RAFFLES INSTITUTION
2023 Year 5 H2 Mathematics Common Test
Questions and Solutions with comments

1 A cubic curve has two stationary points at $(2,\ 0)$ and $\left(-\dfrac{4}{3},\ \dfrac{500}{27}\right)$. Find the equation of the curve. [4]

1	<u>**Method 1**</u> Let the cubic equation be $y = ax^3 + bx^2 + cx + d$. Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx + c$. At $(2,\ 0)$, we have $8a + 4b + 2c + d = 0$ ---- (1) $12a + 4b + c = 0$ ---- (2) At $\left(-\dfrac{4}{3},\ \dfrac{500}{27}\right)$, we have $a\left(-\dfrac{4}{3}\right)^3 + b\left(-\dfrac{4}{3}\right)^2 + c\left(-\dfrac{4}{3}\right) + d = \dfrac{500}{27}$ ---- (3) $3a\left(-\dfrac{4}{3}\right)^2 + 2b\left(-\dfrac{4}{3}\right) + c = 0$ ---- (4) Solving the system of linear equations (1)–(4), we have $a = 1,\ b = -1,\ c = -8$ and $d = 12$. Therefore, the equation is $y = x^3 - x^2 - 8x + 12$. <u>**Method 2**</u> Since $(2,\ 0)$ is a stationary point, then $x = 2$ is a repeated root for the cubic equation. So, we can write the equation as $y = (x-2)^2(ax+b)$. At $\left(-\dfrac{4}{3},\ \dfrac{500}{27}\right)$, $\dfrac{500}{27} = \dfrac{100}{9}\left(-\dfrac{4}{3}a + b\right) \Rightarrow b - \dfrac{4}{3}a = \dfrac{5}{3}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2(x-2)(ax+b) + a(x-2)^2 = (x-2)(3ax + 2b - 2a)$ Since $\left(-\dfrac{4}{3},\ \dfrac{500}{27}\right)$ is a stationary point, $\left(-\dfrac{4}{3} - 2\right)(-4a + 2b - 2a) = 0$ $b - 3a = 0$ Solving, $a = 1,\ b = 3$. Thus, the cubic equation is $y = (x-2)^2(x+3)$	There are a few solutions which did not start with a general form of a cubic equation: ✗ $y = ax^3 + bx^2 + cx$ (missing y-intercept) ✗ $y = (x-2)^2(x+a)$ (assumed that the coeff of x^3 is 1) There are also quite a number of careless mistakes and copying down the wrong values/signs. There are also some who do by hand to solve the systems of linear equation, instead of using the Plysmlt2 app in the GC.

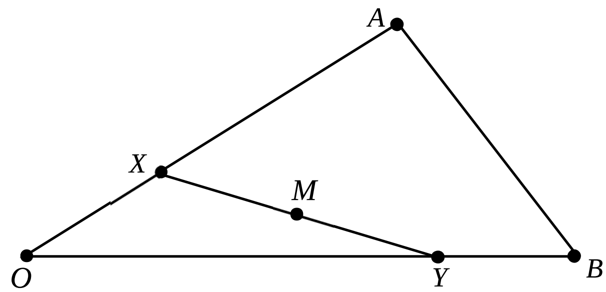
2 Given that the complex numbers w and z satisfy the equations

$$w^{*}+2z=\mathrm{i} \qquad \text{and} \qquad w+(2-\mathrm{i})z=2+6\mathrm{i}$$

find w in the form $a+b\mathrm{i}$, where a and b are real. [4]

2	$w^{*}+2z=\mathrm{i}$ **(**1) $w+(2-\mathrm{i})z=2+6\mathrm{i}$ (2) From (1), substitute $z=\dfrac{\mathrm{i}-w^{*}}{2}$ into (2) $w+(2-\mathrm{i})\left(\dfrac{\mathrm{i}-w^{*}}{2}\right)=2+6\mathrm{i}$ $2w+(2-\mathrm{i})(\mathrm{i}-w^{*})=4+12\mathrm{i}$ $2w-w^{*}(2-\mathrm{i})=3+10\mathrm{i}$ Let $w=a+b\mathrm{i}, \quad a,b\in\mathbb{R}$ $2(a+b\mathrm{i})-(a-b\mathrm{i})(2-\mathrm{i})=3+10\mathrm{i}$ $2a+2b\mathrm{i}+a\mathrm{i}-2a+b+2b\mathrm{i}=3+10\mathrm{i}$ $b+(a+4b)\mathrm{i}=3+10\mathrm{i}$ Compare real parts: $b=3$ (3) Compare imaginary parts: $a+4b=10$ (4) Solve equations (3) and (4) : $a=-2$ and $b=3$ $w=-2+3\mathrm{i}$	

3



With reference to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. It is given that the point X is on OA such that $OX : XA = 1:2$ and the point Y is on OB such that $OY : YB = 3:1$. M is the mid-point of the line segment XY.

(a) Find the vector $\overrightarrow{OM}$ in terms of **a** and **b**. [1]

Point N lies on the line AB such that O, M and N are collinear.

(b) Find the ratio $AN : NB$. [4]

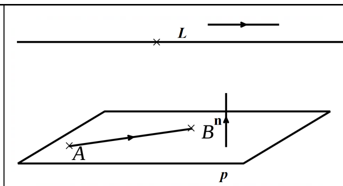
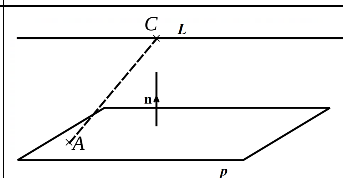
3(a)	$\overrightarrow{OX} = \frac{1}{3}\mathbf{a}$ and $\overrightarrow{OY} = \frac{3}{4}\mathbf{b}$ $\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OX} + \overrightarrow{OY}\right) = \frac{1}{6}\mathbf{a} + \frac{3}{8}\mathbf{b}$	There are a few who has conceptual error: ✗ $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{XY}$
(b)	**<u>Method 1</u>** Since O, M and N are collinear, we can write $\overrightarrow{ON} = k\overrightarrow{OM} = \frac{1}{6}k\mathbf{a} + \frac{3}{8}k\mathbf{b}$, where $k \in \mathbb{R}$ Let $AN : NB = \lambda : 1-\lambda$, where $0 < \lambda < 1$. By Ratio Theorem, $\overrightarrow{ON} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$. Since **a** and **b** are non-parallel and non-zero vectors, **b** : $\frac{3}{8}k = \lambda \Rightarrow k = \frac{8}{3}\lambda$ **a** : $\frac{1}{6}\left(\frac{8}{3}\lambda\right) = 1-\lambda \Rightarrow \lambda = \frac{9}{13}$ $\therefore AN : NB = \frac{9}{13} : \frac{4}{13} = 9:4$ **<u>Method 2</u>** Equation of line OM is $\mathbf{r} = k\left(\frac{1}{6}\mathbf{a} + \frac{3}{8}\mathbf{b}\right), k \in \mathbb{R}$. Equation of line AB is $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}), t \in \mathbb{R}$. At point of intersection N, $\mathbf{a} + t(\mathbf{b} - \mathbf{a}) = k\left(\frac{1}{6}\mathbf{a} + \frac{3}{8}\mathbf{b}\right)$ for some $k, t \in \mathbb{R}$	It is important to note that in order to compare the coefficients of **a** and of **b**, **a** and **b** must be non-parallel and non-zero vectors

	$(1-t)\mathbf{a}+t\mathbf{b}=k\left(\frac{1}{6}\mathbf{a}+\frac{3}{8}\mathbf{b}\right)$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and non-zero vectors, $\frac{3}{8}k=t \Rightarrow k=\frac{8}{3}t$ $\frac{1}{6}\left(\frac{8}{3}t\right)=1-t \Rightarrow t=\frac{9}{13}$ $\overrightarrow{ON}=\mathbf{a}+\frac{9}{13}(\mathbf{b}-\mathbf{a}) \quad \Rightarrow \quad \overrightarrow{AN}=\frac{9}{13}\overrightarrow{AB}$ $\therefore\ AN:NB=9:4$	

4 A plane p is parallel to the line L with equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, $t \in \mathbb{R}$ and passes through the points $A(5, -4, 1)$ and $B(-3, 4, 2)$.

(a) Find a cartesian equation of p. [3]

(b) Find the exact distance between L and p. [3]

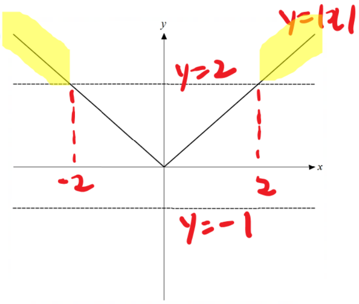
4(a)	Vector parallel to p = $\overrightarrow{BA} = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ -1 \end{pmatrix}$ Since p is parallel to the line L, a vector perpendicular to p = $\begin{pmatrix} 8 \\ -8 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \\ 32 \end{pmatrix} = 2\begin{pmatrix} 5 \\ 3 \\ 16 \end{pmatrix}$ Equation of p is $\mathbf{r} \cdot \begin{pmatrix} 5 \\ 3 \\ 16 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 \\ 16 \end{pmatrix} \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 5 \\ 3 \\ 16 \end{pmatrix} = 29$ Cartesian equation of p is $5x + 3y + 16z = 29$.	 It is important to check your work. Quite a few students made careless mistakes or copy the wrong value.
(b)	Consider $C(1, 2, -1)$ which is a point on L. $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}$ Distance between L and $p = \dfrac{\left\| \overrightarrow{AC} \cdot \begin{pmatrix} 5 \\ 3 \\ 16 \end{pmatrix} \right\|}{\sqrt{5^2 + 3^2 + 16^2}}$ $= \dfrac{\left\| \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 3 \\ 16 \end{pmatrix} \right\|}{\sqrt{290}}$ $= \dfrac{34}{\sqrt{290}}$ or $\dfrac{17\sqrt{290}}{145}$	 Some students approached the question by finding the foot of perpendicular from A to L, note that this would not give the distance between the line and the plane.

5 **Do not use a calculator in answering this question.**

By completing the square, or otherwise, show that $4x^2-10x+13$ is always positive. Hence solve the inequality $\frac{9x^2-15x+3}{x^2-x-2}<5$. [5]

Hence solve the inequality $\frac{9x^2-15|x|+3}{2+|x|-x^2}\le -5$. [3]

5	<u>**Method 1**</u> $4x^2-10x+13=4\left(x^2-\frac{5}{2}x\right)+13$ $=4\left[\left(x-\frac{5}{4}\right)^2-\left(\frac{5}{4}\right)^2\right]+13$ $=4\left(x-\frac{5}{4}\right)^2+\frac{27}{4}$ Since $\left(x-\frac{5}{4}\right)^2 \ge 0 \ \forall x\in\mathbb{R}$, $4\left(x-\frac{5}{4}\right)^2+\frac{27}{4}>0$ so $4x^2-10x+13$ is always positive. <u>**Method 2**</u> Since the discriminant of $4x^2-10x+13$ is $(-10)^2-4(4)(13)=-108<0$, and the coefficient of x^2 (4) is positive, so $4x^2-10x+13>0$ for all $x\in\mathbb{R}$.	Most students are able to do the complete the square correctly.
	$\frac{9x^2-15x+3}{x^2-x-2}<5$ ----------(*) $\Rightarrow \frac{9x^2-15x+3}{x^2-x-2}-5<0$ $\Rightarrow \frac{9x^2-15x+3-5(x^2-x-2)}{x^2-x-2}<0$ $\Rightarrow \frac{4x^2-10x+13}{x^2-x-2}<0$ Since $4x^2-10x+13$ is always positive, then the inequality is reduced to $x^2-x-2<0$ $(x+1)(x-2)<0$ The solution is $-1<x<2$.	The question is mostly well done

	$\dfrac{9x^2-15\lvert x\rvert+3}{2+\lvert x\rvert-x^2}\le -5 \quad --(\#)$ $-\dfrac{9x^2-15\lvert x\rvert+3}{x^2-\lvert x\rvert-2}\le -5$ $\dfrac{9x^2-15\lvert x\rvert+3}{x^2-\lvert x\rvert-2}\ge 5$  Thus, $\lvert x\rvert\le -1$ or $\lvert x\rvert\ge 2$. Note that $\lvert x\rvert< -1$ has no solution. $x=2$ and $x=-2$ cannot be a solution to equation (#). Thus, $\lvert x\rvert\ge 2$ is reduced to $\lvert x\rvert>2$ So, the solution to inequality (#) is $x<-2$ or $x>2$.	There is need to rewrite the inequality to look like inequality (*). After rewriting, notice that the inequality is ≥ 5 instead of <5. Thus, the solution should be a complement of the original solution.

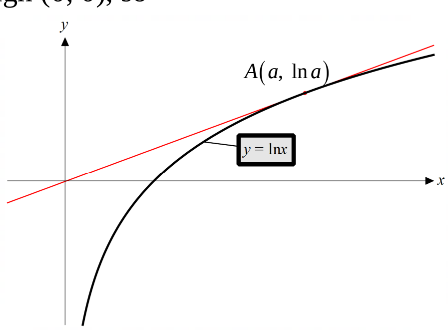
6 **(a)** Use differentiation to find the x-coordinate of the stationary point of the curve

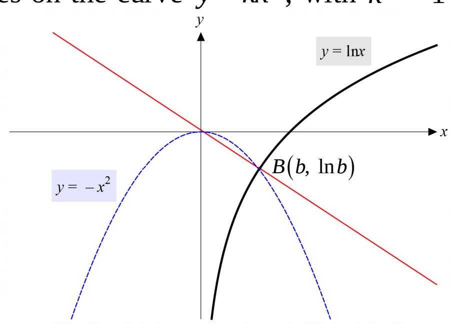
$$y = x^2 - p^2 \ln\left(\frac{x}{q}\right),$$

where p and q are positive and negative constants respectively, and determine the nature of the stationary point. [4]

(b) **(i)** The tangent to the curve $y = \ln x$ at the point $A(a,\ \ln a)$ passes through the origin. Find the value of a. [2]

(ii) The normal to the curve $y = \ln x$ at the point $B(b,\ \ln b)$ passes through the origin. Show that B lies on the curve $y = kx^2$, where k is a constant to be determined. [2]

6(a)	$y = x^2 - p^2\ln\left(\frac{x}{q}\right) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - p^2\left(\frac{q}{x}\right)\left(\frac{1}{q}\right) = 2x - \frac{p^2}{x}.$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 2x - \frac{p^2}{x} = 0$ $\Rightarrow 2x^2 = p^2$ $\Rightarrow x = \pm\sqrt{\frac{p^2}{2}} = -\frac{p}{\sqrt{2}}$ $(\text{since } p > 0 \text{ and } x < 0)$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 + \frac{p^2}{x^2} \Rightarrow \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right\vert_{x=-\frac{p}{\sqrt{2}}} = 2 + \frac{p^2}{\left(-\frac{p}{\sqrt{2}}\right)^2} = 4 > 0.$ Therefore, at $x = -\frac{p}{\sqrt{2}}$, the stationary point is a minimum.	Quite a number of solutions rewrote $\ln\left(\frac{x}{q}\right) = \ln x - \ln q$ which is <u>invalid</u> for this question because q is negative which makes $\ln q$ not to exist. In other words, in order for $\ln\left(\frac{x}{q}\right)$ to exist, x has to be negative since q is negative. Other common mistake: ✗ $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\frac{x}{q}\right)\right) = \frac{1}{\left(\frac{x}{q}\right)}$
(b)(i)	**<u>Method 1</u>** $y = \ln x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ At $A(a,\ \ln a)$, equation of the tangent is $y - \ln a = \frac{1}{a}(x - a)$. Tangent passes through (0, 0), so $0 - \ln a = \frac{1}{a}(0 - a)$ $-\ln a = -1$ $\ln a = 1$ $a = \mathrm{e}.$ 	Quite a few scripts wrote the equation of the tangent line as $y - \ln a = \frac{1}{x}(x - a)$ $\Rightarrow y = -\frac{a}{x} + 1 + \ln a$ This is NOT correct because this is NOT an equation of a straight line. Being a tangent line, it has to be a straight line.

	Method 2 $y=\ln x \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{x}$ Gradient of the curve at $A(a,\ \ln a)=\dfrac{1}{a}$ Since tangent at $A(a,\ \ln a)$ passes through (0, 0), gradient of tangent = $\dfrac{\ln a}{a}$ Hence $\dfrac{\ln a}{a}=\dfrac{1}{a} \Rightarrow a=\mathrm{e}$	
(b) (ii)	**Method 1** At $B(b,\ \ln b)$, equation of the normal is $y-\ln b=-b(x-b)$. Normal passes through (0, 0), so $0-\ln b=-b(0-b) \Rightarrow -\ln b=b^2$ $\Rightarrow \ln b=-b^2$. Thus, when the normal of the curve passes through the origin, this equation $\ln b=-b^2$ is true. Hence the coordinates of $B(b,\ \ln b)$ satisfy the equation $y=-x^2$, and so B lies on the curve $y=kx^2$, with $k=-1$ (shown).  **Method 2** $y=\ln x \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{x}$ Gradient of the normal at $B(b,\ \ln b)= -b$ Since normal at $B(b,\ \ln b)$ passes through (0, 0), gradient of normal = $\dfrac{\ln b}{b}$ Thus $\dfrac{\ln b}{b}=-b \Rightarrow \ln b=-b^2$ Hence the coordinates of $B(b,\ \ln b)$ satisfy the equation $y=-x^2$, and so B lies on the curve $y=kx^2$, with $k=-1$ (shown).	In this question, it is crucial to show that the equation of the normal reduces to the form of $y=kx^2$, instead of starting from the form $y=kx^2$ and deducing the expressions for x, y and value for k.

7 **(a)** The parametric equations of a curve are

$$x = \tan^n \theta\ ,\ \ y = \sec^n \theta\ ,$$

where n is an integer greater than 2 and $0 < \theta < \dfrac{\pi}{2}$.

Find $\dfrac{dy}{dx}$, expressing your answer as a single trigonometric function in terms of n and θ. [3]

(b) Given that $2y - \tan^{-1} y - x = 0$, find $\dfrac{dy}{dx}$, expressing your answer in terms of y, simplifying your answer.

Hence show that $\dfrac{d^2y}{dx^2} = \dfrac{ry\left(1+y^2\right)}{\left(1+2y^2\right)^s}$, where r and s are integers to be determined.

[5]

7(a)	$x = \tan^n \theta \Rightarrow \dfrac{dx}{d\theta} = n\tan^{n-1}\theta\sec^2\theta.$ $y = \sec^n \theta \Rightarrow \dfrac{dy}{d\theta} = n\sec^{n-1}\theta\sec\theta\tan\theta$ $= n\sec^n\theta\tan\theta.$ $\dfrac{dy}{dx} = \dfrac{n\sec^n\theta\tan\theta}{n\tan^{n-1}\theta\sec^2\theta}$ $= \dfrac{\sec^{n-2}\theta}{\tan^{n-2}\theta}$ $= \left(\dfrac{\sec\theta}{\tan\theta}\right)^{n-2}$ $= \operatorname{cosec}^{n-2}\theta$ or $\dfrac{1}{\sin^{n-2}\theta}$	There are some poor indices skills or conceptual understanding: ✗ $\dfrac{\tan\theta}{\tan^{n-1}\theta} = \dfrac{1}{\tan^n\theta}$ ✗ $\operatorname{cosec}\theta = \sin^{-1}\theta$ Although $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, but it is not so right to write $\operatorname{cosec}^{n-2}\theta = \sin^{2-n}\theta$. Because since $n > 2$, so $2-n<0$. We do not write $\sin^{\text{negative power}}\theta$ and the only negative power that we can write is -1, and in such case $\sin^{-1}\theta$ represent the inverse sine function instead of the cosec function.
(b)	**<u>Method 1</u>** $2y - \tan^{-1} y - x = 0$ Differentiate with respect to x, we have $2\dfrac{dy}{dx} - \dfrac{1}{1+y^2}\dfrac{dy}{dx} - 1 = 0$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2-\dfrac{1}{1+y^2}} = \dfrac{1}{\dfrac{2\left(1+y^2\right)-1}{1+y^2}} = \dfrac{1+y^2}{1+2y^2}$ **<u>Method 2</u>** $x = 2y - \tan^{-1} y$ $\dfrac{dx}{dy} = 2 - \dfrac{1}{1+y^2} = \dfrac{2\left(1+y^2\right)-1}{1+y^2} = \dfrac{1+2y^2}{1+y^2}.$ Therefore $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+2y^2}$ -------- (*)	

	Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{ry\left(1+y^2\right)}{\left(1+2y^2\right)^s}$. Differentiate (*) with respect to x, we have $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=\dfrac{\mathrm{d}}{\mathrm{d}y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right).\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $=\left[\dfrac{\left(1+2y^2\right)\left(2y\right)-\left(1+y^2\right)\left(4y\right)}{\left(1+2y^2\right)^2}\right].\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $=\dfrac{-2y}{\left(1+2y^2\right)^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $=\dfrac{-2y}{\left(1+2y^2\right)^2}\left(\dfrac{1+y^2}{1+2y^2}\right)$ $=\dfrac{-2y\left(1+y^2\right)}{\left(1+2y^2\right)^3}$, where $r=-2$, $s=3$ (shown).	Quite a number of students did not apply the chain rule correctly or forgot to append $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

8 Functions f and g are defined by

$$\mathrm{f}: x \mapsto \left|2-\mathrm{e}^{x}\right|, \quad \text{for } x \in \mathbb{R},\ x \geq -5,$$

$$\mathrm{g}: x \mapsto x^2-2x-2, \quad \text{for } x \in \mathbb{R},\ x < -1.$$

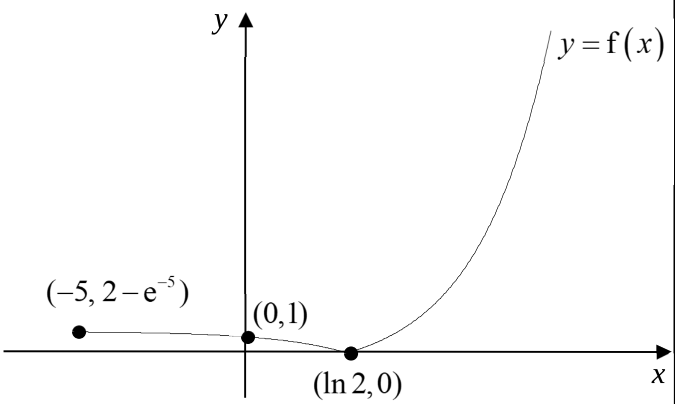
(a) Sketch the graph of $y=\mathrm{f}(x)$. [2]

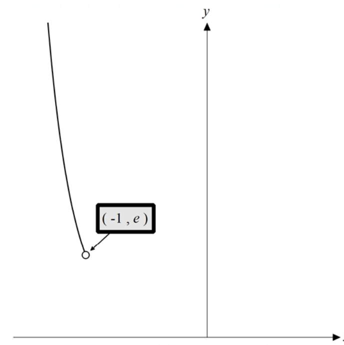
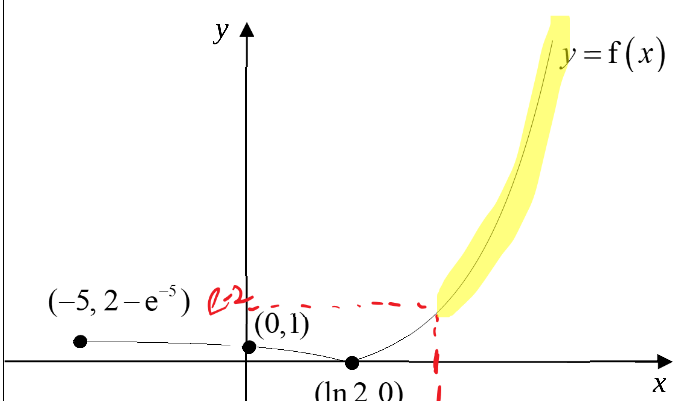
(b) Does f have an inverse? Justify your answer. [1]

(c) Find $\mathrm{g}^{-1}(x)$ and state its domain. [3]

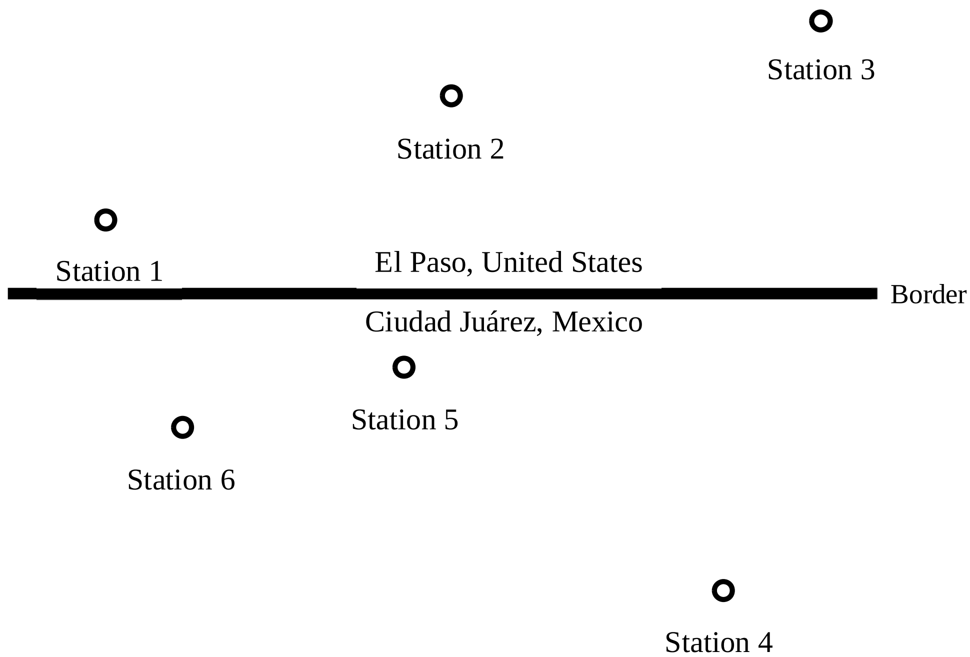
(d) Explain why the composite function fg exists. Find an expression for $\mathrm{fg}(x)$ and write down the domain of fg. [3]

(e) State the range of fg. [1]

8(a)		In this question, the domain of f is $[-5,\infty)$, and so the coordinates of the end-point $(-5, 2-\mathrm{e}^{-5})$ should be indicated. On the other hand, writing the coordinates of the end-point at $(-5,2)$ is not correct. When correct to 3 s.f., the y-coordinate is 1.99.
(b)	Since the line $y=1$ intersects the curve $y=\mathrm{f}(x)$ at two points, f is not one-one and hence f does not have an inverse.	To show that an inverse function does not exist, you just need to specify **one specific horizontal line** that cuts the graph of f at two or more points. Some wrote the following lines which are not correct: ✗ $y=k,\ k \in \mathrm{R_f}=[0,\infty)$ [y = 2 cuts f at only 1 point] ✗ $y=k,\ k \in [0, 2-\mathrm{e}^{-5}]$ [y = 0 cuts f at only 1 point]
(c)	Let $y=x^2-2x-2=(x-1)^2-3,\ x<-1$ $(x-1)^2=3+y$ $x=1\pm\sqrt{3+y}=1-\sqrt{3+y} \quad (\because x<-1)$ $\therefore \mathrm{g}^{-1}(x)=1-\sqrt{3+x}$ Domain of g^{-1} = Range of g = $(1,\infty)$	Provide the reason on the choice of the expression for x.

(d)	Range of g = $(1,\infty)$ Domain of f = $\left[-5,\infty\right)$ Since $R_g \subseteq D_f$, fg exists. $\mathrm{fg}(x)=\mathrm{f}\left(x^2-2x-2\right)=\left\|2-\mathrm{e}^{x^2-2x-2}\right\|=\mathrm{e}^{x^2-2x-2}-2$ $D_{fg}=D_g=(-\infty,-1)$	It is important to state the range of g and domain of f explicitly. This is to let the reader/marker knows what they really are. The graph below is the graph of $y=\mathrm{e}^{x^2-2x-2}$ for $x<-1$ (since the domain of fg is $(-\infty,-1)$). It can be seen that $\mathrm{e}^{x^2-2x-2}>2$ for x < -1. So, we can reduce $\left\|2-\mathrm{e}^{x^2-2x-2}\right\|$ to $\mathrm{e}^{x^2-2x-2}-2$. 
(e)	$D_g=(-\infty,-1)\xrightarrow{g}(1,\infty)\xrightarrow{f}(\mathrm{e}-2,\infty)$ Note that $\mathrm{f}(1)=\|2-\mathrm{e}\|=\mathrm{e}-2$, since $2<\mathrm{e}$. Range of fg = $(\mathrm{e}-2,\infty)$ 	A few wrote the range to be $(2-\mathrm{e},\infty)$

9 "Border Tuner" is a large-scale, participatory art installation designed to interconnect the cities of El Paso in Texas, United States (US) and Ciudad Juárez in Chihuahua, Mexico. Powerful "arms" of light make "bridges of light" that open live sound channels for communication across the US-Mexico border. There are six interactive stations for the participants, three placed in El Paso and three in Ciudad Juárez as shown in the schematic diagram below.



Each of the interactive "Border Tuner" stations features a microphone, a speaker and a large wheel or dial. As a participant turns the dial, an "arm" of light that follows the movement of the dial is created. When two such "arms" of light meet in the sky and intersect, a bidirectional channel of sound opens automatically between the people at the two remote stations, allowing for communication between the people at these two stations. (Source: https://www.bordertuner.net/concept)

Referring the point at Station 1 as the origin O and the horizontal ground as the xy-plane, the coordinates of Stations 2, 4 and 5 are $(132, 36, 2)$, $(274, -119, 0)$ and $(120, -32, 6)$ respectively.

Alice, Sofia and Juan are at Stations 2, 4 and 5 respectively. Alice's "arm" of light passes through a point with coordinates $(142, 26, 22)$ while Sofia's "arm" of light is in the direction of $\begin{pmatrix} -7 \\ k \\ 8 \end{pmatrix}$, where k is a constant.

(a) Given that Alice's "arm" of light is represented by a line l, find a vector equation for l. [2]

(b) Find the acute angle that Alice's "arm" of light makes with the horizontal ground. [2]

(c) Find the value of k so that a channel of communication opens between Alice and Sofia. [4]

(d) Juan directs his "arm" of light such that it is perpendicular to Alice's "arm" of light and he is also able to communicate with Alice. Find a cartesian equation of the line representing Juan's "arm" of light. [5]

9(a)	Direction vector of l $= \begin{pmatrix}142\\26\\22\end{pmatrix} - \begin{pmatrix}132\\36\\2\end{pmatrix} = \begin{pmatrix}10\\-10\\20\end{pmatrix} = 10\begin{pmatrix}1\\-1\\2\end{pmatrix}$ Vector equation of l is $\mathbf{r} = \begin{pmatrix}132\\36\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \lambda \in \mathbb{R}$, Or $\mathbf{r} = \begin{pmatrix}142\\26\\22\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \lambda \in \mathbb{R}$.	A few copied the vectors wrongly from the question or making arithmetic mistakes, such as × $\begin{pmatrix}146\\26\\22\end{pmatrix}, \begin{pmatrix}136\\36\\2\end{pmatrix}$ × $26-36=10$
(b)	**<u>Method 1</u>** The acute angle between Alice's "arm" of light and the normal vector of the horizontal ground is $\cos^{-1}\dfrac{\left\|\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{1^2+1^2+2^2}} = \cos^{-1}\dfrac{2}{\sqrt{6}} \approx 35.264^\circ$ Thus, the angle between Alice's "arm" of light and the horizontal ground is $90^\circ - 35.264^\circ \approx 54.7^\circ$ **<u>Method 2</u>** Required acute angle = $\sin^{-1}\dfrac{\left\|\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{1^2+1^2+2^2}} = \sin^{-1}\dfrac{2}{\sqrt{6}} = 54.7^\circ$ **<u>Method 3</u>** Required acute angle = $\cos^{-1}\dfrac{\left\|\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\|}{\sqrt{1^2+1^2+2^2}\sqrt{1^2+1^2+0^2}} = \cos^{-1}\dfrac{2}{\sqrt{12}} \approx 54.7^\circ$	
(c)	From **(a)**, equation of line representing Alice's "arm" of light is $\mathbf{r} = \begin{pmatrix}132\\36\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \lambda \in \mathbb{R}$. Equation of line representing Sofia's "arm" of light is $\mathbf{r} = \begin{pmatrix}274\\-119\\0\end{pmatrix} + \mu\begin{pmatrix}-7\\k\\8\end{pmatrix},\ \mu \in \mathbb{R}$.	

	If a channel of communication opens between Alice and Sofia, the lines intersect. At point of intersection, $\begin{pmatrix}132\\36\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}274\\-119\\0\end{pmatrix}+\mu\begin{pmatrix}-7\\k\\8\end{pmatrix}$ for some $\lambda,\mu\in\mathbb{R}$. $\Rightarrow\begin{cases}\lambda+7\mu=142\\-\lambda-\mu k=-155\\2\lambda-8\mu=-2\end{cases}$ From GC, $\lambda=51,\ \mu=13,\ \mu k=104\ \Rightarrow k=8$.	
(d)	**<u>Method 1</u>** Let Juan's position be $J(120,-32,6)$ and F be the foot of perpendicular from J to l: $\mathbf{r}=\begin{pmatrix}132\\36\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \lambda\in\mathbb{R}$. Since F is on l, $\overrightarrow{OF}=\begin{pmatrix}132\\36\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. Thus $\overrightarrow{JF}=\overrightarrow{OF}-\overrightarrow{OJ}=\begin{pmatrix}12\\68\\-4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$. Since JF is perpendicular to l, $\overrightarrow{JF}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$ $\left[\begin{pmatrix}12\\68\\-4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$ $-64+6\lambda=0\quad\Rightarrow\quad\lambda=\frac{32}{3}$ Hence $\overrightarrow{JF}=\begin{pmatrix}12\\68\\-4\end{pmatrix}+\frac{32}{3}\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}68/3\\172/3\\52/3\end{pmatrix}=\frac{4}{3}\begin{pmatrix}17\\43\\13\end{pmatrix}$. Equation of the line representing Juan's "arm" of light is $\mathbf{r}=\begin{pmatrix}120\\-32\\6\end{pmatrix}+t\begin{pmatrix}17\\43\\13\end{pmatrix},\ t\in\mathbb{R}$. Cartesian equation of the line is $\frac{x-120}{17}=\frac{y+32}{43}=\frac{z-6}{13}$.	

	Method 2 Let the equation of the line representing Juan's "arm" of light be $\mathbf{r}=\begin{pmatrix}120\\-32\\6\end{pmatrix}+t\begin{pmatrix}a\\b\\c\end{pmatrix},\ t\in\mathbb{R}$. Equation of line representing Alice's "arm" of light is $\mathbf{r}=\begin{pmatrix}132\\36\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix},\ \lambda\in\mathbb{R}$. Since the lines intersect each other, $\begin{pmatrix}120\\-32\\6\end{pmatrix}+t_0\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}132\\36\\2\end{pmatrix}+\lambda_0\begin{pmatrix}1\\-1\\2\end{pmatrix}$ for some $t_0,\lambda_0\in\mathbb{R}$. $t_0\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}12\\68\\-4\end{pmatrix}+\lambda_0\begin{pmatrix}1\\-1\\2\end{pmatrix}$ --- (1) Also, as the lines are perpendicular to each other, $\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$ ---(2) $t_0\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$ $\left[\begin{pmatrix}12\\68\\-4\end{pmatrix}+\lambda_0\begin{pmatrix}1\\-1\\2\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$ using (1) $\lambda_0=\frac{32}{3}$ Thus $t_0\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}12\\68\\-4\end{pmatrix}+\frac{32}{3}\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}\frac{68}{3}\\\frac{172}{3}\\\frac{52}{3}\end{pmatrix}=\frac{4}{3}\begin{pmatrix}17\\43\\13\end{pmatrix}$ Hence the equation of the line representing Juan's "arm" of light is $\mathbf{r}=\begin{pmatrix}120\\-32\\6\end{pmatrix}+t\begin{pmatrix}17\\43\\13\end{pmatrix},\ t\in\mathbb{R}$. Cartesian equation of the line is $\frac{x-120}{17}=\frac{y+32}{43}=\frac{z-6}{13}$.	One can also proceed from this point forward by using this value of λ_0 to obtain the point of intersection of the lines $F\left(\frac{428}{3},\frac{76}{3},\frac{22}{3}\right)$ and hence the direction vector of Juan's "arm" of light given by $\overrightarrow{JF}=\begin{pmatrix}68/3\\172/3\\52/3\end{pmatrix}=\frac{4}{3}\begin{pmatrix}17\\43\\13\end{pmatrix}$