RAFFLES INSTITUTION
H2 Mathematics 9758
2025 Year 6 Term 2 Revision (Summary)

Topic: Permutations and Combinations, Probability

<u>Summary for Permutations and Combinations</u>

Definitions - A **permutation** is an ***ordered arrangement*** of objects
- A **combination** is a ***selection of objects*** in which the order of selection ***does not matter***.

Add or multiply?

Mutually exclusive operations → Add (**Case** 1, 2, 3 etc…)

Doing one operation **followed by** another to get to the result. → Multiply (**Step** 1, 2, 3 etc…)

Permutations

Given	Objects taken	No of Permutations
n distinct objects	n	$n!$
n distinct objects	r ***no repetitions***	${}^{n}P_{r}$ or ${}^{n}C_{r}\,r!$
n distinct objects	r ***with repetitions***	n^{r}
n objects, not all distinct (n_1 *of type 1,* n_2 *of type 2, …,* n_k *of type k, where* $n=n_1+n_2+\ldots+n_k$)	n	$\dfrac{n!}{n_1!n_2!\ldots n_k!}$
n objects, not all distinct	r	***No direct way to calculate.*** Need to consider different cases. Involves combinations & permutations. See section on ***Combinations***.

Useful Techniques when dealing with restrictions:

1. Grouping or Slotting.
   (a) Grouping ("must be together", "cannot be separated" etc.).
   Eg. No of ways to arrange the letters a, b, c, d, e, f, g such that a, b & c are together $=(5!)\times(3!)$
   [abc, d, e, f, g: 5 items → 5! ways, within abc → 3! ways]
   (b) Slotting ("cannot be together", "must be separated" etc.)
   Eg. No of ways to arrange the letters a, b, c, d, e, f, g such that a, b & c are not adjacent to each other $=(4!)\times\left({}^{5}P_{3}\right)$
   [arrange d, e, f, g first → 4! ways, slot and permute a, b, c → ${}^{5}P_{3}$ ways]

2. Taking Complement ("two items cannot be together" or "at least 1").
   Use this technique with care. Note that for Eg in 1(b) above, a, b, c not adjacent to each other is NOT the complement of abc together.

Combinations

Notation: $^{n}C_{r}$ can also be written as $\binom{n}{r}$

Note:

- $\binom{n}{r} = \binom{n}{n-r} = \dfrac{n!}{r!(n-r)!}$. E.g. $\binom{9}{2} = \binom{9}{7}$
- $^{n}P_{r} = {^{n}C_{r}}\,r!$

No of ways to select ***r objects***	Given : ***n distinct*** objects $\binom{n}{r}$	Given: ***n*** objects, not all distinct ***No direct way to calculate.*** Need to consider different cases: (i) Start with case where the r selected objects are ***all distinct.*** (ii) Next, consider case(s) with pair(s) of identical objects, and so on. (iii) **No of ways = Sum of the different cases**
Eg: To choose 3 letters (***arrangements*** not required)	Given: Letters a,b,c,d,e,f No of ways = $\binom{6}{3} = 20$	Given: a, a, a, b, b, c Case 1: All distinct (abc) – 1 way. Case 2: Contains an identical pair (aab, aac, bba, bbc) – 2 x 2 = 4 ways. Case 3: All identical (aaa) – 1 way. Total no of ways = 1 + 4 +1 = 6
Eg. Find the no of 3 letter codes (***arrangements*** to be considered)	Given: Letters a,b,c,d,e,f No of ways $= \binom{6}{3}(3!)$ $= 120$ OR: $^{6}P_{3} = 6 \times 5 \times 4 = 120$	Given: a, a, a, b, b, c Case 1: All distinct (abc) – 1x3! = 6 ways. Case 2: Contains an identical pair (aab, aac, bba, bbc) – $2 \times 2 \times \dfrac{3!}{2!} = 12$ ways. Case 3: All identical (aaa) – 1 way. Total no of ways = 6 + 12 +1 = 19

Remark: Before doing any calculations, it is important to establish if

- the problem involves ***permutations only***, or ***combinations only***, or ***both***.
- the objects or items given are ***all distinct***. (see Eg in above table).
- ***repetitions*** are allowed.

Circular Permutations – Arranging *n* distinct objects in a Circle

- In row arrangement, such as in a queue, there is a *first* & a *last* position. No of ways = $n!$
- In a circle, such as a round table, there is no starting or ending point.
  No of ways = $\frac{n!}{n} = (n-1)!$
  - No. of ways to seat 2 couples and a boy at a round table = $(5-1)! = 24$
  - No. of ways to seat 2 couples and a boy at a round table if each couple is seated together = $(3-1)!2!2! = 8$
    [couple 1, couple 2, boy round the round → $(3-1)!$ ways, each couple → $2!$ ways]
- If the seats of the round table are numbered, we will perform the necessary calculations assuming the seats are not numbered, and multiply the value with the number of seats.
  - No. of ways to seat 2 couples and a boy at a round table with numbered seats = $(5-1)!\ \times 5 = 120$
  - No. of ways to seat 2 couples and a boy at a round table if each couple is seated together, and the seats are numbered = $(3-1)!2!2!\ \times 5 = 40$

Reminder: Questions on P&C may involve probability. Read the questions carefully.

<u>Summary for Probability</u>

If the sample space S consists of a finite number of **equally likely** outcomes, then the probability of event A written as $P(A)$ is defined as

$$\mathrm{P}(A) = \frac{\text{no. of elements in } A}{\text{no. of elements in } S} = \frac{\mathrm{n}(A)}{\mathrm{n}(S)}.$$

In layman definition, $\text{required probability} = \dfrac{\text{outcomes we are interested in}}{\text{total possible outcomes}}$.

Important Results

1. $0 \le \mathrm{P}(A) \le 1$, answers should always be between 0 and 1 inclusive AND exact where possible.
2. $\mathrm{P}(A') = 1 - \mathrm{P}(A)$
3. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

   With the help of venn diagram, others (besides Results 2 and 3) can also be easily deduced. Some examples as follow:
   $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(A' \cap B)$
   $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B')$
   or any others involving $A' \cup B, A \cup B', A' \cup B', A \cap B, A' \cap B, A \cap B', A' \cap B'$ and even 3 sets A, B, C.

4. If A and B are **mutually exclusive**, then $\mathrm{P}(A \cap B) = \mathrm{P}(\phi) = 0$, thus $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

Conditional Probability

If A and B are two events and $\mathrm{P}(B) \neq 0$, then the probability of A, given that B has already occurred, is written as $\mathrm{P}(A \mid B)$ and is calculated using the formula:

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Note:

1. $\mathrm{P}(B|A) = \dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$, where $\mathrm{P}(A) \neq 0$.
2. In general, $\mathrm{P}(A|B) \neq \mathrm{P}(B|A)$.
3. $\mathrm{P}(A|B)$ can be considered as the probability of occurrence of event $A \cap B$ with respect to the sample space B i.e. reduced sample space.

   Example (part of **AJC Prelim 9740/2008/02/Q7**):
   3 machines A, B and C produce 25%, 35% and 40% respectively of the golf balls manufactured by a factory. These balls are either yellow or white. Of the balls produced by A and B, 20% and 30% respectively are yellow. It is known that the probability of picking a yellow ball is 0.355. If 3 balls are picked randomly, find the probability that at least 1 is yellow given that all the balls picked are from machine A.

   Using reduced sample space, $\mathrm{P}(\text{at least 1 is yellow}|\text{all from A}) = 1 - 0.8^3 = 0.488$

Independent Events

Two events A and B are said to be **independent** if the occurrence of A does not affect that of B and vice versa.

To prove independence of events A and B, we can show either one of the following

1. $\mathrm{P}(A \mid B) = \mathrm{P}(A)$
2. $\mathrm{P}(B \mid A) = \mathrm{P}(B)$
3. $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

It can be easily shown that if A and B are independent, then the following pairs are also independent A and B', A' and B', A' and B

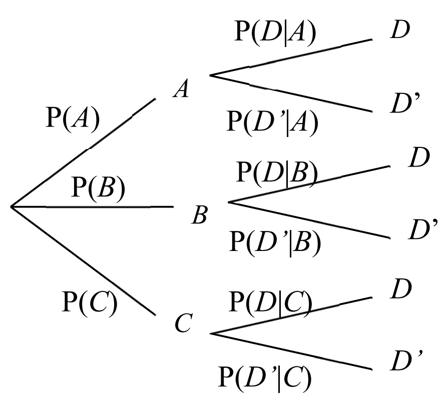
We <u>DO NOT assume independence</u> unless we have proven it or condition is given in the question.

Common <u>mistakes</u>:

Mixed up the tests of "mutually exclusive" and "independence".
Applying $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ even though A and B are <u>NOT independent</u>.

Techniques in calculating probabilities

1. Systematic listing – example 9740/2014/P2/Q10.
2. Venn diagram – example 9740/2015/P2/Q9. This question involves 3 sets.
3. Tree diagram – though a useful technique, many students are lazy to draw the diagram and hence made careless mistakes.



$\mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) = 1$

$\mathrm{P}(A \cap D) = \mathrm{P}(A) \times \mathrm{P}(D \mid A)$

4. P&C – do not be confused when applying this technique. The counting for the denominator has to be consistent with that of the numerator. If order matters for the denominator, then it must matter for the numerator.

Reminder: Probability could easily be embedded in another Statistics question due to its nature. Read question carefully.