrt{5}$ or 6.04 (3 sf.) units	
Question 31 [2015 EOY, Q11]	marks
A man, at point *M*, is 80 metres at a bearing of $200°$ from an observation deck, *D*, which stands 25 metres tall. He begins to walk to a point *P* which is due East of *D* at a bearing of $050°$.  Ignoring the man's height, calculate (i) the angle of elevation of the top of the deck from the man at *M*, **<u>Solution:</u>** Angle of elevation $=\tan^{-1}\frac{25}{80}$ $=17.4°$ (1 d.p.)	 [2]

(ii) the distance he walks when the angle of elevation of the top of the deck from his position is the greatest, and **<u>Solution:</u>** $\measuredangle DMP$ $= 50° - (200° - 180°)$ $= 30°$ Distance he walks $= 80 \cos 30°$ m $= 40\sqrt{3}$ or 69.3 (3 s.f.) m	[3]
(iii) the distance *DP*. **<u>Solution:</u>** By Sine Rule, $\dfrac{DP}{\sin 30°} = \dfrac{80 \text{ m}}{\sin(180° - 30° - 110°)}$ $\Rightarrow DP = \dfrac{80}{2 \sin 40°}$ m $\Rightarrow DP = 62.2$ (3 s.f.) m	[2]