HL MATHEMATICS ANALYSIS & APPROACHES - TOPIC 1: NUMBER & ALGEBRA

ARITHMETIC & GEOMETRIC PROGRESSIONS

A.P.: common difference between consecutive terms, d

1st term: a, common difference: d

No. of terms: n, last term: l

$u_n = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$

To prove A.P.: show $u_{n+1} - u_n = d$

or $u_n - u_{n-1} = d$

G.P.: common ratio between consecutive terms, r

1st term: a, common ratio: r

No. of terms: n

$u_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n - 1)}{r-1}$

Sum to infinity $S_\infty = \frac{a}{1-r}$ exists only if $|r| < 1$.

Use of GDC to generate and sum sequences, including recursive functions

Applications: compound interest, annual depreciation, population growth.

SIGMA NOTATION

Used in summation: $\sum_{r=0 \,\to\, \text{starting term}}^{n \,\to\, \text{ending term}}$

Try to write out terms in sum to see the relationship/pattern (if AP/GP/otherwise)

Examples: A.P. $\sum_r kr + d$

$\sum_{r=1}^{5}(2r+3) = 5 + 7 + 9 + 11 + 13$

A.P: 1st term $a=5$, $d=2$, $n=5$

G.P. $\sum_r k^{cr+d}$

$\sum_{r=2}^{7} 3^r = 9 + 27 + \ldots + 3^7$

G.P: 1st term $a=9$, $r=3$, $n=6$

Sum of constant terms

$\sum_{r=1}^{n} k = \underbrace{k + k + \ldots + k}_{n \text{ "}k\text{"s}} = nk$ (no. of terms × constant)

No. of terms in sum = Top − bottom + 1

e.g. $\sum_{r=n}^{m} \rightsquigarrow = m - n + 1$

EXPONENTS & LOGARITHMS

Definition: $\log_a y = x$

$\Rightarrow y = a^x$

Laws of Logarithms:

(1) $\log_a m + \log_a n = \log_a mn$

(2) $\log_a m - \log_a n = \log_a \left(\frac{m}{n}\right)$

(3) $\log_a m^n = n \log_a m$

(4) $\log_a 1 = 0$

(5) $\log_a a = 1$

(6) $a^{\log_a x} = x$

(7) Change of base: $\log_a x = \frac{\log_b x}{\log_b a}$

Recall: $\ln e = 1$

$\ln e^x = x$, $x \ln e = x$

Laws of Exponents:

(1) $x^n \cdot x^m = x^{m+n}$

(2) $(x^n)^m = x^{mn}$

(3) $(xy)^n = x^n y^n$

(4) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

(5) $\frac{x^m}{x^n} = x^{m-n}$

(6) $x^0 = 1$

PERMUTATIONS & COMBINATIONS

No. of ways to choose r objects from n

$= {}^nC_r = \frac{n!}{r!(n-r)!}$

No. of ways to permute r objects from n

$= {}^nP_r = \frac{n!}{(n-r)!}$

BINOMIAL THEOREM (POSITIVE INDEX)

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + b^n$

General term $T_{r+1} = {}^nC_r a^r b^{n-r}$

where ${}^nC_r = \frac{n!}{r!(n-r)!}$

Pascal's Triangle & Binomial Coefficients

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
...

useful for Paper 1

Note: Symmetry!

BINOMIAL THEOREM (NEGATIVE/FRACTIONAL INDEX)

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots+\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r+\dots$$

Infinite series valid only for $|x|<1$. where $n \in \mathbb{Q}$ or $\mathbb{Z}^-$

Useful Results: $(1-x)^{-1} = 1+x+x^2+\dots+x^r\dots$ for $|x|<1$

$(1+x)^{-1} = 1-x+x^2-\dots+(-1)^r x^r\dots$ for $|x|<1$

Ascending powers of x:

$$(\circlearrowleft+x)^{p/q} = \left(\circlearrowleft\left(1+\frac{x}{\circlearrowleft}\right)\right)^{p/q} = \circlearrowleft^{p/q}\left(1+\frac{x}{\circlearrowleft}\right)^{p/q}$$

Descending powers of x:

$$(1+ax)^{p/q} = \left(ax\left(1+\frac{1}{ax}\right)\right)^{p/q} = (ax)^{p/q}\left(1+\frac{1}{ax}\right)^{p/q}$$

PARTIAL FRACTIONS

"Cover-Up" Rule: 1) Let $\frac{ax+b}{(x-p)(x-q)} = \frac{A}{x-p}+\frac{B}{x-q}$

2) Sub. $x=p$ into LHS, covering up $(x-p)$ factor

$A = \frac{ap+b}{p-q}$

3) Similarly, sub. $x=q$ into LHS, covering $(x-q)$

$B = \frac{aq+b}{q-p}$

PROOFS

1) Direct proof

2) ~~Indirect proof - Proof by Contrapositive~~
~~To prove "If P, then Q", we will prove "If not Q, then not P."~~

3) Indirect proof - Proof by Contradiction

Example: Prove that $\sqrt{2}$ is irrational.

(i) Suppose $\sqrt{2}$ is rational, i.e. $\sqrt{2} = \frac{p}{q}$ where p and q have no common factors.

(ii) $p = \sqrt{2}q \Rightarrow p^2 = 2q$

p^2 is a multiple of 2, and so p is a multiple of 2 i.e. $p = 2k$ for some $k \in \mathbb{Z}$

(iii) $2k = \sqrt{2}q \Rightarrow 4k^2 = 2q^2$

$\Rightarrow q^2 = 2k^2$

q^2 is a multiple of 2, and so q is a multiple of 2.

(iv) p and q have common factor of 2 (contradiction)

$\therefore$ Thus $\sqrt{2}$ is irrational.

4) Disproving a statement using a Counter-example.

MATHEMATICAL INDUCTION

- A type of proof
- Possible non-exhaustive topics: Summation, Divisibility, Inequalities, Differentiation, Complex No.s, Trigonometry
- Ensure proper presentation!

Example:

Let P_n be the proposition that $\sum_{r=1}^{n} 2^r = 2^{n+1}-2$ for $n \in \mathbb{Z}^+$

$n=1$: LHS $= 2^1 = 2$

RHS $= 2^{1+1}-2 = 4-2 = 2$

Since LHS = RHS, P_1 is true. ✓

$n=k$: Assume P_k is true for some $n=k$, $k \in \mathbb{Z}^+$

i.e. $\sum_{r=1}^{k} 2^r = 2^{k+1}-2$ is true. ✓

$n=k+1$: To prove P_{k+1} is true:

i.e. $\sum_{r=1}^{k+1} 2^r = 2^{k+2}-2$ is true.

LHS $= \sum_{r=1}^{k+1} 2^r$

$= \sum_{r=1}^{k} 2^r + 2^{k+1}$ (using Induction Hypothesis)

$= 2^{k+1}-2+2^{k+1}$

$= 2(2^{k+1})-2$

$= 2^{k+2}-2 =$ RHS ✓

Thus P_{k+1} is true whenever P_k is true. Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$ i.e. $\sum_{r=1}^{n} 2^r = 2^{n+1}-2$ is true. □

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

Manual rref → upper triangular matrix

Make these entries zeroes (row operations)

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

For Paper 2 GDC rref

(A) Unique solutions (intersect at a point) $\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix}$

(B) Infinitely many solutions (intersect at a line) $\begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(C) No Solutions

$\begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 0 & c \end{pmatrix}$ (System is inconsistent)

↳ 3 planes do not intersect

L MATHEMATICS ANALYSIS & APPROACHES - TOPIC 1: NUMBER & ALGEBRA (cont'd)

COMPLEX NUMBERS

Definition: $i^2 = -1$ or $i = \sqrt{-1}$

$z = x + iy$ where $Re(z) = x$, $Im(z) = y$

Conjugate of z: $z^* = x - iy$

Addition & Subtraction

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$
$= (x_1 + x_2) + i(y_1 + y_2)$

$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$
$= (x_1 - x_2) + i(y_1 - y_2)$

Multiplication & Division

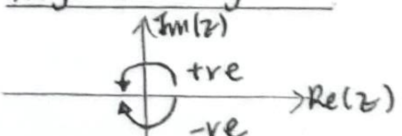
$z_1 \times z_2 = (x_1 + iy_1)(x_2 + iy_2)$
$= x_1x_2 + i(x_1y_2 + x_2y_1) + i^2y_1y_2$
$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$ multiply by conjugate of denominator

$= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$

→ "real-ise" the denominator

Argand Diagram



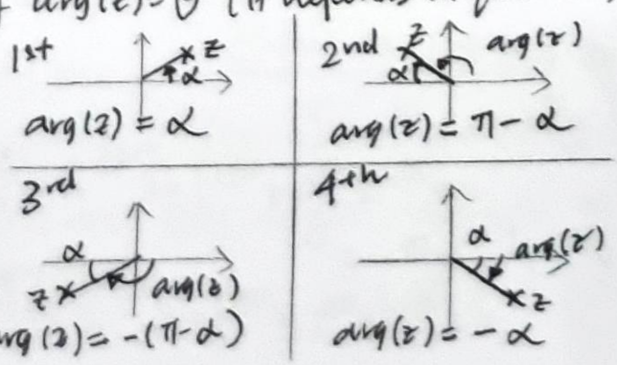
$|z| = r = \sqrt{x^2 + y^2}$ = modulus of z

$arg(z) = \theta$ = angle z makes with reference to positive real axis

↳ positive (clockwise sence)

Range of θ: $-\pi < \underset{\theta}{arg(z)} \leq \pi$

Always sketch Argand diagram in order to get the correct value of $arg(z) = \theta$ (it depends on quadrant)



3 FORMS OF COMPLEX NUMBERS

1) Cartesian Form

$z = x + iy$ where $Re(z) = x$, $Im(z) = y$

2) Modulus-Argument/Polar Form (Trigo)

$z = r(\cos\theta + i\sin\theta)$
$= r \text{ cis } \theta$ where $r = |z|$ (modulus) and $\theta = arg(z)$ (argument)

3) Euler (Exponential) Form

$z = re^{i\theta}$ where $r = |z|$, $\theta = arg(z)$

Need to know how to convert between forms.

Geometrical Interpretation

* Complex Number $i = e^{i\frac{\pi}{2}}$
Multiplication by $i \rightarrow iz$
is a $\frac{\pi}{2}$/90° anti-clockwise rotation of the complex number z about the origin.
* Complex Number $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$
Multiplication by $(1+i) \rightarrow (1+i)z$
is a $\frac{\pi}{4}$/45° anti-clockwise rotation of the complex number z about the origin and a scale factor of $\sqrt{2}$.

Use of GDC

GDC may be used to calculate modulus, argument, 4 operations of complex numbers in P2 & P3.

Properties of Modulus & Argument

For $z = x + iy = r \text{ cis } \theta$

(a) $|z^*| = |z|$

(b) $arg(z^*) = -arg(z)$

(c) $|z|^2 = zz^* = z^*z$

Then $z^* = x - iy = r \text{ cis}(-\theta)$

For $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$

(a) $|z_1z_2| = r_1r_2 = |z_1||z_2|$

(b) $arg(z_1z_2) = \theta_1 + \theta_2 = arg(z_1) + arg(z_2)$

(c) $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$

(d) $arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = arg(z_1) - arg(z_2)$

De Moivre's Theorem

$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

where $z = r(\cos\theta + i\sin\theta)$

i.e. $|z^n| = r^n = |z|^n$

and $\arg(z^n) = n\theta = n\arg(z)$

Expressing $\sin n\theta, \cos n\theta, \tan n\theta$

- in terms of $\sin\theta, \cos\theta, \tan\theta$
- use Binomial Expansion

$\cos 5\theta = Re(\cos 5\theta + i\sin 5\theta)$

$= Re(c+is)^5$ letting $c = \cos\theta$, $s = \sin\theta$

$= Re(c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5)$

$= c^5 - 10c^3s^2 + 5cs^4$

Similarly, $\sin 5\theta = Im(c+is)^5$

$= 5c^4s - 10c^2s^3 + s^5$

$\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta} = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$ divide throughout by c^5

$= \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

Expressing $\sin^n\theta, \cos^n\theta$

- in terms of multiple angles of sine & cosine

$$\boxed{\begin{array}{l} z + \frac{1}{z} = 2\cos\theta \text{ and } z - \frac{1}{z} = 2i\sin\theta \\ z^n + \frac{1}{z^n} = 2\cos n\theta \text{ and } z^n - \frac{1}{z^n} = 2i\sin n\theta \end{array}}$$

$(2\cos\theta)^5 = (z + \frac{1}{z})^5$

$32\cos^5\theta = z^5 + 5z^3 + 10z + 10(\frac{1}{z}) + 5(\frac{1}{z^3}) + (\frac{1}{z^5})$

$= (z^5 + \frac{1}{z^5}) + 5(z^3 + \frac{1}{z^3}) + 10(z + \frac{1}{z})$

$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\therefore \cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

$(2i\sin\theta)^5 = (z - \frac{1}{z})^5$

$32i\sin^5\theta = z^5 - 5z^3 + 10z - 10(\frac{1}{z}) + 5(\frac{1}{z^3}) - (\frac{1}{z^5})$

$= (z^5 - \frac{1}{z^5}) - 5(z^3 - \frac{1}{z^3}) + 10(z - \frac{1}{z})$

$= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$

$\therefore \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Roots of Unity

Solutions of $z^n = 1$.

Form n-sided polygon with internal radius 1 with angle $\frac{2\pi}{n}$ between adjacent vertices

1. Write 1 as e^{i0}.
2. Add $2k\pi$ to argument where $k = 0, \pm1, \pm2, \dots$ (n values of k)
3. Take power $\frac{1}{n}$ on both sides.
4. List values of roots for each value of k.

Example: $z^5 = e^{i0}$

$z^5 = e^{i(2k\pi)}$, $k = 0, \pm1, \pm2$

$(z^5)^{\frac{1}{5}} = e^{i(\frac{2k\pi}{5})}$, $k = 0, \pm1, \pm2$

$\therefore z = e^{i0}, e^{i(\pm\frac{2\pi}{5})}, e^{i(\pm\frac{4\pi}{5})}$

Solution of $z^n = x + iy$

1. Convert $x + iy \rightarrow re^{i\theta}$
2. Add $2k\pi$ to argument where $k = 0, \pm1, \pm2, \dots$ (n values of k)
3. Take power $\frac{1}{n}$ on both sides
4. List values of roots for each value of k.

Example: $z^3 = 2 + 2i$

$z^3 = 2\sqrt{2}\, e^{i(\frac{\pi}{4})}$

$z^3 = 2\sqrt{2}\, e^{i(\frac{\pi}{4} + 2k\pi)}$, $k = 0, \pm1$

$(z^3)^{1/3} = (2^{3/2} e^{i(\frac{\pi}{4} + 2k\pi)})^{1/3}$

$\therefore z = 2^{1/2}e^{i\frac{\pi}{12}}, 2^{1/2}e^{i(\frac{3\pi}{4})}, 2^{1/2}e^{i(\frac{-7\pi}{12})}$

Conjugate Roots of Equations

For polynomial equations with all real coefficients, if complex roots occur, they occur in conjugate pairs.

Useful Results

$e^{i\theta} + 1 = e^{i\theta} + e^{i0}$

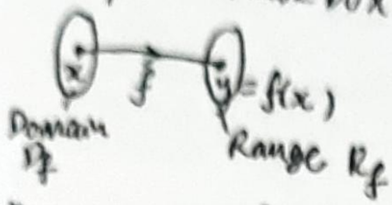
$= e^{i\theta/2}(e^{i\theta/2} + e^{-i\theta/2}) = e^{i\theta/2}\, 2\cos\frac{\theta}{2}$

$e^{i\theta} - 1 = e^{i\theta} - e^{i0}$

$= e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2}) = e^{i\theta/2}\, 2i\sin\frac{\theta}{2}$

FUNCTIONS

- Concept: "black box"

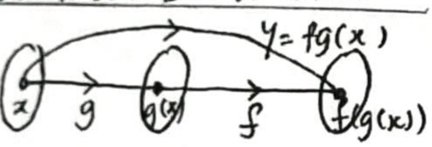


- Presentation: $f: x \mapsto$ or $f(x) =$
- Range of function R_f → look at y-values

ODD/EVEN FUNCTIONS

- Odd function: symmetry about origin
$f(-x) = -f(x)$
Examples: $y = x^3$, $y = \sin x$
- Even function: symmetry about y-axis
$f(x) = f(-x)$
Examples: $y = x^2$, $y = \cos x$

COMPOSITE FUNCTIONS



- Composite function $y = f(g(x))$
- Domain of fg = Domain of g
- Composite function $y = f(g(x))$ exists only if $R_g \subseteq D_f$
- Range of fg = Range of f when the Domain of f is restricted to Range of g.
  ↳ or find range by sketching $y = fg(x)$.

ONE-TO-ONE FUNCTIONS

- Check if function is one-to-one by using Horizontal Line Test:

Any line $y = k$, $k \in$ Range of f, cuts the graph of $y = f(x)$ at one point only.

- Just need any counter-example to disprove.

INVERSE FUNCTIONS

- Inverse functions $y = f^{-1}(x)$ exists only if $y = f(x)$ is one-to-one.
- How to find inverse functions:
(1) Let $y = f(x)$.
(2) Make x the subject of the formula.
(3) Change all y's to x in final form.
- Check that $D_f = R_{f^{-1}}$ and $R_f = D_{f^{-1}}$

Example:

Find the inverse of $f(x) = (x-2)^2 + 5$, $x \leq 2$

(1) $y = (x-2)^2 + 5$

(2) $(x-2)^2 = y - 5$

$x - 2 = \pm\sqrt{y-5}$

$x = 2 \pm \sqrt{y-5}$

(3) $f^{-1}(x) = 2 \pm \sqrt{x-5}$

But D_f is $x \leq 2$

$R_{f^{-1}} = D_f \Rightarrow x \leq 2$

$\therefore f^{-1}(x) = 2 - \sqrt{x-5}$, $x \geq 5$ ($\because D_{f^{-1}} = R_f$)

RELATIONSHIP BETWEEN f AND f^{-1}

$y = f^{-1}(x)$ is a reflection of $y = f(x)$ about the line $y = x$.

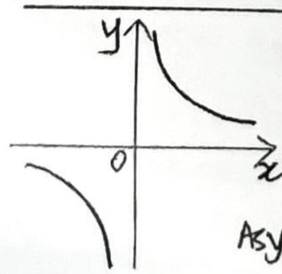
$D_f = R_{f^{-1}}$ and $R_f = D_{f^{-1}}$

SELF-INVERSE FUNCTION

A self-inverse function is one which is the inverse of itself, i.e. $f^{-1}(x) = f(x)$.

Example: $f(x) = \frac{x}{x-1}$, $x > 1$

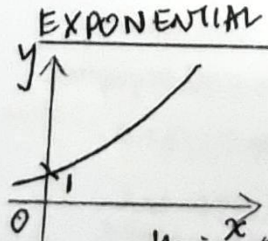
RECIPROCAL FUNCTION



$y = \frac{1}{x}$, $x \neq 0$
no x & y intercepts

Asymptotes: $x = 0$, $y = 0$

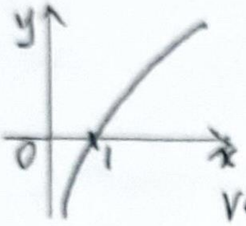
EXPONENTIAL FUNCTION



$y = e^x$, $y = a^x$, $a > 0$
general shape
(0, 1) y-intercept
Horizontal asymptote: $y = 0$

HL MATHEMATICS ANALYSIS & APPROACHES - TOPIC 2: FUNCTIONS (cont'd)

LOGARITHMIC FUNCTION



$y = \log_a x$, $x > 0$
$y = \ln x$, $x > 0$
x-intercept: $(1, 0)$
Vertical asymptote: $x = 0$

NOTE: For $0 < x < 1$, y-values are negative.

RELATIONSHIP BETWEEN EXPONENTIAL & LOGARITHMIC

$a^x = e^{x \ln a}$ $\quad \log_a a^x = x$

$a^{\log_a x} = x$, $x > 0$ $\quad e^{\ln x} = x$, $x \ln e = x$

- $y = \ln x$ and $y = e^x$ are inverse functions
- Their graphs are reflections in the line $y = x$.

GRAPHING TECHNIQUES

- Use of GDC to sketch, zoom in, zoom out, key in domain of graph, find min/max/intersections/intercepts, horizontal/vertical asymptotes, symmetry, consider domain/range.
- Restrict domain of graph in plotting $\boxed{\text{Ctrl}}$ $\boxed{=}$ "|" and "≤" and "≥"

GRAPHS OF FUNCTIONS

1) $y = |f(x)|$ - Keep 1st & 2nd quadrants
   - reflect 3rd & 4th quadrants about x-axis.

2) $y = f(|x|)$ - Keep 1st & 4th quadrants
   - Reflect about y-axis.

3) $y = \frac{1}{f(x)}$ given $y = f(x)$

Plotting reciprocal graph with original graph

$y = f(x)$	$y = \frac{1}{f(x)}$
$f(x)$ increases	$\frac{1}{f(x)}$ decreases
$f(x)$ decreases	$\frac{1}{f(x)}$ increases
(a, b) maximum	$(a, \frac{1}{b})$ minimum
(a, b) minimum	$(a, \frac{1}{b})$ maximum
x-intercept: $x = a$	$x = a$ vertical asymptote
$x = a$ vertical asymptote	x-intercept: $x = a$
$y = k$ horizontal asymptote	$y = \frac{1}{k}$ horizontal asymptote

4) $y = (f(x))^2$ given $y = f(x)$.

All y-values are squared.

$y = f(x)$	$y = (f(x))^2$
$f(x)$ increases	$(f(x))^2$ increases
$f(x)$ decreases	$(f(x))^2$ decreases
$b > 0$:	$b > 0$:
(a, b) maximum	(a, b^2) maximum
(a, b) minimum	(a, b^2) minimum
$b < 0$:	$b < 0$:
(a, b) maximum	(a, b^2) minimum
(a, b) minimum	(a, b^2) maximum
$x = a$ vertical asymptote	$x = a$ vertical asymptote
$y = b$ horizontal asymptote	$y = b^2$ horizontal asymptote

Note: $f(x) > 1$, then $f(x) < (f(x))^2$
$0 < f(x) < 1$, then $f(x) > (f(x))^2$

5) $y = f'(x)$ given $y = f(x)$ (vice versa)

- Consider gradient of $y = f(x)$ by drawing tangents to curve at different points.
- $f'(x) = 0$ at min/max turning points.

TRANSFORMATION OF GRAPHS

(1) Translations: $y = f(x) \pm b$ (up/down)
$y = f(x \pm a)$ (left/right)

(2) Reflections: $y = -f(x)$ (in x-axis)
$y = f(-x)$ (in y-axis)

(3) Vertical stretch with scale factor p:
$y = p f(x)$

(4) Horizontal stretch with scale factor $\frac{1}{q}$:
$y = f(qx)$

(5) Remember: "x is a rebel!"

(6) Composite Transformations:
Order: $y = \underset{3}{c}\, f(\underset{2}{b}x + \underset{1}{a}) + \underset{4}{d}$

Know how to geometrically describe transformations!

RATIONAL FUNCTIONS

$$y = \frac{ax+b}{cx+d}$$

How to find asymptotes of rational functions

(1) Long division / "create denominator in numerator"

(2) $y = k + \frac{e}{cx+d}$ } this is a proper fraction: degree of numerator < degree of denominator

k → Horizontal asymptote: $y=k$

$cx+d$ → vertical asymptote: $x = \frac{-d}{c}$

Example: $y = \frac{x+7}{2x-5}$

(1) $y = \dfrac{\frac{1}{2}(2x-5)+\frac{19}{2}}{2x-5} = \frac{1}{2} + \frac{19}{2(2x-5)}$

(2) Horizontal asymptote: $y=\frac{1}{2}$

vertical asymptote: $x = \frac{5}{2}$

You may go on to find x/y intercepts for sketch.

Example: $y = \frac{4}{3x-2} = 0 + \frac{4}{3x-2}$

Horizontal asymptote: $y=0$; vertical asymptote: $x=\frac{2}{3}$

NOTE: You may perform a series of transformations from the reciprocal graph $y=\frac{1}{x}$, $x\neq 0$ to obtain sketch.

REMAINDER & FACTOR THEOREM

If a polynomial $P(x)$ is divided by $(x-a)$, then the remainder is $P(a)$.

If a polynomial $P(x)$ is divided by $(ax-b)$, then the remainder is $P(\frac{b}{a})$, provided $a \neq 0$.

$(x-a)$ is a factor of polynomial $P(x) \Leftrightarrow P(a)=0$.

* Need to know how to factorize $P(x)$, using long division / synthetic division
* Need to know how to distinguish between the root of a polynomial eqn & a factor of a polynomial.

FUNDAMENTAL THEOREM OF ALGEBRA

Every polynomial of degree n has at most n roots.

THEORY OF EQUATIONS

Quadratic equation: $ax^2+bx+c=0$

Sum of roots $\alpha+\beta = -\frac{b}{a}$

Product of roots $\alpha\beta = \frac{c}{a}$

Cubic equation: $ax^3+bx^2+cx+d=0$

Sum of roots $\alpha+\beta+\gamma = -\frac{b}{a}$

Product of roots $\alpha\beta\gamma = (-1)^3\frac{d}{a}$

Useful Results:

$\alpha\beta+\beta\gamma+\alpha\gamma = \frac{c}{a}$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

To find new cubic polynomial:

(1) Find relation between the new roots (y) and old roots (x).

(2) Substitute $x=f(y)$ into the original polynomial equation

OR: Expand using new roots.

SOLUTION OF POLYNOMIAL INEQUALITIES

TABLE METHOD

Example: $(x+7)(x-2)(x-4) > 0$

	$x<-7$	$-7<x<2$	$2<x<4$	$x>4$
$x+7$	−	+	+	+
$x-2$	−	−	+	+
$x-4$	−	−	−	+
$(x+7)(x-2)(x-4)$	−	+	−	+

$\therefore$ Solution is $-7<x<2$ or $x>4$ //

RATIONAL INEQUALITIES

* Do not cross-multiply!
* Bring all to LHS
* Apply Table method.

MODULUS INEQUALITIES

* Use of GDC to solve
* Do not need to know analytical methods.

HL MATHEMATICS ANALYSIS & APPROACHES - TOPIC 3 : GEOMETRY & TRIGONOM

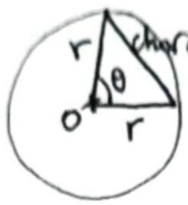
VOLUME & SURFACE AREA

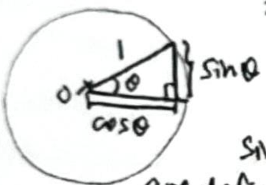
3D SOLIDS such as pyramid, cone, sphere, cylinder etc.

CIRCULAR MEASURE

- Radians : $\pi \text{ rad} = 180°$
- chord
- Sector (slice of pizza)
- Segment (crust)
- Arc length $s = r\theta$
- Area of sector $= \frac{1}{2}r^2\theta$
- Area of segment = Area of sector - Area of $\triangle$

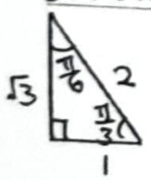
$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$





Need to know how $\sin\theta$, $\cos\theta$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$ are defined in terms of unit circle.

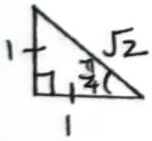
SPECIAL ANGLES



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$ $\quad \sin\frac{\pi}{6} = \frac{1}{2}$

$\tan\frac{\pi}{3} = \sqrt{3}$ $\quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$



$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\tan\frac{\pi}{4} = 1$

Need to know these special angles in all other combinations in 2nd/3rd/4th quadrants.

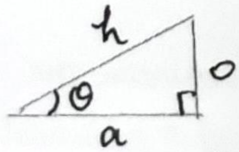
RECIPROCAL TRIGONOMETRIC RATIOS

$\sec\theta = \frac{1}{\cos\theta}$ $\quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $\quad \cot\theta = \frac{1}{\tan\theta}$

PYTHAGOREAN IDENTITIES

Pythagoras' Theorem

$o^2 + a^2 = h^2$



Identities: $\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

COMPOUND ANGLE IDENTITIES

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$\tan(A\pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

DOUBLE ANGLE IDENTITIES

$\sin 2x = 2\sin x\cos x$

$\cos 2x = 2\cos^2 x - 1$

$= 1 - 2\sin^2 x$

$= \cos^2 x - \sin^2 x$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

HALF ANGLE IDENTITIES

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\cos x = 2\cos^2\frac{x}{2} - 1$

$= 1 - 2\sin^2\frac{x}{2}$

$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

SOLUTION OF TRIGO EQUATIONS

1. Find basic angle α.
2. Decide which quadrants $\begin{array}{c|c} S & A \\ \hline T & C \end{array}$
3. 1st: $x = \alpha$, 2nd: $x = \pi - \alpha$

   3rd: $x = \pi + \alpha$, 4th: $x = 2\pi - \alpha$

Examples: $2\sin x = 1$, $0 \le x \le 2\pi$

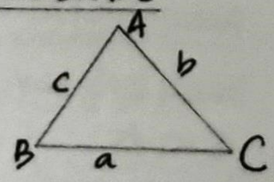
$2\sin 2x = 3\cos x$, etc.

SOLUTION OF TRIANGLES

Cosine Rule :

$c^2 = a^2 + b^2 - 2ab\cos C$

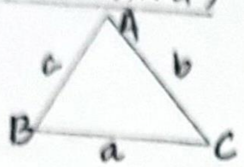
or $\cos C = \frac{a^2+b^2-c^2}{2ab}$



SOLUTION OF TRIANGLES (cont'd)

Sine Rule:

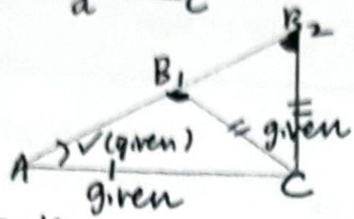
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Ambiguous case

Given: AC, BC, $\angle BAC$

2 possible Δs formed



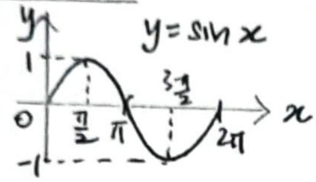
$\angle B_2 \rightarrow$ acute ; $\angle B_1 \rightarrow$ obtuse

TRIGONOMETRIC GRAPHS

$y = \sin x$

Period $= 2\pi$

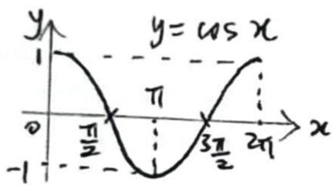
Range is $-1 \leq y \leq 1$



$y = \cos x$

Period $= 2\pi$

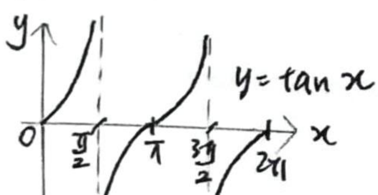
Range is $-1 \leq y \leq 1$



$y = \tan x$

Period $= \pi$

Range is $\mathbb{R}$



Asymptotes are $x = \pm\frac{k\pi}{2}$, k odd

TRANSFORMATION OF TRIGO GRAPHS

$y = a\sin(b(x+c)) + d$

Period of graph $= \frac{2\pi}{b}$ (for sin, cos)

or $= \frac{\pi}{b}$ (for tan)

Amplitude (sin, cos) $= a$

OTHER APPLICATIONS OF TRIGONOMETRY

(1) BEARINGS
- Always draw $\overset{N}{\uparrow}$ for reference
- clockwise from North (3 digits)

(2) ANGLES OF ELEVATION & DEPRESSION
- Always draw diagram
- look for 90°, use of $\tan\theta$ (usually)

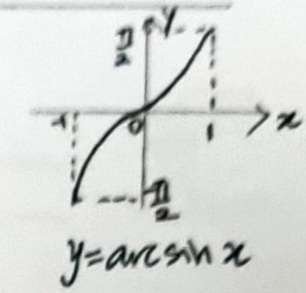
(3) 3D TRIGO - look for 90°, use TOA CAH SOH.

INVERSE TRIGO FUNCTIONS

$x \mapsto \arcsin x$

Domain $= [-1, 1]$

Range $= [-\frac{\pi}{2}, \frac{\pi}{2}]$

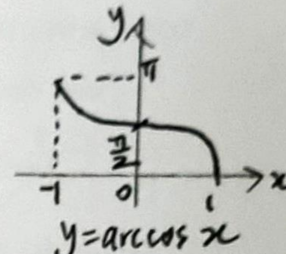


$y = \arcsin x$

$x \mapsto \arccos x$

Domain $= [-1, 1]$

Range $= [0, \pi]$



$y = \arccos x$

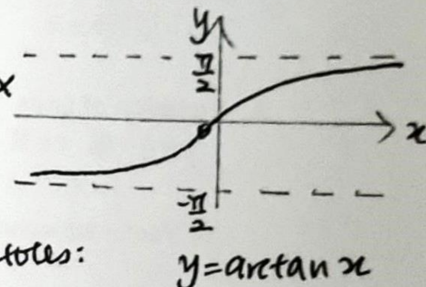
$x \mapsto \arctan x$

Domain $= \mathbb{R}$

Range $=]-\frac{\pi}{2}, \frac{\pi}{2}[$

Horizontal asymptotes:

$x = \pm\frac{\pi}{2}$.



$y = \arctan x$

* Need to know the domains, principal ranges and graphs for the above 3 inverse trigo functions.

NOTE: Graphs have to be one-to-one for inverse functions to exist. Principal ranges ensure that.

IMPORTANT TRIGO RELATIONSHIPS

$\cos\theta = \sin(\frac{\pi}{2} - \theta)$

$\sin\theta = \cos(\frac{\pi}{2} - \theta)$

$\tan\theta = \cot(\frac{\pi}{2} - \theta)$

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

$\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

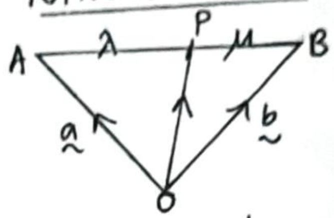
$\tan(\pi + \theta) = \tan\theta$

$\sin(2\pi - \theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos\theta$

$\tan(2\pi - \theta) = -\tan\theta$.

RATIO THEOREM

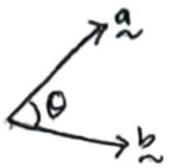


$$\overrightarrow{OP} = \frac{\mu \underset{\sim}{a} + \lambda \underset{\sim}{b}}{\mu + \lambda}$$

TOPIC 4: VECTORS – SUMMARY OF FORMULAE

1. **Unit Vectors**

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$$

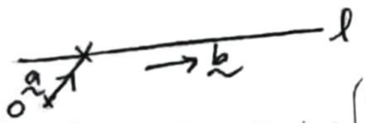


2. **Scalar or Dot Product**

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

3. **Equation of lines**

$l : \underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{b},\ t \in \mathbb{R}$



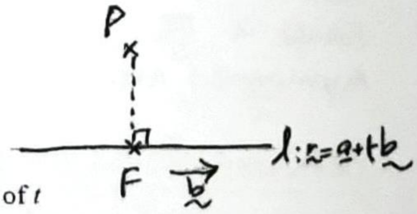
(a) Vector equation of line: $l : \underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{b},\ t \in \mathbb{R} \Rightarrow l : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\ t \in \mathbb{R}$

(b) Parametric equation of line: $x = a_1 + tb_1,\ y = a_2 + tb_2,\ z = a_3 + tb_3,\ t \in \mathbb{R}$

(c) Cartesian equation of line: (make t the subject of the formula and equate from parametric)

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

4. **Finding Foot of Perpendicular from point to line**



(i) F lies on the line l: $\overrightarrow{OF} = \underset{\sim}{a} + t\underset{\sim}{b}$ for some $t \in \mathbb{R}$

(ii) Find the vector $\overrightarrow{PF}$: $\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \underset{\sim}{a} + t\underset{\sim}{b} - \underset{\sim}{p}$

(iii) Use the fact that $\overrightarrow{PF} \cdot \underset{\sim}{b} = 0$: $[\underset{\sim}{a} + t\underset{\sim}{b} - \underset{\sim}{p}] \cdot \underset{\sim}{b} = 0 \Rightarrow$ find value of t

(iv) Substitute value of t to find coordinates of F.

5. **Parallel, Intersecting or Skew Lines**

(i) Check for parallel lines (if direction vectors are parallel, then parallel)

(ii) If non-parallel, check for intersection
- Use GDC to solve system of linear equations (rref)
- Solving manually: use 2 of 3 equations, check 3rd equation for consistency
- Find coordinates of point of intersection

(iii) If non-parallel and non-intersecting, then SKEW lines.

6. **Vector ~~of~~ or Cross Product**

$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta\,\hat{\underset{\sim}{n}}$ where $\hat{\underset{\sim}{n}}$ is a unit vector perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

COLLINEARITY

3 points A, B, C are collinear if $\vec{AB} = k\vec{AC}$, $k \in \mathbb{R}$

COPLANAR

4 points A, B, C, D are coplanar if $\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0$

7. Formulae for Vector Product

(i) Area of triangle $= \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$

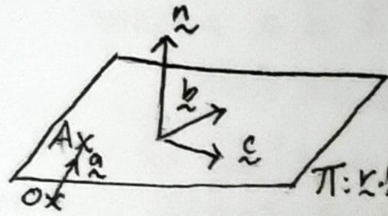
(ii) Area of parallelogram $= |\underset{\sim}{a} \times \underset{\sim}{b}|$

(iii) Volume of parallelepiped $= |\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$

(iv) Volume of tetrahedron $= \frac{1}{6}|\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})|$

8. Equation of planes

$\Pi : \underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n} = d$



$\Pi : \underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$

(a) (Normal) Vector equation of plane:

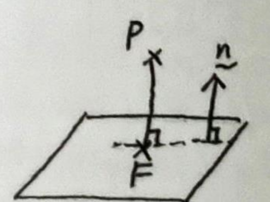
$$\Pi : \underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n} = d \quad \Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = an_1 + bn_2 + cn_3 = d$$

(b) Cartesian equation of plane: $ax + by + cz = d$

(c) Vector equation of plane: (where $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are non-parallel vectors in the plane)

$\Pi : \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}, \quad \lambda, \mu \in \mathbb{R}$

9. Finding Foot of Perpendicular from point to plane



$\Pi : \underset{\sim}{r} \cdot \underset{\sim}{n} = d$

(i) F lies on the plane Π: $\overline{OF} \cdot \underset{\sim}{n} = d$

(ii) Find the equation of the line $PF : l : \underset{\sim}{r} = \underset{\sim}{p} + t\underset{\sim}{n}, \ t \in \mathbb{R}$

(iii) F lies on the line PF : $\overline{OF} = \underset{\sim}{p} + t\underset{\sim}{n}$ for some $t \in \mathbb{R}$

(iv) Use the fact that $\overline{OF} \cdot \underset{\sim}{n} = d$: $(\underset{\sim}{p} + t\underset{\sim}{n}) \cdot \underset{\sim}{n} = d \quad \Rightarrow$ find value of t

(v) Substitute value of t to find coordinates of F.

10. Angles Formulae (for acute angle, mod numerator)

(i) Angle between two lines: $\cos\theta = \frac{\underset{\sim}{b_1} \cdot \underset{\sim}{b_2}}{|\underset{\sim}{b_1}||\underset{\sim}{b_2}|}$

(ii) Angle between line and plane: $\sin\theta = \frac{\underset{\sim}{b} \cdot \underset{\sim}{n}}{|\underset{\sim}{b}||\underset{\sim}{n}|}$

(iii) Angle between two planes: $\cos\theta = \frac{\underset{\sim}{n_1} \cdot \underset{\sim}{n_2}}{|\underset{\sim}{n_1}||\underset{\sim}{n_2}|}$

$\perp$ DISTANCE FROM O TO Π

Plane equation $\Pi : \underset{\sim}{r} \cdot \underset{\sim}{n} = D$

$\perp$ distance from O to Π is $\frac{|D|}{|\underset{\sim}{n}|}$

11. Intersection of 3 Planes: (use GDC rref to solve)

(a) Unique solution (point of intersection → must find coordinates)

(b) Infinite number of solutions (line of intersection → find vector equation of line)

(c) No solution (last row is not all zeroes)

GDC syntax

rref $\boxed{\text{MENU}}$ $\boxed{7}$ $\boxed{1}$ $\boxed{1}$

linSolve $\boxed{\text{MENU}}$ $\boxed{3}$ $\boxed{2}$

DISTANCE BETWEEN PARALLEL PLANES

$\Pi_1 : \underset{\sim}{r} \cdot \underset{\sim}{n} = D_1$

$\Pi_2 : \underset{\sim}{r} \cdot \underset{\sim}{n} = D_2$

$\perp$ distance between Π_1 & Π_2

$= \frac{|D_1 - D_2|}{|\underset{\sim}{n}|}$

DESCRIPTIVE STATISTICS

POPULATION vs SAMPLE

MEAN μ vs $\bar{x}$

VARIANCE σ^2 vs s^2

CORRELATION COEFFICENT ρ vs r

CALCULATION OF $\bar{x}$ AND σ_x

Sample mean $\bar{x}$ is a measure of location of the data.

Variance σ_x^2 or s^2 (or SD σ_x / s) is a measure of spread of the data.

*CALCULATION USING GDC

Key in data (use mid-class interval for grouped data)

x	f
⋮	⋮

$\boxed{\text{MENU}}\ \boxed{4}\ \boxed{1}\ \boxed{1}$
One-variable statistics

Read off $\bar{x}$ (sample mean) and σ_x (sample standard deviation)

*CALCULATION USING FORMULAE

$$\bar{x} = \frac{\Sigma x}{n} \quad \text{or} \quad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

$$\sigma_x^2 = \frac{\Sigma x^2 - n\bar{x}^2}{n} = \frac{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}{n} = \frac{\Sigma (x_i - \bar{x})^2}{n}$$

$$= \frac{\Sigma f_i (x_i - \bar{x})^2}{\Sigma f_i} = \frac{\Sigma f_i x_i^2}{n} - \bar{x}^2$$

*MEAN, MEDIAN and MODE

Need to know each of the above measures of central tendency

*GDC - BOX PLOTS

1) "Lists & spreadsheets"
2) Key in data

x
⋮

3) Highlight data (A→▲)
4) $\boxed{\text{MENU}}\ \boxed{3}\ \boxed{9}$ Quick Graph
5) $\boxed{\text{DOC}}\ \boxed{5}\ \boxed{8}$ Ungroup
6) $\boxed{\text{MENU}}\ \boxed{1}\ \boxed{2}$ Boxplot

Outliers
beyond
$Q_3 + 1.5 \times IQR$
$Q_1 - 1.5 \times IQR$

SAMPLING TECHNIQUES

(1) PROBABILITY SAMPLING
- Simple random sampling
- Stratified random sampling
- Systematic sampling

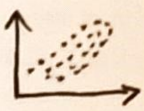
(2) NON-PROBABILITY SAMPLING
- Quota sampling
- Convenience sampling

* Know how to describe each method and briefly mention pros & cons.

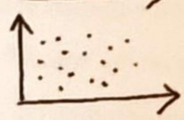
CORRELATION & REGRESSION

Population product moment correlation coefficient

$$\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$$

$\rho > 0$  As x increases, y increases

$\rho < 0$  As x increases, y decreases

$\rho = 0$  No linear correlation between x & y

(1) If X, Y are independent, then $\rho = 0$
(2) If X, Y are linearly related, then $\rho = \pm 1$

Sample product moment correlation coefficient

$$r = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\Sigma (x_i - \bar{x})^2 \Sigma (y_i - \bar{y})^2}} = \frac{\Sigma x_i y_i - n\bar{x}\bar{y}}{\sqrt{(\Sigma x_i^2 - n\bar{x}^2)(\Sigma y_i^2 - n\bar{y}^2)}}$$

* Line of y on x: $y - \bar{y} = b(x - \bar{x})$

where $b = \dfrac{\Sigma x_i y_i - n\bar{x}\bar{y}}{\Sigma x_i^2 - n\bar{x}^2}$

* Line of x on y: $x - \bar{x} = d(y - \bar{y})$

where $d = \dfrac{\Sigma x_i y_i - n\bar{x}\bar{y}}{\Sigma y_i^2 - n\bar{y}^2}$

* Both regression lines pass through $(\bar{x}, \bar{y})$

* Relation: $r^2 = bd$

* GDC: $\boxed{\text{MENU}}\ \boxed{4}\ \boxed{1}\ \boxed{3}$

LINEAR REGRESSION (cont'd)

- Comment on validity of regression lines in predicting data (EXTRAPOLATION?)
- Comment on sign and strength of correlation coefficient r.

Range of $\lvert r\rvert$	Adjective (strength)
$\lvert r\rvert = 1$	perfect
$0.95 \le \lvert r\rvert < 1$	very strong
$0.87 \le \lvert r\rvert < 0.95$	strong
$0.5 \le \lvert r\rvert < 0.87$	moderate
$0.1 \le \lvert r\rvert < 0.5$	weak
$0 \le \lvert r\rvert < 0.1$	no

PROBABILITY

Definition: $P(A) = \dfrac{n(A)}{n(\mathcal{E})}$

where $\mathcal{E}$ is the sample space.

$0 \le P(A) \le 1$ for any event A

Complement: $P(A') = 1 - P(A)$

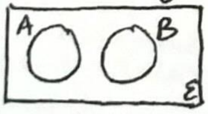
Can use P & C to calculate $n(A)$ & $n(\mathcal{E})$

Addition Law:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Mutually Exclusive events are events that cannot occur simultaneously.

i.e. $P(A \cap B) = 0$



So we have

$P(A \cup B) = P(A) + P(B)$

Conditional Probability:

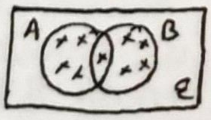
$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ a problem of reduced sample space

Look out for "given"/condition in qn.

Bayes' Theorem:

$$P(B_i|A) = \frac{P(B_i) \times P(A|B_i)}{P(A)} = \frac{P(B_i) \times P(A|B_i)}{\sum_{i=1}^{m} P(B_i) \times P(A|B_i)}$$

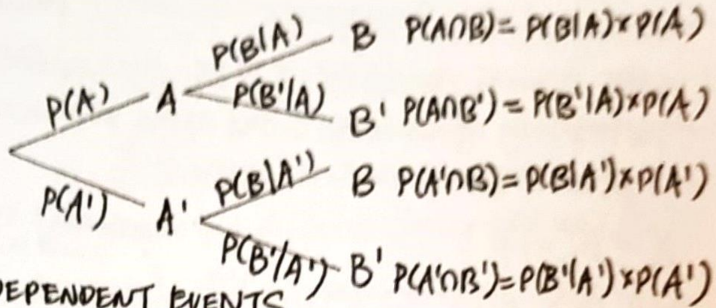
Exhaustive events include all possible outcomes in the sample space $\mathcal{E}$. $P(A \cup B) = 1$



Probability Tree Diagrams:

Sum across branches add up to 1.

Second event branches (and beyond) → conditional probability



INDEPENDENT EVENTS

Events A & B are independent if

$P(A \cap B) = P(A) \times P(B)$; $P(A|B) = P(A)$ or $P(B|A) = P(B)$

Total Probability Theorem: $P(A) = \sum_{i=1}^{m} P(A \cap B_i) = \sum_{i=1}^{m} P(B_i) \times P(A|B_i)$ for $B_1, B_2, \dots B_m$ mutually exclusive exhaustive events.

DISCRETE RANDOM VARIABLE

X is a discrete random variable if it can only assume discrete values and $\sum_{\text{all } x} P(X=x) = 1$.

EXPECTATION (DRV)

Mean $E(X) = \sum_{\text{all } x} x P(X=x)$

$E(g(X)) = \sum_{\text{all } x} g(x) P(X=x)$

VARIANCE (DRV)

$E(X^2) = \sum_{\text{all } x} x^2 P(X=x)$

$Var(X) = E(X^2) - (E(X))^2$

PROBABILITY DENSITY FUNCTION (pdf)

$f(x) = P(X=x)$

CUMULATIVE DISTRIBUTION FUNCTION (CDF)

$F(x_n) = P(X \le x_n) = \sum_{x=x_1}^{x_n} P(X=x)$

LINEAR TRANSFORMATIONS on X

$E(aX+b) = aE(X) + b$

$Var(aX+b) = a^2 Var(X)$

BINOMIAL DISTRIBUTION

X is a Binomial random variable if
- total of n repeated trials
- each trial has 2 outcomes: "success"/"failure"

(Success is what you define it to be - Mrs KOOZM)
- probability of success is the same in each trial
- repeated trials are independent

$X \sim B(n, p)$

pdf $P(X=k) = {}^nC_k\, p^k(1-p)^{n-k}$, $k = 0, 1, 2, \ldots n$ (bounded by 0 and n)

CDF $F(x) = P(X \le x) = \sum_{k=0}^{x} {}^nC_k\, p^k(1-p)^{n-k}$, $x = 0, 1, 2, \ldots n$

$E(X) = np$, $Var(X) = npq = np(1-p)$

*USE OF GDC

(1) $P(X=k) \rightarrow y = \text{binompdf}(n, p, k)$

$\boxed{\text{MENU}}\ \boxed{5}\ \boxed{5}$

(2) $P(a \le X \le b) \rightarrow y = \text{binomCdf}(n, p, a, b)$

(3) To find MODE (most likely value) for X

Graphs: $y = \text{binompdf}(n, p, x)$

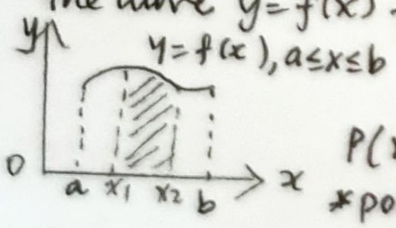
Table: Scroll to find value(s) of x with highest probability value

(4) To find no. of trials n given $P(X \le x) = k$

$n \rightarrow \text{invBinomN}(k, p, x)$

CONTINUOUS RANDOM VARIABLE

Probability is represented by area under the curve $y = f(x)$ for continuous random variable X.



$\int_{\text{all } x} f(x)\,dx = 1$

$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,dx$

*point probabilities $P(X=x) = 0$

EXPECTATION (CRV)

Mean $E(X) = \int_{\text{all } x} x f(x)\,dx$

$E(g(X)) = \int_{\text{all } x} g(x) f(x)\,dx$

VARIANCE (CRV)

$E(X^2) = \int_{\text{all } x} x^2 f(x)\,dx$

$Var(X) = E(X^2) - (E(X))^2$

MODE (CRV)

value of X for which f(x) is greatest in range of X.

CUMULATIVE DISTRIBUTION FUNCTION

$F(t) = P(X \le t) = \int_a^t f(x)\,dx$

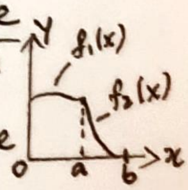
$= 1 - \int_t^b f(x)\,dx$

p.d.f. $f(x) = \frac{d}{dx} F(x)$ → CDF

MEDIAN m: $\int_a^m f(x)\,dx = 0.5$ OR $F(m) = 0.5$

*Multi-part pdf Example

pdf $f(x) = \begin{cases} f_1(x), & 0 \le x < a \\ f_2(x), & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$



CDF:

$F_1(x) = \int_0^x f_1(x)\,dx$

$F_2(x) = \int_a^x f_2(x)\,dx + F_1(a)$

OR $= 1 - \int_x^b f_2(x)\,dx$

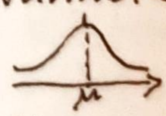
*Combine 2 functions & present final answer.

NORMAL DISTRIBUTION

- a continuous random variable

$X \sim N(\mu, \sigma^2)$

μ Mean = Median = Mode



Symmetry about $x = \mu$

(a) Calculation of Probability

$X \sim N(\mu, \sigma^2)$

$P(a \le X \le b) \rightarrow \text{normCdf}(a, b, \mu, \sigma)$

*key in 1000/-1000 for "∞/-∞".

(b) Calculation of Boundary

$X \sim N(\mu, \sigma^2)$

Boundary value $a \rightarrow \text{invNorm}(\text{area value}, \mu, \sigma)$

*always use Left Hand value of area.

(c) Calculation of Unknown μ, σ

Need to standardize using

$Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$

Convert: $P(X \le x_1) = a$

↓

$P\left(Z \le \frac{x_1 - \mu}{\sigma}\right) = a$

$\frac{x_1 - \mu}{\sigma} \rightarrow \text{invNorm}(a, 0, 1)$

Form simultaneous equations to solve for unknown.

$\boxed{\text{MENU}}\ \boxed{3}\ \boxed{2}$ linsolve in GDC.

HL MATHEMATICS ANALYSIS & APPROACHES – TOPIC 5 : CALCULUS

LIMITS & FIRST PRINCIPLES

As $x \to a$, $f(x) \to l$.

Equivalently, $\lim_{x \to a} f(x) = l$

Algebra of limits:

1. $\lim_{x \to a} c = c$, where c is a constant
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} c \times f(x) = c \times \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$ provided the limits exist
5. $\lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$
8. $\lim_{x \to a} f(x) = f(\lim_{x \to a} x)$ provided $f(x)$ is continuous at $x = a$.

Useful techniques for Rational Functions

(a) Long division → proper fraction

(b) Divide throughout by highest power in denominator

L'Hôpital's Rule (please state when using)

If (a) $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\pm\infty$

(b) $g'(x) \neq 0$ for every $x \in]a,b[\setminus \{c\}$

(c) $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists,

then $\boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}}$

Special case (when $f \times g = 0 \times \infty$)

- write $f \times g$ as $\frac{f}{\frac{1}{g}}$ or $\frac{g}{\frac{1}{f}}$

Example: $\lim_{x \to -\infty} xe^{x} = \lim_{x \to -\infty} \frac{x}{e^{-x}}$

★ L'Hôpital's Rule ★ $= \lim_{x \to -\infty} \frac{1}{(-e^{-x})}$

$= \lim_{x \to -\infty} -e^{-x} = 0$

Special case (usually for e)

- $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$ if f is continuous at $\lim_{x \to a} g(x)$.

Example: $\lim_{x \to \infty} x^{\frac{1}{x}} = \lim_{x \to \infty} e^{\ln(x^{1/x})} = \lim_{x \to \infty} e^{\frac{1}{x}\ln(x)}$

$= e^{\lim_{x \to \infty} \frac{1}{x} \ln x} = e^{0} = 1.$

First Principles Differentiation

Derivatives from "first principles"

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

(1) Always write down $f(x)$ and $f(x+h)$

(2) Simplify and expand in order to "cancel" h in denominator in expression.

(3) Substitute $h = 0$ into expression

(4) Check by differentiating original $f(x)$.

Example: Find $f'(x)$ for $f(x) = x^3$ by using first principles.

$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h}\right)$ step 1

$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h}$ step 2

$= \lim_{h \to 0} 3x^2 + 3xh + h^2$ step 3 $= 3x^2$

Step 4 : $y = x^3$, $\frac{dy}{dx} = 3x^2$.

EQUATION OF TANGENTS & NORMALS

$y = f(x)$

(1) Find $\frac{dy}{dx}$ or $f'(x)$.

(2) Substitute x value to find (a) gradient (b) y-value.

(3) Use $y - y_1 = m(x - x_1)$ to find equation

NOTE: Read question properly.

Gradient of normal, $m_{normal} = \frac{-1}{m_{tangent}}$

INCREASING/DECREASING FUNCTIONS

A function $f(x)$ is an increasing function if $\frac{dy}{dx} > 0$ or $f'(x) > 0$.

A function $f(x)$ is a decreasing function if $\frac{dy}{dx} < 0$ or $f'(x) < 0$.

HIGHER DERIVATIVES

2nd derivative: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$

3rd derivative: $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x)$

nth derivative: $\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = f^{(n)}(x)$

HL MATHEMATICS ANALYSIS & APPROACHES - TOPIC 5: CALCULUS (cont'd)

COMMON FORMS (INTEGRALS)

(1) $\int f'(x)\,[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

(2) $\int f'(x)\,e^{f(x)}\,dx = e^{f(x)} + C$

(3) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

INTEGRATION BY SUBSTITUTION

(1) Let $u = f(x) \Rightarrow x = f^{-1}(u)$

(2) Find $\frac{dx}{du}$.

(3) $\int_{x_1}^{x_2} g(x)\,dx = \int_{u_1}^{u_2} \underbrace{g(f^{-1}(u))}_{\text{replace } x \text{ by } f^{-1}(u)} \left(\frac{dx}{du}\right) du$ * change limits *

(4) Integrate in new variable, u

(5) Definite integral → get numerical answer

Indefinite integral → back-substitute final answer to x.

INTEGRATION BY PARTS

L log   I inverse trigo   A algebra   T trigo   E exponential

u ←

$$\int u \frac{dv}{dx}\,dx = [uv] - \int v \frac{du}{dx}\,dx$$

(1) Identify L.I.A.T.E.

(2) Choose u, the other part is $\frac{dv}{dx}$.
Write down $\frac{du}{dx}$ and v.

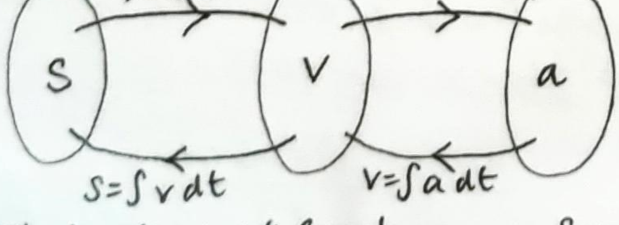
(3) Apply formula.

Examples: $\int x \sin x\,dx$, $\int \ln x\,dx$, $\int \arcsin x\,dx$

Repeated integration by parts: $\int x^2 \sin x\,dx$, $\int e^x \sin x\,dx$

KINEMATICS

Displacement s, Velocity v, Acceleration a



$= v\frac{dv}{ds}$ ↓ when v is a function of s and not t.

Velocity-Time v-t Graph
- gradient $a = \frac{dv}{dt}$
- area under graph = displacement
- Min/max velocity at $a = 0$

Speed (scalar) $= |v|$

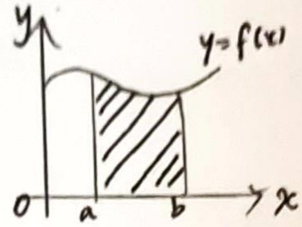
Distance (scalar) $= \int_{t_1}^{t_2} |v|\,dt$

Average speed $= \frac{\text{Total Distance}}{\text{Total Time}} = \frac{\int |v|\,dt}{t}$

AREAS & VOLUMES OF REVOLUTION

* Always sketch the graphs first! *

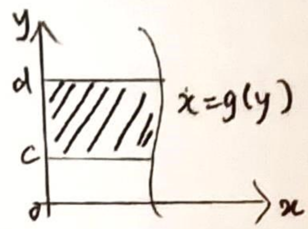
(1) Area under curve (between curve & x-axis)



Area $= \int_a^b y\,dx$

$= \int_a^b f(x)\,dx$

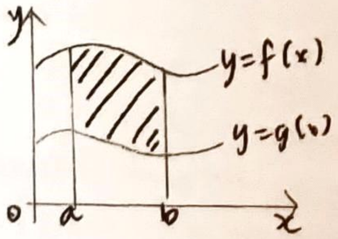
(2) Area under curve (between curve & y-axis)



Area $= \int_c^d x\,dy$

$= \int_c^d g(y)\,dy$

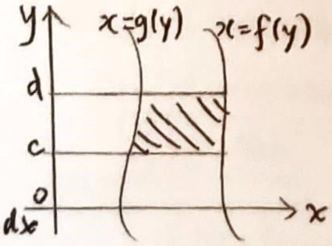
(3) Area between 2 curves (x-axis)



Area = Top − Bottom

$= \int_a^b f(x) - g(x)\,dx$

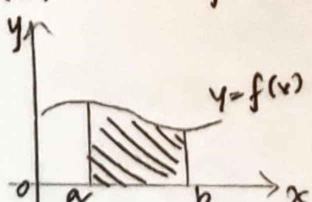
(4) Area between 2 curves (y-axis)



Area = Right − Left

$= \int_c^d f(y) - g(y)\,dy$

(5) Volume of solid of revolution (x-axis)

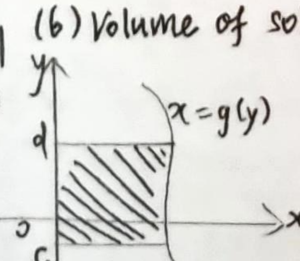


Volume

$= \pi \int_a^b y^2\,dx$

$= \pi \int_a^b (f(x))^2\,dx$

(6) Volume of solid of revolution (y-axis)



Volume

$= \pi \int_c^d x^2\,dy$

$= \pi \int_c^d (g(y))^2\,dy$

IL MATHEMATICS ANALYSIS & APPROACHES - TOPIC 5: CALCULUS (cont'd)

DERIVATIVES

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
a^x	$a^x \ln a$
$\log_a x$	$\frac{1}{x \ln a}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

CHAIN RULE

$y = f(u(x))$, $\frac{dy}{dx} = f'(x) \times \frac{du}{dx}$

PRODUCT RULE

$y = uv$

$\frac{dy}{dx} = uv' + vu'$ or $u\frac{dv}{dx} + v\frac{du}{dx}$

QUOTIENT RULE

$y = \frac{u}{v}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

RATE OF CHANGE

Use of chain Rule to link variables

$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{dy}{dA} \times \frac{dA}{dx} \times \frac{dx}{dt}$

STATIONARY POINTS

- Occur at $\frac{dy}{dx} = 0$ or $f'(x) = 0$
- Determine the nature of stationary points

1st Derivative Test:

x	a^-	a	a^+
$\frac{dy}{dx}$	+	0	−

$\Rightarrow$ max

x	a^-	a	a^+
$\frac{dy}{dx}$	−	0	+

$\Rightarrow$ min

2nd Derivative Test:

$\frac{d^2y}{dx^2} < 0 \Rightarrow$ max $\quad \frac{d^2y}{dx^2} > 0 \Rightarrow$ min

CONCAVITY

- Concave upwards

  If $\frac{dy}{dx}$ is increasing, i.e. $\frac{d^2y}{dx^2} > 0$

- Concave downwards

  If $\frac{dy}{dx}$ is decreasing, i.e. $\frac{d^2y}{dx^2} < 0$

POINTS OF INFLEXION

- Definition: the point at which concavity changes
- Usually $\frac{d^2y}{dx^2} = 0$ at the point
- Confirm by considering

x	a^-	a	a^+
$\frac{d^2y}{dx^2}$	+	0	−

OR

x	a^-	a	a^+
$\frac{d^2y}{dx^2}$	−	0	+

- $\frac{d^2y}{dx^2}$ must change sign across the point.
- If $\frac{dy}{dx} = 0$, then stationary point of inflexion.
- If $\frac{dy}{dx} \neq 0$, then non-stationary point of inflexion.

IMPLICIT DIFFERENTIATION

(1) Apply $\frac{d}{dx}$ to both sides of equation.

(2) Terms in $y \rightarrow$ differentiate as usual but multiply by $\frac{dy}{dx}$

(3) Make $\frac{dy}{dx}$ the subject of the formula.

CHAIN RULE FOR HIGHER DERIVATIVES

Question wants you to find $\frac{d^2y}{dt^2}$ but $\frac{dy}{dt}$ is an expression in terms of x, not t.

$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$ → an expression in x, not t

$= \frac{d}{dx}\left(\frac{dy}{dt}\right) \times \frac{dx}{dt}$

INDEFINITE INTEGRATION

Refer to table of derivatives.
Remember to add "+C" for arbitrary constant.
Do NOT confuse integration with differentiation!

DEFINITE INTEGRALS

$\frac{d}{dx} F(x) = f(x)$

$\int_a^b f(x)\,dx = F(b) - F(a)$

lower limit, upper limit

NOTE: Paper 2/3 questions which do not ask for exact answers, use GDC!

$\boxed{\text{MENU}}$ $\boxed{4}$ $\boxed{2}$ or $\int_{\square}^{\square} \square \, d\square$

DIFFERENTIAL EQUATIONS

- Solve by integration
- Know the difference between general solution (+c) and particular solution.

(1) Direct Integration

$$\boxed{\frac{dy}{dx} = f(x)}$$

$$y = \int f(x)\,dx$$

(2) Variable Separable

$$\boxed{\frac{dy}{dx} = f(x)g(y)}$$

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Example: $3x\frac{dy}{dx} = 5y^2$

$$\int \frac{1}{y^2}\,dy = \frac{5}{3}\int \frac{1}{x}\,dx$$

$$-\frac{1}{y} = \frac{5}{3}\ln|x| + C$$

$$y = -\frac{1}{\frac{5}{3}\ln|x| + C}$$

(3) Homogeneous Differential Equations

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$ (x & y terms are of the same degree)

Steps: (a) Let $y = vx$

(b) $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(c) Change DE in x & y to DE in x & v

(d) Back substitute: $v = \frac{y}{x}$.

Example:

$$\frac{dy}{dx} = \frac{x+2y}{x}$$

$$v + x\frac{dv}{dx} = \frac{x+2vx}{x}$$

$$v + x\frac{dv}{dx} = 1+2v$$

$$x\frac{dv}{dx} = 1+v$$

$$\int \frac{1}{1+v}\,dv = \int \frac{1}{x}\,dx$$

$$\ln|1+v| = \ln|x| + C$$

$$1+v = Ax$$

$$1+\frac{y}{x} = Ax$$

$$y = Ax^2 - x$$

(4) Integrating Factor

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$ mixture of x & y terms: $P(x)y$

Steps: (a) Integrating factor $I(x) = e^{\int P(x)\,dx}$

(b) Multiply throughout by $I(x)$

$\Rightarrow$ LHS $= \frac{d}{dx}(I(x)y)$, RHS $= Q(x)I(x)$

(c) Integrate both sides directly w.r.t. x.

(4) Integrating Factor (cont'd)

Example: $\frac{dy}{dx} + 3x^2y = 6x^2$

$$I(x) = e^{\int 3x^2\,dx} = e^{x^3}$$

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

$$\frac{d}{dx}(e^{x^3}y) = 6x^2e^{x^3}$$

$$\int \frac{d}{dx}(e^{x^3}y)\,dx = \int 6x^2e^{x^3}\,dx$$

$$e^{x^3}y = 2e^{x^3} + C \Rightarrow y = 2 + Ce^{-x^3}$$

EULER'S METHOD

★ Know how to use GDC "lists & spreadsheets"

★ refer to pg 64-68 in Y6 Guidebook

★ For working, show table from GDC

Euler's method can be improved by reducing step size.

MACLAURIN'S SERIES

$$f(x) = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

where $f^{(n)}(0)$ is the n^{th} derivative of $f(x)$ evaluated at $x = 0$.

Standard Maclaurin's Expansions (Formula Booklet)

(1) $e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

(2) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^n}{n}$

(3) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$

(4) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

(5) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

★ Recall apple concept

★ Maclaurin's series can be used to
- evaluate limits
- solve differential equations