8 GRAVITATIONAL FIELDS

H2 Physics 9748

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Learning Outcomes

Candidates should be able to:

(a) recall and use Newton's law of gravitation in the form $F = \dfrac{Gm_1m_2}{r^2}$.

(b) derive, from Newton's law of gravitation and the definition of gravitational field strength, the field strength due to a point mass $g = G\dfrac{M}{r^2}$.

(c) recall and use $g = G\dfrac{M}{r^2}$ for the gravitational field strength due to a point mass to solve problems.

(d) show an understanding that near the surface of the Earth, gravitational field strength is approximately constant and is equal to the acceleration of free fall.

(e) define gravitational potential at a point as the work done per unit mass by an external force in bringing a small test mass from infinity to that point.

(f) solve problems using the equation $\phi = -G\dfrac{M}{r}$ for the gravitational potential in the field due to a point mass.

(g) show an understanding that the gravitational potential energy of a system of two point masses is $U_G = -G\dfrac{Mm}{r}$.

(h) recall that gravitational field strength at a point is equal to the negative potential gradient at that point and use this to solve problems.

(i) analyse problems related to escape velocity by considering energy stores and transfers.

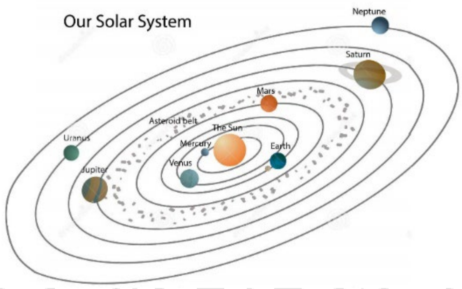
(j) analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes.

(k) show an understanding of satellites in geostationary orbit and their applications.

Introduction

In 1687, Isaac Newton proposed in his ***Philosophiæ Naturalis Principia Mathematica*** that every mass attracts another mass with a force of gravity. According to him, two seemingly unrelated phenomena – the fall of an apple from an apple tree and the orbital motion of the planets around the Sun – are due to the same reason: gravitational attraction. He came up with the Newton's Law of Gravitation. This law is universally valid and applies to any planet in the solar system and even between distant galaxies.

How Newton discovered Newton's Laws of Gravitation



8.1 Gravitational Force *F*

❖ Newton's Law of Gravitation

Newton's Law of Gravitation

It states that two point masses attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The magnitude of the gravitational force F between two particles of masses M and m which are separated by a distance r is given by

$$\boxed{F = \frac{GMm}{r^2}}$$

Formula

where G is the gravitational constant and G = $\mathbf{6.67 \times 10^{-11}\ N\ m^2\ kg^{-2}}$.

Gravitational force is a **vector** quantity.

S.I. unit for gravitational force is the **newton (N)**.

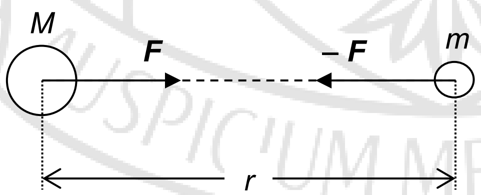


Fig. 8.1

Note:

- This law is applicable only between point masses.
  - Every spherical body with constant density can be considered as a point mass at the centre of the sphere.
- The gravitational forces between two point masses are equal and opposite. As shown in Fig. 8.1, they
  - constitute an action and reaction pair of forces.
  - are attractive in nature, and
  - always act along the line joining the two point masses.
- Newton's law of gravitation is an example of an inverse square law
  - because the magnitude of the force varies inversely with the square of the separation of the two point masses.
- Gravitational force is attractive in nature.
  - Some text/sources indicate this with a negative sign in the formula to highlight that the force is attractive.

Example 1

A man of mass 85.0 kg is standing on the surface of the Earth. The Earth has a mass of 5.98×10^{24} kg and a radius of 6.37×10^{6} m.

Calculate the force that the Earth exerts on the man and the force that the man exerts on Earth.

Solution

Example 2

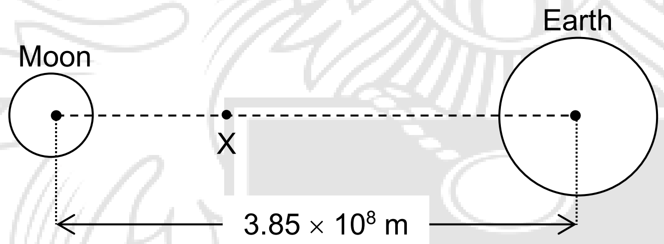
On the surface of the Earth, the gravitational force acting on an object is 45 N. When the object is at a height h above the surface, the gravitational force acting on it is 5 N.

Determine h in terms of R where R is the radius of the earth.

Solution

Example 3

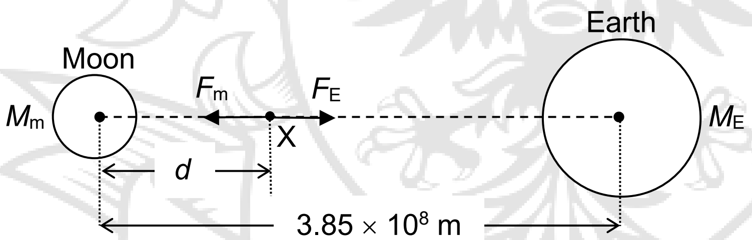
The mass of the Earth is 5.98×10^{24} kg and that of the Moon is 7.35×10^{22} kg. A spacecraft travelling from the Earth to the Moon will reach a point X where it experiences no resultant gravitational force. The distance between the centre of the Earth and the centre of the Moon is 3.85×10^{8} m.



(a) Draw a free-body diagram of the spacecraft at X.
(b) Calculate the distance from X to the centre of the Moon.
(c) Describe the motion of the spacecraft before reaching X and after passing X if its engine is switched off.

Solution

(a)



8.2 Gravitational Field Strength *g*

❖ Gravitational Field

Gravitational Field

A **gravitational field** is a region of space in which a mass placed in that region experiences a gravitational force.

❖ Gravitational Field Lines

A region of gravitational field can be visualised as consisting of an array of imaginary field lines. The gravitational force on a mass placed at a point in a field, acts along the tangent to the field line at that point.

- The direction of the field lines indicates the direction of the gravitational field.

- The density of the field lines indicates its strength.
  - A region with a stronger gravitational field strength will have closer or denser field lines.

- For a point mass or a uniform spherical mass, the field lines are directed towards its centre.

- Fig. 8.2(a) shows the field lines around Earth. Zooming into a region near the surface of Earth, the field lines seem to be parallel to each other and evenly spaced as shown in Fig. 8.2(b).

- Hence, near the surface of the Earth, we can consider the gravitational field to be approximately uniform.

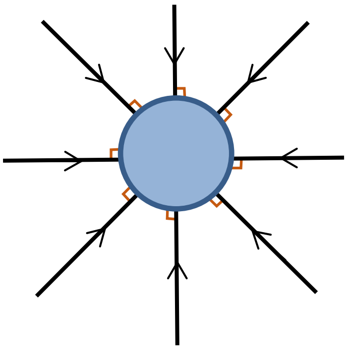


Fig. 8.2(a)
Field lines around the Earth

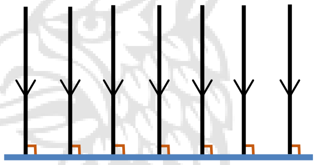


Fig. 8.2(b)
Field lines near the surface of the Earth

❖ Gravitational Field Strength, *g*

Gravitational Field Strength

The **gravitational field strength** at a point in space is defined as the gravitational force experienced per unit mass at that point.

$$\boxed{g = \frac{F}{m}}$$

Formula

❖ Gravitational Field Strength of a Point Mass

Since the gravitational force between two point masses is given by $F = \frac{GMm}{r^2}$,

$$\boxed{g = \frac{GM}{r^2}}$$

Formula

Gravitational field strength is a **vector** quantity.
Gravitational field strength is also known as **gravitational acceleration** or **free fall acceleration**.
S.I. unit for gravitational field strength is **N kg^{-1}** or **m s^{-2}**.

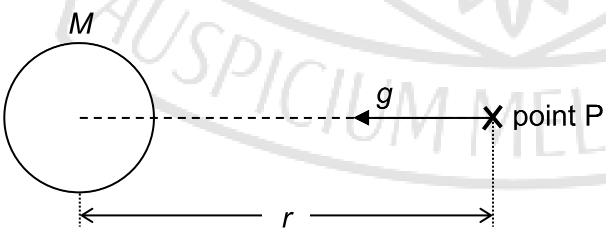


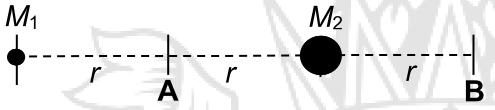
Fig. 8.3

Note:
Gravitational field strength points towards the mass which created it as shown in Fig. 8.3. Some books indicate this by including a negative sign, just like gravitational force.

The resultant field strength at a point due to more than one mass can be found from the **vector sum** of the individual gravitational field strengths due to each mass at that point.

Example 4

Determine the resultant gravitational field strength at points A and B due to the two masses, given that $M_2 > M_1$.



Solution

❖ Gravitational Field Strength of a Uniform Sphere

Variation of *g* with *r*

For a uniform solid sphere of radius R, the variation of the gravitational field strength g with displacement r from the centre of the sphere is shown in Fig. 8.4.

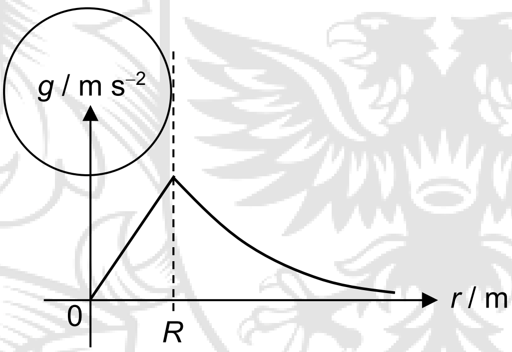


Fig. 8.4

Inside the sphere ($r \leq R$):

- The mass M of the inner sphere of radius r is $M = \rho V = \rho\left(\frac{4}{3}\pi r^3\right)$.
- The gravitational field strength due to the inner sphere varies linearly with distance r (from the shell theorem):

$$g = \frac{GM}{r^2} = \frac{G}{r^2}\left(\frac{4}{3}\rho\pi r^3\right) = \frac{4}{3}G\rho\pi r$$

Outside the sphere ($r \geq R$):

A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its centre.

Hence, for uniform spheres, planets and stars, we can use $F = \dfrac{GMm}{r^2}$ and $g = \dfrac{GM}{r^2}$ to calculate the gravitational forces and gravitational field strengths <u>outside</u> of them.

In summary:

- The gravitational field strength inside a <u>hollow</u> sphere is zero.
- The gravitational field strength inside a <u>uniform solid</u> sphere varies linearly with distance r from its centre.
- The gravitational field strength outside a <u>uniform solid</u> sphere or <u>hollow</u> sphere varies inversely with r^2.
- The direction of gravitation field strength acts towards the centre of mass M.

[For further reading, refer to the Shell Theorem in **Appendix A**]

8.3 Gravitational Potential Energy U and Gravitational Potential ϕ

❖ Gravitational Potential Energy U

Gravitational Potential Energy

The **gravitational potential energy** of a mass at a point in a gravitational field is defined as the work done by an external force in bringing the mass from infinity to that point.

Consider a particle of mass M placed at a fixed point. The work required to bring this particle from infinity to that point is <u>zero</u> since there is no other masses exerting any gravitational force on it.

Now a second particle of mass m is brought from infinity and placed at a distance r from the first particle of mass M.

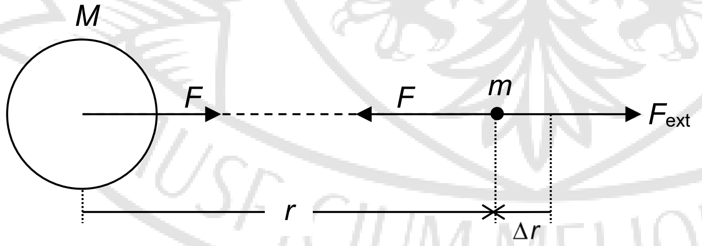


Fig. 8.5

At infinity, the gravitational force due to mass M is zero. The **point at infinity** (i.e. $r = \infty$) is **defined to have zero gravitational potential energy**.

The work that must be done by an external force F_{ext} to move m from infinitiy to a distance r from the centre of M,

$$W = \int_{\infty}^{r} F_{ext} dr = \int_{\infty}^{r} \frac{GMm}{r^2} dr = \left[-\frac{GMm}{r} \right]_{\infty}^{r} = -GMm\left[\frac{1}{r} - \frac{1}{\infty} \right] = -\frac{GMm}{r}$$

(*the above mathematical derivation is not required in the A level syllabus*)

RECALL: initial energy + work done by an external force = final energy

Since the initial gravitational potential energy of m at infinity is zero, the gravitational potential energy U of m at r must be given by $-\frac{GMm}{r}$.
(Deduced from 0 + W = U)

Hence, the gravitational potential energy U of a system of two particles of masses M and m separated by a distance r is given by

$$\boxed{U = -\frac{GMm}{r}}$$

Formula

Gravitational potential energy is a **scalar** quantity.
S.I. unit for gravitational potential energy is the **joule (J)**.

Note:
- Due to its definition, the **negative sign must be indicated**.
- For small distances h above the surface of the Earth, $U \approx mgh$.
  [For the derivation, refer to **Appendix B**]

Relationship between Force and Potential Energy

We have learnt that for a field of force, the relationship between the force F and the potential energy U for one-dimensional motion is given by $F = -\frac{dU}{dx}$.

Hence, the gravitational force F acting on a mass m in the presence of another mass M is thus the **negative of the gravitational potential energy gradient**.

F and gravitational potential energy U is related by the following expression:

$$\boxed{F = -\frac{dU}{dr}}$$

Formula

- The magnitude of the force at the point distance r from M is equal to the numerical value of the gradient of the potential energy curve at r.

- The negative sign indicates that the force points in the direction of decreasing potential energy.

Note: $-\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = GMm\frac{d}{dr}\left(\frac{1}{r}\right) = GMm\left(-\frac{1}{r^2}\right) = -\frac{GMm}{r^2} = F$

❖ Gravitational Potential ϕ

Gravitational Potential

The **gravitational potential** at a point in a gravitational field is defined as the work done per unit mass by an external force in bringing a small test mass from infinity to that point.

$$\boxed{\phi = \frac{U}{m}}$$

Formula

At a distance r from a point mass M, $U = -\frac{GMm}{r}$, therefore

$$\boxed{\phi = -\frac{GM}{r}}$$

Formula

S.I. unit for gravitational potential is **J kg^{-1}**.

Fig. 8.6 shows the variation of the gravitational potential with distance from a spherical mass.

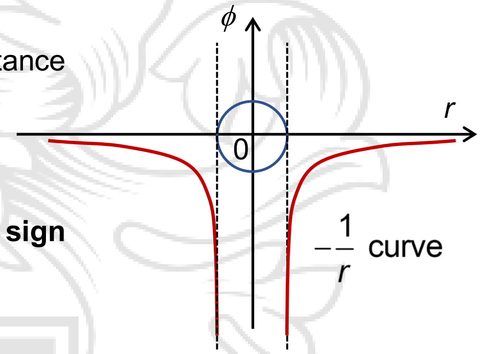


Fig. 8.6

Note:

- Gravitational potential is a **scalar** quantity. Hence, the **negative sign** is not an indication of direction and **cannot be omitted**.

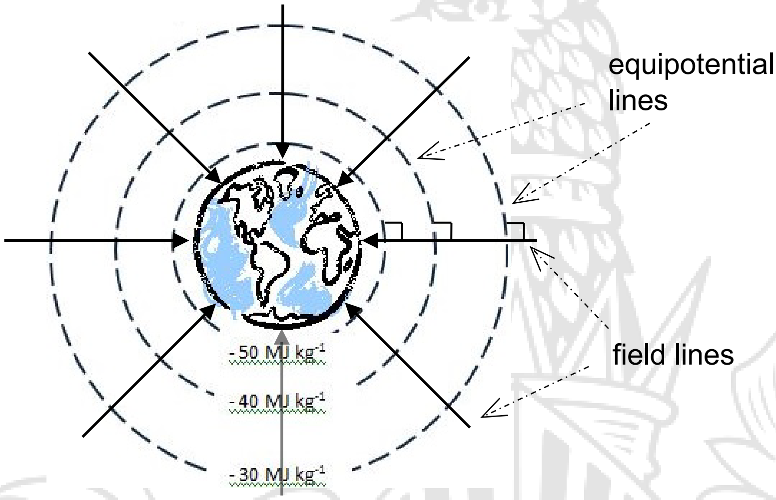
- **At infinity, the gravitational potential is defined to be zero**.
  $\Rightarrow$ This means that the potential has the highest value at infinity and potential is negative at other locations.
  $\Rightarrow$ The potential decreases from infinity towards mass M (i.e. potential becomes more negative).

- The total gravitational potential at a point due to two or more masses can be found by adding the individual potentials at that point due to each mass.

Equipotential Lines

- Points at a fixed distance from a mass form an equipotential surface.

- Equipotential surfaces of equal interval are further apart as we move away from a spherical mass as shown in Fig. 8.7.



In the diagram, equipotential lines are used to represent the equipotential surfaces.

Fig. 8.7

Why is gravitational potential (or potential energy) negative in value?

NOTE!!

- Gravitational potential (or potential energy) at infinity is zero.

- Since gravitational force is attractive in nature, to bring a mass from infinity to a point in the gravitational field, the direction of the external force is opposite to the direction of displacement of the mass. (We can imagine the external force supporting the mass as it is being lowered towards the Earth.)

- This results in negative work done per unit mass (or work done) by the external force.

Hence, based on its definition, gravitational potential (or potential energy) has a negative value.

Relationship between Field Strength and Potential

Gravitational field strength g is the **negative gravitational potential gradient**.

g and gravitational potential ϕ is related by the following expression:

Formula

$$\boxed{g = -\frac{d\phi}{dr}}$$

The negative sign indicates that the field strength points in the direction of decreasing potential.

Note: $F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) \Rightarrow \frac{F}{m} = -\frac{d}{dr}\left(-\frac{GM}{r}\right) \Rightarrow g = -\frac{d\phi}{dr}$

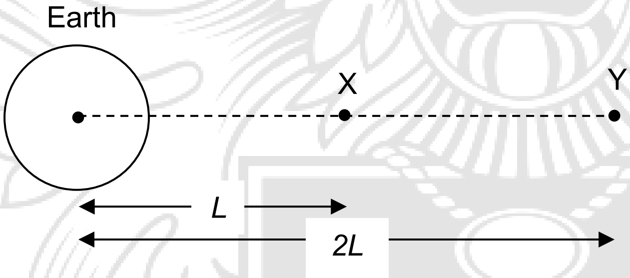
Example 5

The mass of the Earth is 5.98×10^{24} kg and its radius is 6.37×10^{6} m. Calculate the change in the gravitational potential as an object is moved from the surface of the Earth to a point 1200 m above the surface.

Solution

Example 6

The figure shows two points X and Y at distances L and $2L$ from the centre of the Earth. The gravitational potential at X is -8.0 kJ kg^{-1}.



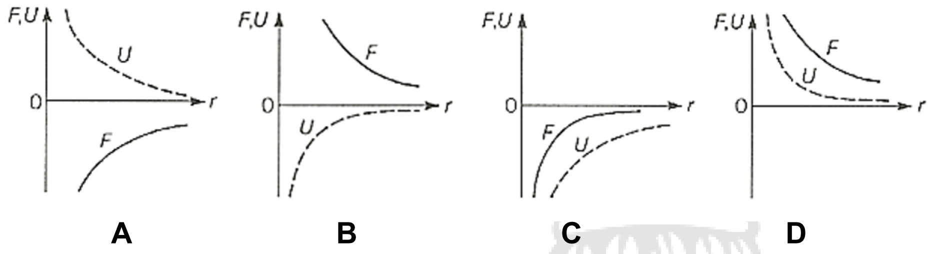
(a) Determine the work done against gravity when a 2.0 kg mass is taken from X to Y.

(b) Explain whether the work done in **(a)** depends on the actual path taken from X to Y.

Solution

Example 7 [N95/I/7]

Which diagram shows the variation of gravitational force F on a point mass, and of gravitational potential energy U of the mass, with its distance r from another point mass?



Solution

8.4 Graphs of $\phi - r$ and $g - r$ between Two Masses

Potential and Field strength between Two Masses

Gravitational field and potential between 2 point masses simulation

The potential ϕ at a point between two masses is the algebraic sum (i.e. scalar addition) of the individual potentials at that point due to each mass.

The field strength g at a point between two masses is the vector sum (i.e. vector addition) of the individual field strengths at that point due to each mass.

g and ϕ is related by: $g = -\dfrac{d\phi}{dr}$. The negative sign indicates that the field strength points in the direction of decreasing potential, as g is a vector.

The variation with displacement r, measured from the centre of A, of the total gravitational potential ϕ and the corresponding resultant gravitational field strength g between the surfaces of spherical masses A and B along the line joining their centres is shown in Fig. 8.8.

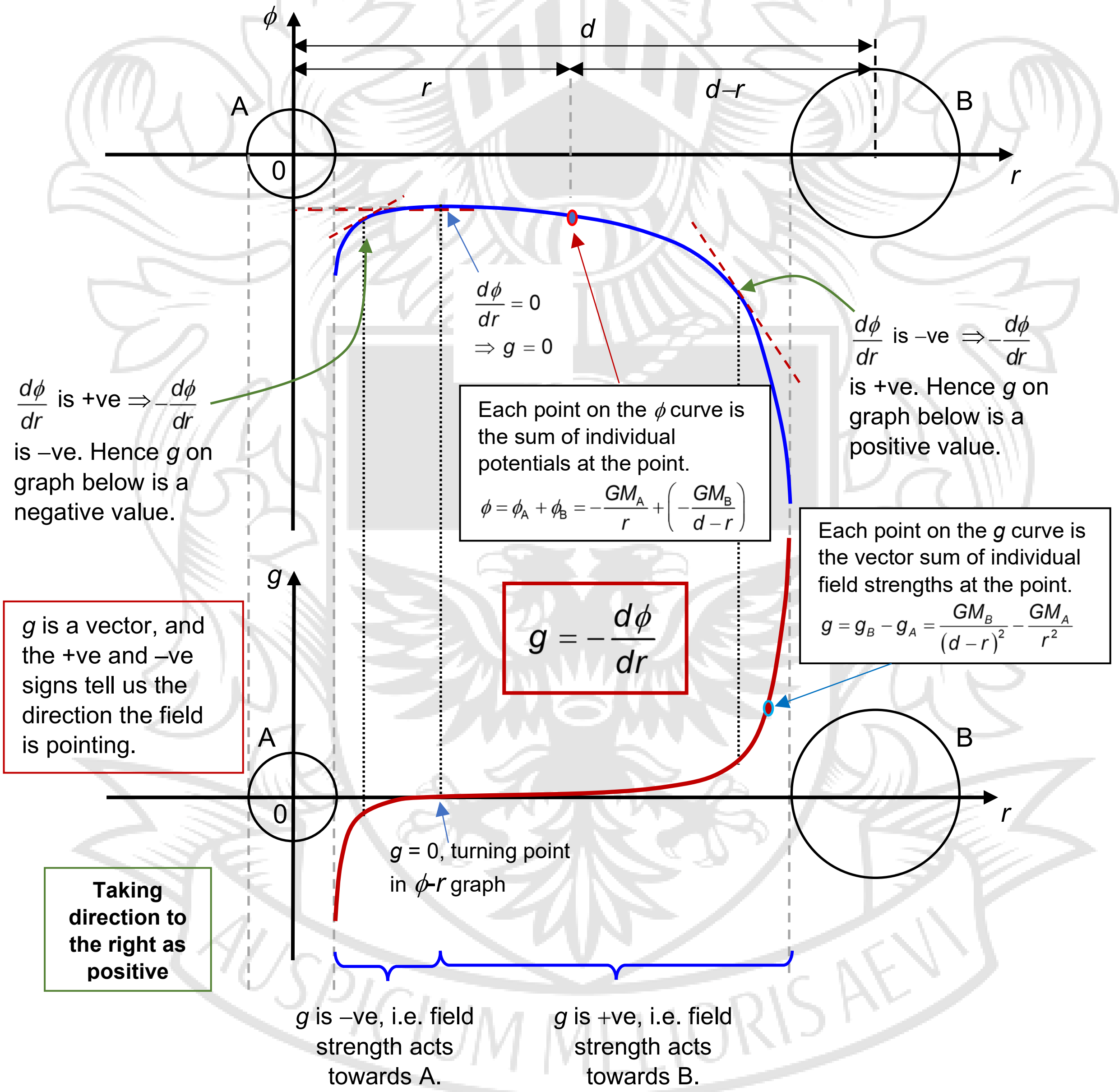


Fig. 8.8

8.5 Escape Velocity

Have you ever wondered if it is possible to throw a body into the air at such a high velocity that it escapes the Earth's gravitational pull and never falls back to Earth? This is plausible because the acceleration of free fall does not remain at 9.81 m s^{-2} but decreases as the body moves away from the Earth.

As a body moves away from Earth, there is energy transfer from kinetic energy to gravitational potential energy. For the body to escape from the Earth's gravitational influence and reach infinity, it has to be projected from the surface with a sufficiently high speed such that the decrease in the body's kinetic energy is sufficient for the increase in gravitational potential energy as the body moves from the Earth's surface to infinity.

NOTE!!

The **minimum speed** needed for the body to ***just*** escape from the gravitational influence of a massive body to infinity is called **escape velocity or escape speed**.

To determine the escape velocity of a body of mass *m*, let us consider the amount of energy the body needs to escape from the massive body's gravitational influence and ***just*** reach infinity.

At infinity, the body's G.P.E. (E_P) is defined as zero. If the body has sufficient initial kinetic energy to ***just*** reach infinity, its K.E. (E_K) decreases to zero at infinity. So the total energy store (E_T) of the body at infinity is $E_T = E_P + E_K = 0$.

From the Principle of conservation of energy, the total energy store of a body should remain constant throughout its motion. This means that **any body with a total energy store of zero will be able to *just* reach infinity and *stop* there**.

Any body with a total energy store of more than zero will be able to go beyond infinity, and completely escape the massive body's gravitational field.

Consider a body of mass *m* being projected vertically with a velocity *v* from the surface of a massive body of mass *M* and radius *R*.

The initial G.P.E. and K.E. of *m* at the surface of *M* are

$$E_P = -\frac{GMm}{R}$$

$$E_K = \frac{1}{2}mv^2$$

Total energy of *m*, $E_T = E_P + E_K = -\frac{GMm}{R} + \frac{1}{2}mv^2$

The minimum energy required for *m* to ***just*** reach infinity is when

$$E_{T,\min} = 0$$

$$-\frac{GMm}{R} + \frac{1}{2}m{v_{\min}}^2 = 0$$

Therefore the **escape velocity** or the minimum speed required for the body to *just* reach infinity is

$$v_{min} = \sqrt{\frac{2GM}{R}}$$

Note:

- The escape velocity is independent of the mass of the body.
- The escape velocity for Earth is 11.2 km s^{-1}.

Example 8 [RI/H2/Prelims2010/II/3 modified]

The mass of the Earth is 5.98×10^{24} kg and its radius is 6370 km. Determine the minimum kinetic energy and escape speed required to project a spacecraft of mass 2550 kg from the surface of the Earth so that it completely escapes from the gravitational attraction of the Earth. Ignore air resistance.

Solution

8.6 Rotation of Earth

❖ Circular Motion (recap)

For any object of mass m to move in uniform circular motion of radius r, there must be a resultant force F_c acting on the object which is directed towards the centre of the circle. This is known as the centripetal force, which can be expressed as

$$F_c = ma_c = \frac{mv^2}{r} = mr\omega^2$$

where v is the linear velocity and ω is the angular velocity.

The time taken for the object to make one complete circle is known as the period T, which is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$

❖ Apparent Weight and Free Fall Acceleration

Fig. 8.9 shows a body of mass m hanging from a spring balance at two places on the Earth – the Equator and the North Pole.

In both cases there are two forces acting on the body – gravitational force F_G (true weight) and force T due to the spring.

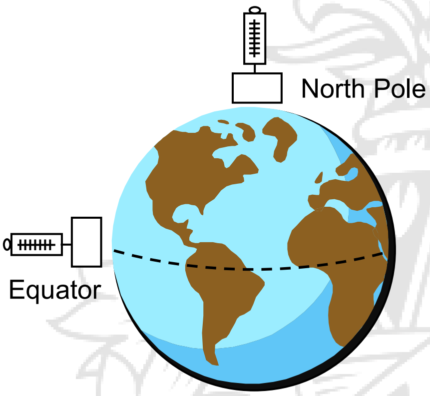


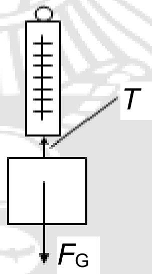
Fig. 8.9

If the Earth is taken to be a uniform sphere, then F_G is the same at both the equatorial and polar regions. However, T will be different because the body undergoes circular motion at the Equator.

<u>**Resultant force F_R at the polar region**</u>

By Newton's second law, resultant force

$$F_R = F_G - T$$



Since there is no circular motion,

$$F_R = 0$$

Therefore,

$$T = F_G$$

The spring balance indicates F_G which is the true weight of the body.

Hence,

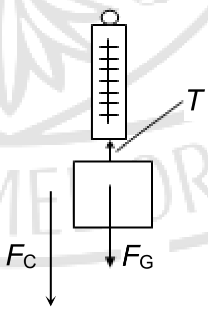
$$mg_{freefall} = mg$$
$$g_{freefall} = g$$

The free fall acceleration $g_{freefall}$ at the polar region is the true gravitational field strength g of the Earth at that region.

<u>**Resultant force F_R at the Equator**</u>

By Newton's second law, resultant force

$$F_R = F_G - T$$



Since the object undergoes circular motion,

$$F_R = F_C = F_G - T$$

where F_C is the centripetal force.

Therefore,

$$T = F_G - F_C$$

The spring balance indicates a value <u>less</u> than the true weight F_G of the body. This value is known as the **apparent weight.**

Hence,

$$mg_{freefall} = mg - ma_C$$
$$g_{freefall} = g - a_C$$

where g_{freefall} is the free fall acceleration,
a_c is the centripetal acceleration.

The free fall acceleration g_{freefall} at the Equator is <u>less</u> than the true gravitational field strength g of the Earth at the Equator.

At the Equator, gravitational force is responsible for both the centripetal force to keep the body moving in a uniform circular motion and for accelerating the body towards Earth (acceleration of free fall).

NOTE!!

Example 9

The radius of the Earth is 6370 km and the mass of the Earth is 5.98×10^{24} kg.

(a) Calculate the centripetal acceleration of an object placed at the Equator.

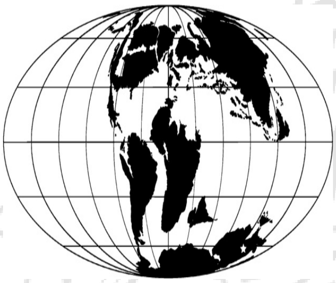
(b) If a 5.0 kg mass is placed on a weighing scale at the Equator, what would the scale read in newtons?

Solution

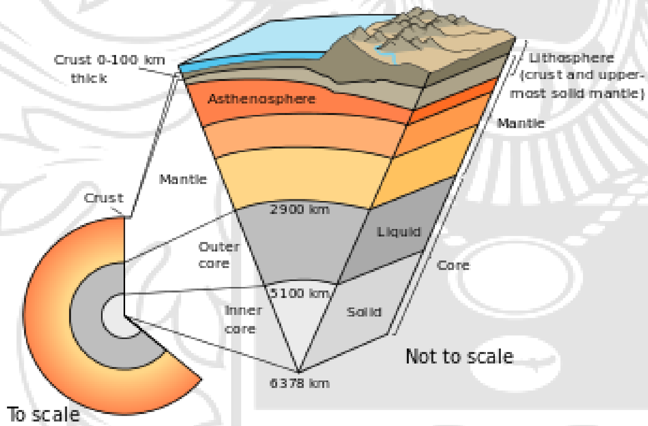
❖ Factors affecting Gravitational Field Strength of Earth

Near the Earth's surface, the gravitational field strength is approximately constant and is equal to the acceleration of free fall of 9.81 m s^{-2}, which is an average value. The precise strength of Earth's gravity varies with location.

1. The gravitational field strength over the Earth's surface varies as the result of its lack of spherical symmetry. Its polar diameter is about 40 km less than its equatorial diameter.



2. The Earth's density is not uniform. There are local variations in the Earth's gravitational field caused by mountains and trenches, as well as the presence of minerals and oil deposits.



Due to the combination of the non-spherical symmetry of Earth and the rotation of the Earth, the effective gravitational field strength at sea level increases from about 9.780 m s^{-2} at the Equator to about 9.832 m s^{-2} at the poles. So a body will weigh about 0.5 % more at the poles than at the Equator.

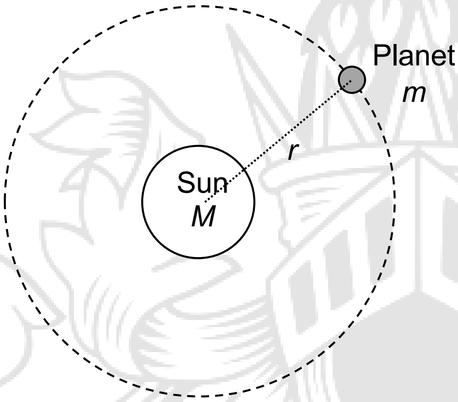
8.7 Circular Orbits

❖ Planetary Circular Orbits

Kepler's Third Law

Kepler's third law shows that there is a precise mathematical relationship between a planet's distance from the Sun and the amount of time it takes to revolve around the Sun. It was this law that inspired Newton, who came up with three laws of his own to explain why the planets move as they do.

Consider a planet of mass *m* moving with linear speed *v* in a circular orbit of radius *r* about the Sun of mass *M*.



The only force acting on the planet is the gravitational force on the planet by the Sun. Hence, this **gravitational force F_G provides the centripetal force F_C** necessary for the circular motion of the planet round the Sun.

Applying Newton's second law of motion,

$$F_G = F_C$$

$$\frac{GMm}{r^2} = mr\omega^2$$

$$\frac{GMm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T^2 = \frac{4\pi^2}{GM}r^3$$

$$T^2 \propto r^3$$

where *T* is the orbital period and *r* is the orbital radius.

This relationship is known as Kepler's third law which states that the <u>square of the periods of revolution of the planets are directly proportional to the cubes of their mean distances from the Sun</u>.
The period is <u>independent of the mass</u> that is orbiting around the Sun.

NOTE!!

Kepler's third law may also be applied to orbiting satellites. For planets or satellites describing circular orbits about the <u>same central body</u>, the square of the period is proportional to the cube of the radius of the orbit.

Example 10 [MI/H2/Prelims 2010/I/14d modified]

Mercury is 5.79×10^{10} m from the Sun and it takes 0.241 Earth years for Mercury to make one revolution around the Sun. If Neptune is 450×10^{10} m from the Sun, calculate the period, in Earth years, of Neptune around the Sun.

Solution

❖ Satellites Motion

What is a satellite?

A satellite is a moon, planet or machine that orbits a planet or star. For example, Earth is a natural satellite because it orbits the Sun. Likewise, the moon is a natural satellite because it orbits Earth.

There are also thousands of artificial, man-made satellites that orbit Earth. Some satellites take pictures of the planet that help meteorologists predict weather and track hurricanes while others take pictures of other planets, the Sun, black holes, dark matter or faraway galaxies to help scientists better understand the solar system and universe.

There are also some satellites that are used mainly for communications such as beaming TV signals and phone calls around the world. A group of more than 20 satellites make up the Global Positioning System (GPS). If you have a GPS receiver, these satellites can help pinpoint your exact location.

Energy of a Satellite

Consider a satellite of mass m orbiting with a speed v round the Earth of mass M in a circular orbit of radius r.

Since the satellite is in circular orbit, **the gravitational force on it provides the centripetal force**.

$$F_G = F_C$$

$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$

$$v^2 = \frac{GM}{r}$$

The kinetic energy E_K of the satellite is

$$E_K = \frac{1}{2}mv^2$$

$$\boxed{E_K = \frac{1}{2}\frac{GMm}{r}}$$

Formula

The potential energy E_P of the satellite is

$$\boxed{E_P = -\frac{GMm}{r}}$$

Formula

The total energy E_T of the satellite is the sum of the kinetic energy and potential energy of the satellite.

$$E_T = E_P + E_K = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$$

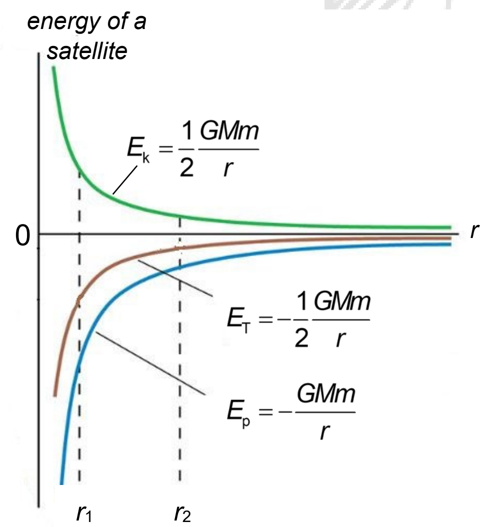
$$\boxed{E_T = -\frac{1}{2}\frac{GMm}{r}}$$

Formula

Bodies that are bounded have a negative total energy.

Bodies with $E_T \geq 0$ will be able to reach infinity and escape.

The graph shows the variation with r, the different forms of energy of a satellite in a circular orbit.



A satellite orbiting a massive object is a bound system.

Total energy in a bound system is negative, reflecting the fact that the satellite cannot escape.

Points on the graph along the dotted lines r_1 and r_2 indicate the corresponding values of the different energy forms that the satellite possesses at orbital radii of r_1 and r_2 respectively.

Example 11

The Earth is a uniform sphere of radius 6370 km and mass 5.98×10^{24} kg. An object orbits at an altitude of 300 km above the Earth.

(a) Determine its linear velocity.
(b) Briefly explain why the orbital plane of any satellite includes the centre of the Earth.

Solution

Geostationary Orbits

A **geostationary satellite** is one that remains at a fixed position in the sky as viewed from any location on the Earth's surface.

A geostationary satellite must satisfy the following conditions:

1. Its orbital period is the same as that of the Earth about its axis of rotation (24 hrs).

2. Its direction of rotation is the same as that of the Earth about its axis of rotation (from West to East).

3. Its plane of orbit lies in the same plane as the equator (i.e. the satellite is above the equator).

Geostationary orbit animation

To determine the radius *r* of a geostationary orbit:
The gravitational force on the satellite provides the centripetal force.

$$F_G = F_C$$

$$\frac{GMm}{r^2} = mr\omega^2$$

$$\frac{GMm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$r = \left(\frac{T^2GM}{4\pi^2}\right)^{\frac{1}{3}}$$

$$= \left(\frac{(24\times60\times60)^2(6.67\times10^{-11})(5.98\times10^{24})}{4\pi^2}\right)^{\frac{1}{3}}$$

$$= 42250 \text{ km}$$

Advantages of a geostationary satellite

1. There is continuous surveillance of the region under it.

2. It is easy for the ground station to communicate with it as it is permanently within view. Ground-based antennas can remain fixed in one direction. Hence, geostationary satellites are often used for communication purposes.

3. Due to its high altitude, the satellite can transmit and receive signals over a large area.

Disadvantages of a geostationary satellite

Geostationary (GEO) satellites orbit at a high altitude compared to the Low Earth Orbit (LEO) and Medium Earth Orbit (MEO) satellites. A LEO is normally at an altitude of less than 1000 km. A MEO comprises a wide range of orbits anywhere between LEO and GEO.

This leads to

1. a significant loss of signal strengths,

2. poorer resolution in imaging satellites,

3. time-lag in telecommunication.

Example 12

Which of the following quantities are *not necessarily* the same for satellites that are in geostationary orbits around the Earth?

A. angular velocity
B. centripetal acceleration
C. kinetic energy
D. period of orbit
E. radius of orbit
F. mass
G. momentum

Solution

Example 13

A satellite of mass 2400 kg is placed in a geostationary orbit at a distance of 4.23×10^7 m from the centre of the Earth.

(a) Calculate
1. the angular velocity of the satellite,
2. the speed of the satellite,
3. the acceleration of the satellite,
4. the force of attraction between the Earth and the satellite,
5. the mass of the Earth.

(b) Explain why a geostationary satellite
1. must be placed vertically above the equator,
2. must move from west to east.

(c) Explain why such satellites are often used for telecommunications.

Solution

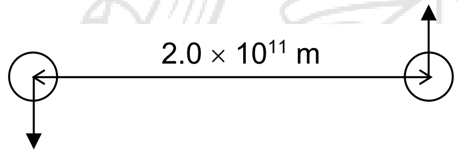
❖ Binary Star System

A binary star system consists of two stars that are gravitationally bound. The two stars orbit around the center of mass of its system.

Astrophysicists find binary systems to be useful in determining the mass of the individual stars involved. When two bodies orbit one another, their masses can be calculated very precisely. The data collected from binary stars allow astrophysicists to extrapolate the relative mass of similar single stars.

Example 14

The figure below shows a binary star system which consists of two stars, each of mass 4.0×10^{30} kg, separated by 2.0×10^{11} m. The stars rotate about the centre of mass of the system.



(a) Label on the above diagram, with a letter X, a point where the gravitational field strength is zero. Explain why you have chosen this point.
(b) Determine the gravitational potential at X.
(c) Calculate the force on each star due to the other star.
(d) Calculate the linear speed of each star in the system.

Solution

Example 15

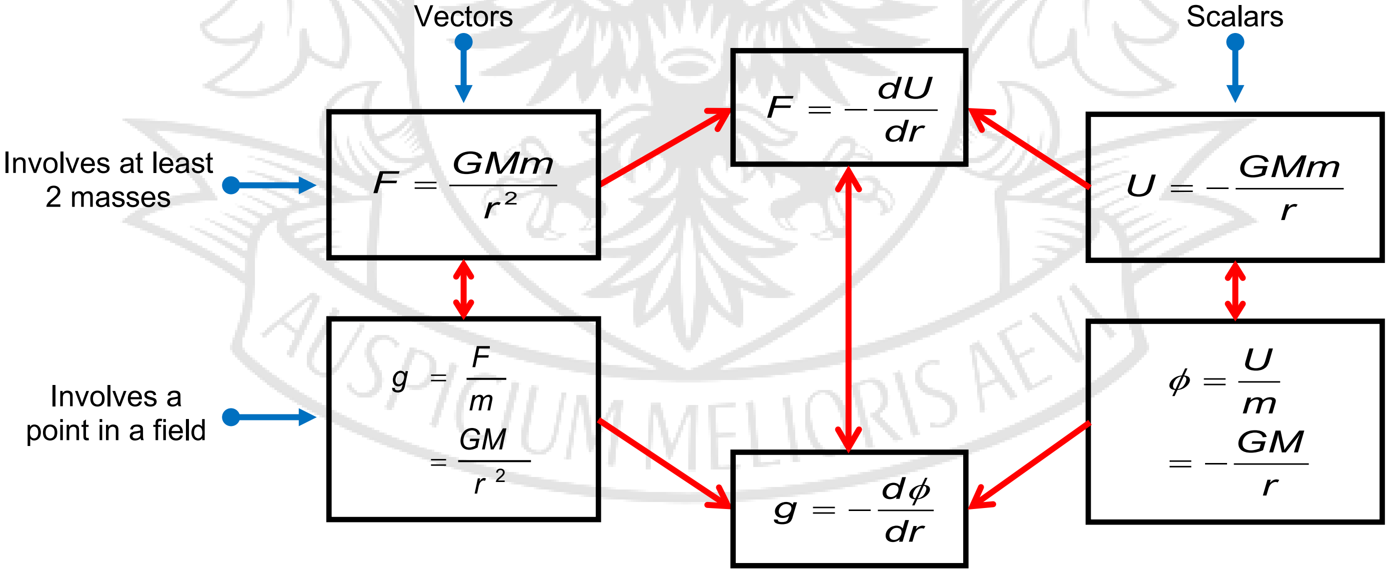
Two stars of masses M and m, separated by a distance d, revolve in circular orbits about their centre of mass. Show that each star has a period given by

$$T^2 = \frac{4\pi^2 d^3}{G(M+m)}.$$

Solution

Summary

- Gravitational force: $F = \dfrac{GMm}{r^2}$
- Gravitational field strength: $g = \dfrac{GM}{r^2}$
- Gravitational potential energy: $U = -\dfrac{GMm}{r}$
- Gravitational potential: $\phi = -\dfrac{GM}{r}$
- Relationship between F and g: $g = \dfrac{F}{m}$
- Relationship between U and ϕ: $\phi = \dfrac{U}{m}$
- Relationship between F and U: $F = -\dfrac{dU}{dr}$
- Relationship between g and ϕ: $g = -\dfrac{d\phi}{dr}$
- For a body to escape from a planet's gravitational influence:
$$-\frac{GMm}{R} + \frac{1}{2}mv^2 \geq 0 \quad \Rightarrow \quad v_{\text{escape}} = \sqrt{\frac{2GM}{R}}$$
- For a body orbiting around a planet with an orbital radius r. Gravitational force provides the centripetal force,
$$\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \Rightarrow \quad v_{\text{orbital}} = \sqrt{\frac{GM}{r}}$$
$$\frac{GMm}{r^2} = mr\omega^2 = mr\left(\frac{2\pi}{T}\right)^2 \quad \Rightarrow \quad T^2 = \frac{4\pi^2}{GM}r^3$$
- These conditions will ensure that a geostationary satellite is always above the same point on the Earth's surface, i.e. its relative position with respect to the Earth remains unchanged:
    1. Same 24-hour orbital period as the Earth.
    2. Same West to East rotation as the Earth.
    3. Plane of orbit lies in the same plane as the equator.



Appendix A

- **Gravitational Field inside a Homogenous Spherical Shell of Mass** (not in syllabus)

The **shell theorem** gives gravitational simplifications that can be applied to objects inside or outside a spherically symmetrical body.

Isaac Newton proved the shell theorem and stated that:

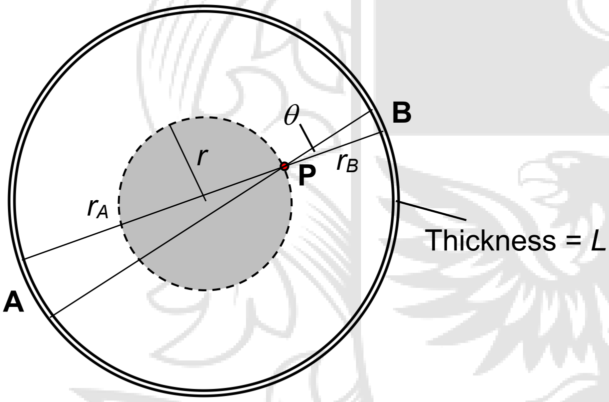
1. A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its centre.
2. If the body is a spherically symmetric shell (i.e. a hollow ball), no net gravitational force is exerted by the shell on any object inside, regardless of the object's location within the shell.

A result of the shell theorem is that inside a solid sphere of constant density, the gravitational force varies linearly with distance from the centre, becoming zero by symmetry at the centre of mass.

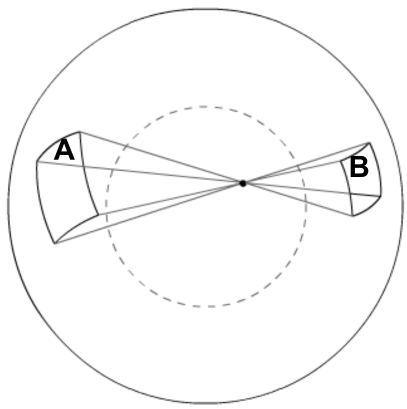
This can be seen as follows: take a point within such a sphere, at a distance from the centre of the sphere. You can then ignore all the shells of greater radius, according to the shell theorem. Only the mass of the sphere within this radius will have effect on the gravitational force at that point.

Proof

Cross-sectional view



3D view to show the areas of A and B



Consider a point P located within a solid sphere a distance r from the centre of the sphere. We will prove that the "effective mass" consists of only the mass enclosed by the shaded sphere within dotted sphere of radius r. The approach is to prove that the rest of the mass (the unshaded portion) cancels out each other's gravitational effect such that their net force on any mass placed at P is zero.

First, consider a thin spherical shell of thickness L. P experiences the gravitational field from every portion of the shell. Two pieces of mass A and B are located on the opposite sides of P as shown. If the angle they subtend at P is θ, then their areas (considering a square of sides = arc length, $r\theta$) are given by $(r_A\theta)^2$ and $(r_B\theta)^2$ respectively. So their masses will be $(r_A\theta)^2L\rho$ and $(r_B\theta)^2L\rho$ respectively, where ρ is the density of the sphere. Hence, their gravitational field strength at P will be $G\frac{(r_A\theta)^2 L\rho}{r_A^2}$ and $G\frac{(r_B\theta)^2 L\rho}{r_B^2}$ respectively.

In both cases the r's cancel away and we get $G\theta^2 L\rho$ for both. So the gravitational field strengths of A and B are equal in magnitude at P. It is also obvious that they are opposite in direction since A and B are on opposite sides of P. So the resultant field at P due to A and B is zero. Apply this argument to every direction around P, we find that the resultant field due to the whole spherical shell is zero.

A solid sphere can be imagined to be made up of layers of spherical shells. If every shell with radius $> r$ produces zero resultant field at P then clearly their combined effect is still zero.

So the effective mass is only those within the dotted shaded sphere.

Appendix B

❖ **Derivation of $\Delta U = mgh$**

When an object of mass m is lifted through height h from point 1 to point 2 near the surface of the Earth, we usually use the formula mgh to determine the gain in gravitational potential energy U. This is a good approximation if g is constant between point 1 and point 2.

Proof:

Take point 1 to be the ground level and point 2 a distance h above it, such that $r_1 = R_E$ and $r_2 = R_E + h$. Then

$$\begin{aligned}
\Delta U &= \left(-\frac{GMm}{r_2}\right) - \left(-\frac{GMm}{r_1}\right) \\
&= \left(-\frac{GMm}{R_E + h}\right) - \left(-\frac{GMm}{R_E}\right) \\
&= -\frac{GMm}{R_E}\left(\frac{R_E}{R_E + h} - 1\right) \\
&= -\frac{GMm}{R_E}\left(\frac{R_E - R_E - h}{R_E + h}\right) \\
&= \frac{GMm}{R_E}\left(\frac{h}{R_E + h}\right) \\
&\approx \frac{GMm}{R_E}\left(\frac{h}{R_E}\right) && \text{since } h << R_E \\
&= m\left(\frac{GM}{R_E^2}\right)h && \text{since } g = \frac{GM}{R_E^2} \\
&= mgh
\end{aligned}$$

NOTETAKING

"What makes planets go around the sun? At the time of Kepler, some people answered this problem by saying that there were angels behind them beating their wings and pushing the planets around an orbit. As you will see, the answer is not very far from the truth. The only difference is that the angels sit in a different direction and their wings push inward."
Richard Feynman