1.	Let P_n be the statement $$(1+x)(1+2x)(1+3x)\cdots(1+nx)\geq 1+\frac{n(n+1)}{2}x$$ for all positive integers *n*. When $n=1$, $\text{LHS}=1+x$ $\text{RHS}=1+\frac{(1)(2)}{2}x=1+x\geq$ (or =) RHS Thus P_1 is true. Suppose P_k is true for some positive integer *k*. Then LHS of P_{k+1} $=(1+x)(1+2x)(1+3x)\cdots(1+kx)(1+(k+1)x)$ $\geq\left(1+\frac{k(k+1)}{2}x\right)(1+(k+1)x)$ $=1+(k+1)x+\frac{k(k+1)}{2}x+\left(\frac{k(k+1)}{2}x\right)((k+1)x)$ $=1+(k+1)\left(1+\frac{k}{2}\right)x+\frac{1}{2}k(k+1)^2x^2$ $>1+\frac{(k+1)(k+2)}{2}x\ \left(\because\frac{1}{2}k(k+1)^2x^2>0\right)$ $=\text{RHS of }\mathrm{P}_{k+1}$ Thus P_k is true implies that P_{k+1} is true. Since P_1 is true and P_k is true implies that P_{k+1} is true, P_n is true for all positive integers *n* by mathematical induction. $\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)$ $=\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\ln\left(1+\frac{3}{n}\right)+\cdots+\ln\left(1+\frac{n}{n}\right)$ $=\ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)$ $\geq\ln\left(1+\frac{n(n+1)}{2}\times\frac{1}{n}\right)$ (use $x=\frac{1}{n}$ in first part) $=\ln\left(1+\frac{n+1}{2}\right)$ $=\ln\left(\frac{n+3}{2}\right)$ $=\ln(n+3)-\ln 2$ (shown)	
2.	(a) $\det(\mathbf{A})\neq 0\Rightarrow\mathbf{A}$ is invertible but $\begin{pmatrix}0&0\\0&0\end{pmatrix}$ is non- invertible.	

OR

$$\det\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \neq 0 \Rightarrow \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \text{ is not in the set.}$$

Since the set does not contain the zero matrix, it is not a subspace of $\mathbf{M}_{2\times 2}(\mathbb{R})$.

(b) Let U denote the set of all 2×2 matrices $\mathbf{B}$ such that $\mathbf{B}^T = -\mathbf{B}$.

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}^T = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = -\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Hence $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \in U$ and U is non-empty.

Consider any two elements $\mathbf{B}$, $\mathbf{C} \in U$.

$(\mathbf{B}+\mathbf{C})^T = \mathbf{B}^T + \mathbf{C}^T = -\mathbf{B} + (-\mathbf{C}) = -(\mathbf{B}+\mathbf{C})$

$\therefore \mathbf{B}+\mathbf{C} \in U$ (Closure under addition)

Consider any element $\mathbf{B} \in U$ and $k \in \mathbb{R}$.

$(k\mathbf{B})^T = k\mathbf{B}^T = k(-\mathbf{B}) = -(k\mathbf{B})$

$\therefore k\mathbf{B} \in U$ (Closure under scalar multiplication)

Hence U is a subspace of $\mathbf{M}_{2\times 2}(\mathbb{R})$.

3.

$$x = \sec\theta \Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta.$$

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = \frac{dy}{dx}\sec\theta\tan\theta.$$

$$\frac{d^2y}{d\theta^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{d\theta}\right)\sec\theta\tan\theta + \frac{dy}{dx}\left(\sec\theta\tan^2\theta + \sec^3\theta\right)$$

$$= \frac{d^2y}{dx^2}\sec^2\theta\tan^2\theta + \frac{dy}{dx}\left(\sec\theta\tan^2\theta + \sec^3\theta\right)$$

$$= \frac{d^2y}{dx^2}\sec^2\theta\left(\sec^2\theta - 1\right) + \frac{dy}{dx}\sec\theta\left(\sec^2\theta - 1 + \sec^2\theta\right)$$

$$= \frac{d^2y}{dx^2}\left(\sec^4\theta - \sec^2\theta\right) + \frac{dy}{dx}\sec\theta\left(2\sec^2\theta - 1\right)$$

$$= \left(x^4 - x^2\right)\frac{d^2y}{dx^2} + x\left(2x^2 - 1\right)\frac{dy}{dx}$$

$$\frac{1}{x}\left(\frac{d^2y}{d\theta^2}\right) = \left(x^3 - x\right)\frac{d^2y}{dx^2} + \left(2x^2 - 1\right)\frac{dy}{dx}.$$

Substituting into the differential equation in x and y:

$$\frac{1}{x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\right)+\frac{ky}{x}=\frac{2}{x^3}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}+ky=\frac{2}{x^2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}+ky=\frac{2}{\sec^2\theta}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}+ky=2\cos^2\theta \text{ (shown)}$$

Characteristic equation:

$$m^2+k=0$$

$$m=\pm\sqrt{k}\mathrm{i} \quad (\because k>0)$$

Complementary function:

$$\begin{aligned} y_c &= A\cos\sqrt{k}\theta+B\sin\sqrt{k}\theta \\ &= A\cos\sqrt{k}\left(\sec^{-1}x\right)+B\sin\sqrt{k}\left(\sec^{-1}x\right). \end{aligned}$$

Since $2\cos^2\theta=\cos 2\theta+1$, particular integral:

$$\begin{aligned} y_p &= C\cos 2\theta+D\sin 2\theta+E \\ y_p' &= -2C\sin 2\theta+2D\cos 2\theta \\ y_p'' &= -4C\cos 2\theta-4D\sin 2\theta. \end{aligned}$$

Substituting into the new differential equation:

$$-4C\cos 2\theta-4D\sin 2\theta+k\left(C\cos 2\theta+D\sin 2\theta+E\right)=\cos 2\theta+1$$

$$C(k-4)\cos 2\theta+D(k-4)\sin 2\theta+kE=\cos 2\theta+1$$

Comparing coefficients:

$kE=1$ $\quad C(k-4)=1$ $\quad$ and $\quad D(k-4)=0$

$E=\frac{1}{k},$ $\quad C=\frac{1}{k-4},$ $\quad D=0 \quad (\because k\neq 4).$

$$\therefore y_p=\frac{1}{k-4}\cos 2\theta+\frac{1}{k}.$$

Thus, $y=A\cos\sqrt{k}\theta+B\sin\sqrt{k}\theta+\frac{1}{k-4}\cos 2\theta+\frac{1}{k}.$

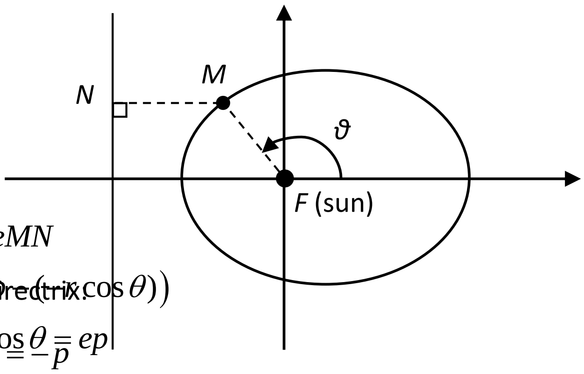
$$\begin{aligned} \text{Now,} \quad \cos 2\theta &= 2\cos^2\theta-1 \\ &= \frac{2}{\sec^2\theta}-1 \\ &= \frac{2}{x^2}-1=\frac{2-x^2}{x^2}. \end{aligned}$$

Thus, the general solution is

$$y = A\cos\sqrt{k}\left(\sec^{-1}x\right) + B\sin\sqrt{k}\left(\sec^{-1}x\right) + \frac{2-x^2}{(k-4)x^2} + \frac{1}{k}$$

where A and B are arbitrary constants.

4. **(i)** Let N be the foot of perpendicular from M to the directrix $x = -p$.



$MF = eMN$

$r = e\left(p - (-r\cos\theta)\right)$

$r - re\cos\theta = ep$

$\therefore r = \dfrac{ep}{1-e\cos\theta}$ (shown, where $k = ep$)

(ii) Let the distance from the sun to the aphelion and that from the sun to the perihelion be d_1 and d_2 respectively.

Then $d_1 = \dfrac{k}{1-e\cos 0} \Rightarrow d_1 = \dfrac{k}{1-e} ---(1)$

and $d_2 = \dfrac{k}{1-e\cos\pi} \Rightarrow d_2 = \dfrac{k}{1+e} ---(2)$

$(1)\div(2): \dfrac{d_1}{d_2} = \dfrac{61}{50} = \dfrac{1+e}{1-e}$

$\Rightarrow 61 - 61e = 50 + 50e$

$\Rightarrow e = \dfrac{11}{111}$ (shown)

If the e becomes smaller than $\dfrac{11}{111}$ but larger than 0, the orbit will get <u>more circular</u>.

(iii) $d_1 = \dfrac{61}{111}(3.04) = \dfrac{k}{1-e}$

$\Rightarrow k = 1.51121316 = 1.51$ (to 3 sf)

$$r = \frac{k}{1-e\cos\theta} \Rightarrow \frac{dr}{d\theta} = \frac{-k\left(e\sin\theta\right)}{\left(1-e\cos\theta\right)^2}$$

Distance travelled in one complete orbit

$$= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\left(\frac{k}{1-e\cos\theta}\right)^2 + \left(\frac{k(e\sin\theta)}{(1-e\cos\theta)^2}\right)^2}\, d\theta$$

$= 9.565819311$ AU

$\therefore$ the orbital period of planet M

$$= \frac{9.565819311}{5.06}$$

$= 1.89$ years

5. **(i)**

$z^7 + 128 = 0$

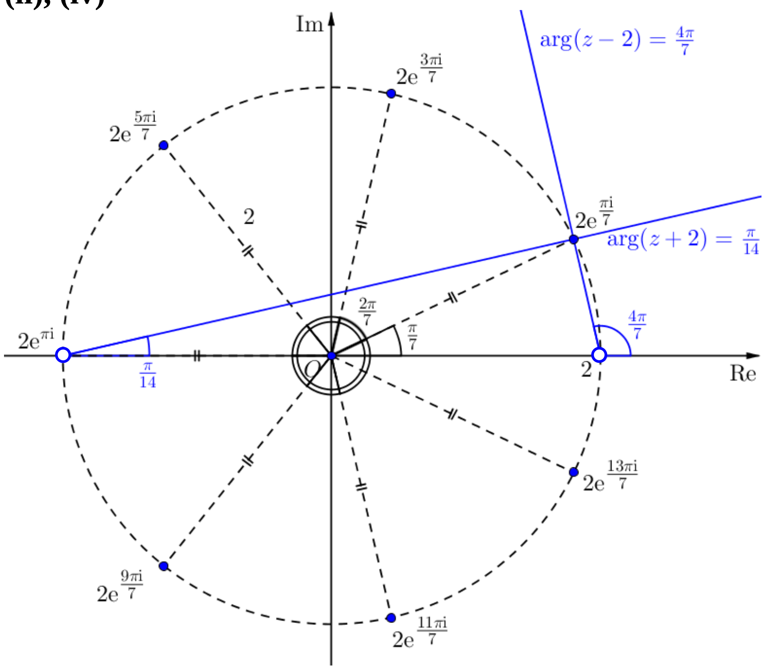
$z^7 = -128$

$z^7 = 128e^{i(2k+1)\pi},\ k \in \mathbb{Z}$

$z = 2e^{i\left(\frac{2k+1}{7}\right)\pi},\ k = 0, 1, 2, 3, 4, 5, 6$

$= 2e^{i\frac{\pi}{7}}, 2e^{i\frac{3\pi}{7}}, 2e^{i\frac{5\pi}{7}}, 2e^{i\pi}, 2e^{i\frac{9\pi}{7}}, 2e^{i\frac{11\pi}{7}}, 2e^{i\frac{13\pi}{7}}$

(ii), (iv)



(iii)

Considering sum of roots of the equation $z^7 + 128 = 0$ and taking real part,

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+\cos\pi+\cos\frac{9\pi}{7}+\cos\frac{11\pi}{7}+\cos\frac{13\pi}{7}=0$$

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+(-1)+\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}=0$$

$$2\left(\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}\right)-1=0$$

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$$

$$1-2\sin^2\left(\frac{\pi}{14}\right)+1-2\sin^2\left(\frac{3\pi}{14}\right)+1-2\sin^2\left(\frac{5\pi}{14}\right)=\frac{1}{2}$$

$$3-2\left[\sin^2\left(\frac{\pi}{14}\right)+\sin^2\left(\frac{3\pi}{14}\right)+\sin^2\left(\frac{5\pi}{14}\right)\right]=\frac{1}{2}$$

$$\therefore \sin^2\left(\frac{\pi}{14}\right)+\sin^2\left(\frac{3\pi}{14}\right)+\sin^2\left(\frac{5\pi}{14}\right)=\frac{3-\frac{1}{2}}{2}=\frac{5}{4}$$

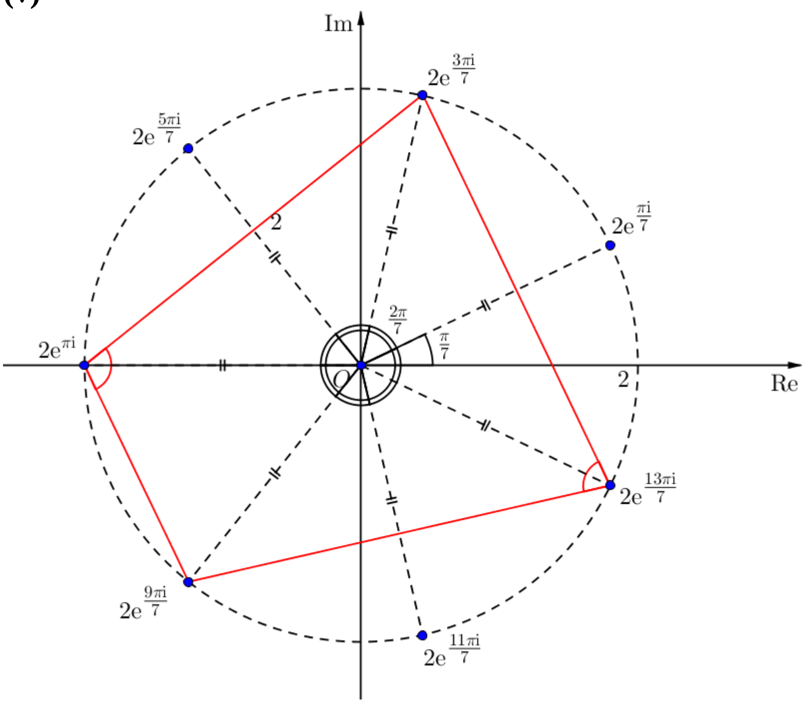
(iv)

Area

$$=\frac{1}{2}(2)^2\left(\sin\left(3\times\frac{2\pi}{7}\right)+\sin\frac{\pi}{7}\right) \text{ or } \frac{1}{2}\times4\times2\sin\frac{\pi}{7}$$

$$=4\sin\frac{\pi}{7} \text{ or } 1.74 \text{ units}^2$$

(v)



$$\arg\left(\frac{(z_1-z_2)(z_3-z_4)}{(z_3-z_2)(z_1-z_4)}\right)$$

$$=\left(\arg(z_1-z_2)-\arg(z_3-z_2)\right)+\left(\arg(z_3-z_4)-\arg(z_1-z_4)\right)$$

	This argument represents the sum of two opposite angles in a cyclic quadrilateral (or equivalently angles in opposite segments) and thus they must add up to π radians (shown). (Note that since z_1, z_2, z_3 and z_4 are any 4 distinct roots, the quadrilateral shown above is not unique.)	
6.	**(i)** H_0: The colour preference is independent of gender H_1: The colour preference is dependent on gender Under H_0, the expected frequencies are	

	White	Green	Black	Blue	Red	Others
Male	92	75.3	65.3	53.3	84	30
Female	46	37.7	32.7	26.7	42	15

Degree of freedom $=(6-1)\times(2-1)=5$

From GC, $\chi^2_{\text{test}}=14.02682385$ and $p\text{-value}=0.0154399223$

Since *p*-value < 0.05, we reject H_0 at the 5% significance level and conclude that there is sufficient evidence to claim that preferences for colours of gaming keyboards differ between male and female gamers.

(ii) The table of contributions are as follows:

	White	Green	Black	Blue	Red	Others
M	0.043 5	0.00147	2.08	0.0020 8	2.01	0.533
F	0.087 0	0.00295	4.17	0.0041 7	4.02	1.07

The largest contributions are from the "Black" and "Red" columns, where there is a significantly larger proportion of males who prefer black colour and a larger proportion of females who prefer red colour.
Specifically, more male prefer black while fewer male prefer red than expected, and more female prefer red while fewer female prefer black than expected.

7. **(i)** To carry out a *t*-test on the data will require the condition that the scores (or difference between the scores) to be normally distributed. This condition might not be met by the data and hence it is not appropriate to use the *t*-test.

(ii) Let X = score after module – score before module
Assume that the distribution of the population of the differences is symmetrical.
H_0: Population median of X is 0
H_1: Population median of X is greater than 0

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
1st	71	70	65	42	74	81	56	69	57	77	81	55
2nd	51	72	46	93	95	31	81	84	88	67	70	82
X	−20	2	−19	51	21	−50	25	15	31	−10	−11	27
R	6	1	5	12	7	11	8	4	10	2	3	9

From table, the sum of negative ranks, $Q=27$.
From MF26, at 1% level of significance with $n=12$,
One-tail critical value of $T\le 9$ $(\text{or } \ge 69)$

Since 27 falls beyond the critical region, we do not reject H_0 and conclude at 1% level of significance that there is insufficient evidence to claim that the population median for the scores in the writing test has improved (i.e. insufficient evidence to conclude that the module is effective).

8.

(i) For a p.d.f., $\int_0^1 \frac{k}{x^2 - x + 1}\,dx = 1$

$$\Rightarrow \quad k\int_0^1 \frac{1}{\left(x-\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}\,dx = 1$$

$$\Rightarrow \quad k\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\tfrac{1}{2}}{\tfrac{\sqrt{3}}{2}}\right)\right]_0^1 = 1$$

$$\Rightarrow \quad \frac{2k}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1 = 1$$

$$\Rightarrow \quad \frac{2k}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right] = 1$$

$$\Rightarrow \quad \frac{2k}{\sqrt{3}}\left[\frac{\pi}{6} + \frac{\pi}{6}\right] = 1$$

$$\Rightarrow \quad k = \frac{3\sqrt{3}}{2\pi}$$

(ii) From symmetry, Mean = Median = Mode = 0.5

(iii) For $0 \le x \le 1$, $\mathrm{P}(X < x) = \frac{3\sqrt{3}}{2\pi}\int_0^x \frac{1}{\left(t-\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}\,dt$

$$= \frac{3}{\pi}\left[\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)\right]_0^x$$

$$= \frac{3}{\pi}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2}$$

Hence $\mathrm{F}(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{3}{\pi}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2} & \text{for } 0 \le x \le 1, \\ 1 & \text{for } x > 1. \end{cases}$

(iv) $\mathrm{E}(Y) = \mathrm{E}\left(50\tan^{-1} X\right) = \frac{3\sqrt{3}}{2\pi}\int_0^1 \frac{50\tan^{-1} x}{x^2 - x + 1}\,dx$

$= 22.03293348 \approx 22.0$ (3 s.f.)

(v) $\mathrm{P}(Y > 25) = \mathrm{P}\left(50\tan^{-1} X > 25\right) = \mathrm{P}\left(X > \tan\left(\frac{1}{2}\right)\right)$

$$= \frac{3\sqrt{3}}{2\pi}\int_{\tan(1/2)}^{1} \frac{1}{x^2 - x + 1}\,\mathrm{d}x = 0.4489963784 \approx 0.449 \text{ (3 s.f.)}$$

9.

(i) For the first design, a 2-sample t-test is more appropriate and the benefit is that it is faster to implement since the cameras only need to go through a single round of data collection.

For the second design, a paired-sample t-test is more appropriate and the benefit is that we can be more confident that any statistical difference is due to the change in batteries since all other factors would be kept relatively constant.

(ii) We need to assume that the battery life for both battery models follows two independent normal distributions with a common variance.

From data,

$$s_p{}^2 = \frac{\sum(x-\bar{x})^2 + \sum(y-\bar{y})^2}{7+8-2} = \frac{961+1369}{13} = 179.23$$

To test $\mathrm{H}_0: \mu_Y = \mu_X$ vs $\mathrm{H}_1: \mu_Y > \mu_X$

Under H_0, $T = \dfrac{\bar{Y} - \bar{X}}{S_p\sqrt{\frac{1}{n_X} + \frac{1}{n_Y}}} \sim t(n_X + n_Y - 2) = t(13)$

```
NORMAL FLOAT AUTO REAL RADIAN MP
        2-SampTTest
 μ1>μ2
 t=2.164877659
 p=0.0247946937
 df=13
 x̄1=801
 x̄2=786
 Sx1=13.9846855
↓Sx2=12.655697
```

From GC, $t = 2.1649$ and $p\text{-value} = 0.0248$.

For such a p-value, we can conclude that there is sufficient evidence at the 10% and 5% levels of significance but not at the 1% level of significance that the new battery model has a longer battery life, making the evidence relatively strong.

(iii) From GC, the 99% CI is (783.7, 818.3).

```
        TInterval
 (783.7,818.3)
 x̄=801
 Sx=13.9846855
 n=8
```

10.

(i) We need to assume that

(1) the arrivals of customers occur independently, and

(2) the average rate of customer arrivals over any time period remains constant.

(ii) Assume that the number of customers arriving and the number of food delivery orders received are independent.
Let S be the random variable denoting the sum of the number of customers arriving and the number of food delivery orders received in a 10-minute interval.
Then $S \sim \text{Po}(2+5) = \text{Po}(7)$.

$\text{P}(S \le 10) = 0.901479206 \approx 0.901$ (3 s.f.)

(iii) The conditions necessary are
(1) the event that each successive interval is busy is independent of all other intervals,and
(2) the probability of each interval being busy is constant.

The conditions are addressed by the independent nature of the Poisson distribution.

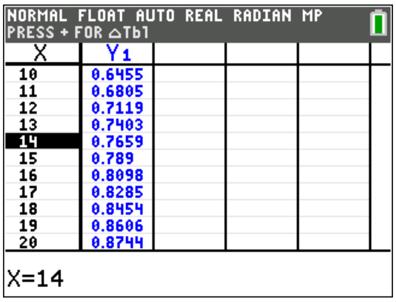
(iv) $\text{P}(S > 10) = 0.098521$

From (b), $B \sim \text{Geo}(0.098521)$

$$\text{E}(B) = \frac{1}{0.098521} = 10.1501415 \approx 10.2 \text{ (3 s.f.)}$$

$$\text{Var}(B) = \frac{1-0.098521}{0.098521^2} = 92.87523101 \approx 92.9 \text{ (3 s.f.)}$$

(v) $\text{P}(B > m) < 0.25 \quad \Rightarrow \quad \text{P}(B \le m) > 0.75$



From GC, least m = 14