SOLUTIONS:

Question 1

Qn		Key Points
1	2025 NJC Promo Q7(a) Consider the equation $$\alpha(\mathbf{u}+\mathbf{v}-\mathbf{w})+\beta(\mathbf{v}+\mathbf{w}-\mathbf{u})+\gamma(\mathbf{w}+\mathbf{u}-\mathbf{v})=0 \quad (*)$$ $$(\alpha-\beta+\gamma)\mathbf{u}+(\alpha+\beta-\gamma)\mathbf{v}+(-\alpha+\beta+\gamma)\mathbf{w}=0.$$ Since **u**, **v** and **w** are linearly independent, we have $$\alpha-\beta+\gamma=0,$$ $$\alpha+\beta-\gamma=0,$$ $$-\alpha+\beta+\gamma=0.$$ By GC, $\alpha=\beta=\gamma=0$. Thus, $(*)$ has only the trivial solution. Therefore, $\mathbf{u}+\mathbf{v}-\mathbf{w}$, $\mathbf{v}+\mathbf{w}-\mathbf{u}$, and $\mathbf{w}+\mathbf{u}-\mathbf{v}$ are linearly independent.	

Question 20

		Key Points
2	$z=\frac{\mathrm{d}y}{\mathrm{d}x}-y \Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x} \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}$. Substitute into the given DE gives $\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}+2(z+y)-3y=2\mathrm{e}^x$ $\frac{\mathrm{d}z}{\mathrm{d}x}+2z+\underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}-y}_{z}=2\mathrm{e}^x$ $\frac{\mathrm{d}z}{\mathrm{d}x}+3z=2\mathrm{e}^x$ Multiply DE by IF = $\mathrm{e}^{\int 3\,\mathrm{d}x}=\mathrm{e}^{3x}$ gives $\frac{\mathrm{d}}{\mathrm{d}x}\left(z\mathrm{e}^{3x}\right)=2\mathrm{e}^{4x}$ $z\mathrm{e}^{3x}=\frac{1}{2}\mathrm{e}^{4x}+C$ on integrating When $x=0$, $z=\frac{\mathrm{d}y}{\mathrm{d}x}-y=2-1=1$. Thus $1=\frac{1}{2}+C \Rightarrow C=\frac{1}{2}$. Therefore $z\mathrm{e}^{3x}=\frac{1}{2}\left(\mathrm{e}^{4x}+1\right)$ $z=\frac{1}{2}\left(\mathrm{e}^{x}+\mathrm{e}^{-3x}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}-y=\frac{1}{2}\left(\mathrm{e}^{x}+\mathrm{e}^{-3x}\right)$ Multiply DE by IF = $\mathrm{e}^{-\int \mathrm{d}x}=\mathrm{e}^{-x}$ gives $\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{-x}\right)=\frac{1}{2}\left(1+\mathrm{e}^{-4x}\right)$ $y\mathrm{e}^{-x}=\frac{1}{2}\left(x-\frac{1}{4}\mathrm{e}^{-4x}\right)+C$ on integrating When $x=0$, $y=1$. Thus $1=-\frac{1}{8}+C \Rightarrow C=\frac{9}{8}$. Therefore	Differentiate Simplify to linear form Obtain PS Use $z=\frac{\mathrm{d}y}{\mathrm{d}x}-y$

	$ye^{-x} = \frac{1}{2}\left(x - \frac{1}{4}e^{-4x}\right) + \frac{9}{8}$ $y = \frac{1}{2}\left(xe^{x} - \frac{1}{4}e^{-3x}\right) + \frac{9}{8}e^{x}$	Obtain final PS (cao)

Question 3

	2025 NJC Promo Q7(b)	Key Points
(a)	Take $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathbf{M}_{2\times 2}(\mathbb{R})$ and $k \in \mathbb{R}$. $\mathrm{F}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]$ $= \mathrm{F}\left[\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}\right]$ $= \int_{-1}^{1} (a+e)x^2 + (b+f+c+g)x + d + h \ \mathrm{d}x$ $= \int_{-1}^{1} ax^2 + (b+c)x + d \ \mathrm{d}x + \int_{-1}^{1} ex^2 + (f+g)x + h \ \mathrm{d}x$ $= \mathrm{F}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] + \mathrm{F}\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]$ $\therefore$ F preserves addition. $\mathrm{F}\left[k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \mathrm{F}\left[\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}\right]$ $= \int_{-1}^{1} kax^2 + (kb+kc)x + kd \ \mathrm{d}x$ $= k\int_{-1}^{1} ax^2 + (b+c)x + d \ \mathrm{d}x$ $= k\mathrm{F}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right].$ $\therefore$ F preserves scalar multiplication. Therefore, F is a linear transformation.	

(b)	$$\mathrm{F}\left[\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right]=0$$ $$\int_{-1}^{1} px^2+(q+r)x+s \; \mathrm{d}x=0$$ $$\left[\frac{px^3}{3}+\frac{(q+r)x^2}{2}+sx\right]_{-1}^{1}=0$$ $$\left(\frac{p}{3}+\frac{q+r}{2}+s\right)-\left(-\frac{p}{3}+\frac{q+r}{2}-s\right)=0$$ $$\frac{2p}{3}+2s=0$$ $$p=-3s$$ $\therefore$ The null space of F is $$\left\{\begin{pmatrix} p & q \\ r & s \end{pmatrix}\middle\| p=-3s,\ q,r\in\mathbb{R}\right\}$$ $$=\left\{\begin{pmatrix} -3s & q \\ r & s \end{pmatrix}\middle\| q,r,s\in\mathbb{R}\right\}$$ $$=\left\{s\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}+q\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}+r\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\middle\| q,r,s\in\mathbb{R}\right\}$$ $$=\operatorname{span}\left\{\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}.$$ Since the three matrices above are linearly independent, a basis for the null space of F is $\left\{\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$.	
(c)	The range space of F is $\mathbb{R}$.	
(d)	For $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ to be antisymmetric, $$\begin{pmatrix} p & q \\ r & s \end{pmatrix}=-\begin{pmatrix} p & q \\ r & s \end{pmatrix}^T=\begin{pmatrix} -p & -r \\ -q & -s \end{pmatrix}$$	

	$\Rightarrow p = s = 0$ and $q = -r$. Thus, antisymmetric matrices are of the form $\begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}$, $r \in \mathbb{R}$.	
	To show $W \subseteq$ null space of F : <u>Method 1</u>: $$\begin{aligned} \mathrm{F}\left[\begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}\right] &= \int_{-1}^{1} 0x^2 + (-r+r)x + 0 \; \mathrm{d}x \\ &= \int_{-1}^{1} 0 \; \mathrm{d}x \\ &= 0. \end{aligned}$$ $\therefore W \subseteq$ null space of F. <u>Method 2</u>: Since $\begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix} = 0\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} - r\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + r\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $W \subseteq$ null space of F.	Note that question is not asking to show that W is a subspace only. It needs you to show that it is a subspace of the nullspace of F. Hence, do not just show that the set of matrices contains the zero matrix and closed under vector addition and scalar multiplication (not required). All we need to show is that W is a subset of the nullspace of F (which is already a subspace) and the "zero vector" is in W, we are done.
	Also, since $W = \text{span}\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$, W is a subspace of $\mathbf{M}_{2\times 2}(\mathbb{R})$. Hence, W is a vector space. Therefore, W is a subspace of the null space of F. The dimension of W is 1.	$W = \text{span}\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$ implies that zero matrix is in W already.

Question 4

		Key Points
(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{3x} - 2y$ $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1$ $y_0 = 0$ $y_1 \approx 0 + 0.5\left(0e^0 - 0\right) = 0$ $y_2 \approx 0 + 0.5\left(0.5e^{3(0.5)} - 0\right) = 1.12042$ $= 1.120$ (3dp)	
(b)	$x_0 = 0,\ x_1 = 1$ $u_1 = 0 + 1\left(0e^0 - 0\right) = 0$ $y_1 \approx 0 + 1\frac{0 + \left(1e^3 - 0\right)}{2} = 10.0427$ $= 10.043$ (3dp)	Initial approximation
(c)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{25}e^{3x} + \frac{3}{5}xe^{3x} - \frac{2}{25}e^{-2x}$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{21}{25}e^{3x} + \frac{9}{5}xe^{3x} + \frac{4}{25}e^{-2x}$ Since $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, the curve of y against x is concave upwards. The approximations in (i) and (ii) are underestimates.	
(d)	When $x = 1, y = \frac{1}{5}e^3 - \frac{1}{25}e^3 + \frac{1}{25}e^{-2} \approx 3.21909$ $\therefore$ (i) gives a better estimate.	

Question 5 (Source : 2025 VJC Promo Qn 11)

(a)	**(a)** $\mathrm{f}(x,y)=7-0.05y^2-3x^{2/3}$ $\nabla \mathrm{f}(x,y)=\begin{pmatrix}\mathrm{f}_x\\ \mathrm{f}_y\end{pmatrix}=\begin{pmatrix}-2x^{-1/3}\\ -0.1y\end{pmatrix}$ When $x=0$, f_x is undefined. Hence points on the canopy where $z=7-0.05y^2$ **(b)(i)** Instantaneous rate of descent $=\lvert D_{\mathbf{u}}\mathrm{f}(x,y)\rvert=\lvert\nabla \mathrm{f}(x,y)\rvert=\sqrt{4x^{-2/3}+0.01y^2}$ **(ii)** $\therefore$ the least instantaneous rate of descent occurs where $x=\pm1$ and $y=0$. Least rate of descent $=\lvert D_{\mathbf{u}}\mathrm{f}(x,y)\rvert=2<3$ $\therefore$ the canopy is considered inefficient in shedding rainwater. **(c)** Length of 2 vertical beams $=\mathrm{f}(a,b)+\mathrm{f}(-a,b)=2\mathrm{f}(a,b)$ $(\because \mathrm{f}$ is even wrt. $x)$ The vertical plane is $y=b$. $\therefore$ the vertical trace is $z=7-0.05b^2-3x^{2/3}$. Length of arc PQ $=\displaystyle\int_{-a}^{a}\sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $=2\displaystyle\int_0^a\sqrt{1+\left(-2x^{-1/3}\right)^2}\,\mathrm{d}x$ (by symmetry of z about $x=0$) $=2\displaystyle\int_0^a\sqrt{1+\frac{4}{x^{2/3}}}\,\mathrm{d}x$ $=2\displaystyle\int_0^a\frac{\sqrt{x^{2/3}+4}}{x^{1/3}}\,\mathrm{d}x$ $\therefore L(a,b)=2\mathrm{f}(a,b)+2\displaystyle\int_0^a\frac{\sqrt{x^{2/3}+4}}{x^{1/3}}\,\mathrm{d}x$ (shown)

(i) $L(a,b)=2\mathrm{f}(a,b)+3\int_0^a \frac{2}{3}x^{-1/3}\sqrt{x^{2/3}+4}\ \mathrm{d}x$

$$=2\mathrm{f}(a,b)+2\left(x^{2/3}+4\right)^{3/2}\Big|_0^a$$

$$=2\left(7-0.05b^2-3a^{2/3}\right)+2(a^{2/3}+4)^{3/2}-16$$

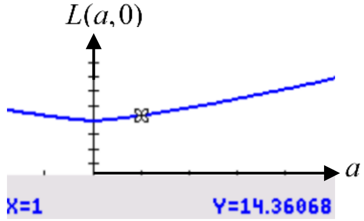
$$=2(a^{2/3}+4)^{3/2}-0.1b^2-6a^{2/3}-2$$

For a fixed a, the largest possible $L(a,b)$ occurs when $b=0$.

$\therefore$ $L(a,0)$ is largest.

(ii) From GC, largest $L(a,0)=2(a^{2/3}+4)^{3/2}-6a^{2/3}-2$ occurs when $a=1$.

$L(1,0)=2(5)^{3/2}-6-2=10\sqrt{5}-8=14.4$ (3sf)

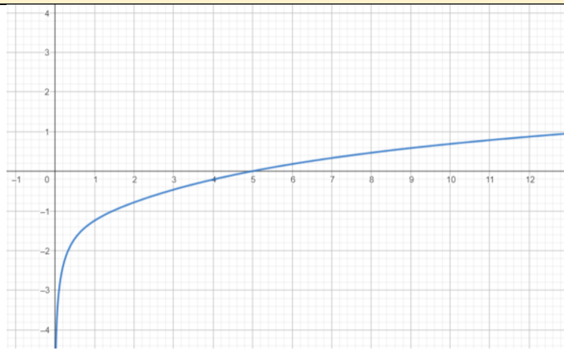


Question 6

		Key Points
(i)	Let P and Q be the points representing the complex numbers z and $z\mathrm{e}^{\mathrm{i}\theta}$ respectively. Then OQ is an anticlockwise rotation of OP about O through an angle θ.	
(ii)	Since $\angle BAC = \frac{\pi}{3}$, $\arg(\gamma-\alpha)-\arg(\beta-\alpha)=\frac{\pi}{3} \Rightarrow \arg\left(\frac{\gamma-\alpha}{\beta-\alpha}\right)=\frac{\pi}{3}$. $\Delta\, ABC$ (in anticlockwise order) is equilateral if and only if $\gamma-\alpha=(\beta-\alpha)\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ from (i).	
(iii)	$\gamma-\alpha=(\beta-\alpha)\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} \Rightarrow \frac{\gamma-\alpha}{\beta-\alpha}=\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ $\Rightarrow \frac{\beta-\alpha}{\gamma-\alpha}=\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$ (take reciprocal) Thus $\frac{\gamma-\alpha}{\beta-\alpha}+\frac{\beta-\alpha}{\gamma-\alpha}=\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}+\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}=2\cos\frac{\pi}{3}=1$.	
(iv)	Let $z=\frac{\gamma-\alpha}{\beta-\alpha}$. Then $\frac{1}{z}=\frac{\beta-\alpha}{\gamma-\alpha}$. By (iii), $\frac{\gamma-\alpha}{\beta-\alpha}+\frac{\beta-\alpha}{\gamma-\alpha}=1 \Rightarrow z+\frac{1}{z}=1$ $\Rightarrow z^2-z+1=0$ (*) So (*) has a root $\frac{\gamma-\alpha}{\beta-\alpha}$. By symmetry, $\frac{\beta-\alpha}{\gamma-\alpha}$ is also a root of (*) $z^2-z+1=0 \Rightarrow z=\frac{1\pm\sqrt{1-4}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$. Since $\arg\left(\frac{\gamma-\alpha}{\beta-\alpha}\right)=\frac{\pi}{3}$, $\frac{\gamma-\alpha}{\beta-\alpha}=\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$ and $\frac{\beta-\alpha}{\gamma-\alpha}=\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$.	Minus 1m if $\frac{\gamma-\alpha}{\beta-\alpha}=\frac{1}{2}-\frac{\sqrt{3}}{2}i$ or $\frac{\beta-\alpha}{\gamma-\alpha}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

	Alternatively, from (ii), $\dfrac{\gamma-\alpha}{\beta-\alpha}=e^{i\frac{\pi}{3}}=\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$. $\dfrac{\beta-\alpha}{\gamma-\alpha}=e^{-i\frac{\pi}{3}}=\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$.	
(v)	$\dfrac{\gamma'-\alpha'}{\beta'-\alpha'}=\dfrac{-\gamma^*-\left(-\alpha^*\right)}{-\beta^*-\left(-\alpha^*\right)}$ $=\dfrac{\gamma^*-\alpha^*}{\beta^*-\alpha^*}$ $=\left(\dfrac{\gamma-\alpha}{\beta-\alpha}\right)^*$ $=\left(e^{i\frac{\pi}{3}}\right)^*$ $=e^{-i\frac{\pi}{3}}$	

Question 7

		Key Points
(i)	 Let $\mathrm{f}(x)=\mathrm{e}^{-x}+\ln\left(\frac{x}{5}\right)$ $\mathrm{f}(1)=-1.2415<0$ $\mathrm{f}(5)=0.0067379>0$ Since $\mathrm{e}^{-x}+\ln\left(\frac{x}{5}\right)$ is continuous for $x>0$, by the Intermediate Value Theorem, there is at least one real root in $[1,5]$. $\mathrm{f}'(x)=-\mathrm{e}^{-x}+\frac{1}{x}$ $=\frac{-x+\mathrm{e}^{x}}{x\mathrm{e}^{x}}$ >0 for $x>0$ Since f is strictly increasing for $x>0$, there is exactly one real root.	IVT Gradient Strictly increasing
(ii)	$x_1=\frac{1\lvert\mathrm{f}(5)\rvert+5\lvert\mathrm{f}(1)\rvert}{\lvert\mathrm{f}(1)\rvert+\lvert\mathrm{f}(5)\rvert}$ ≈ 4.978409 $=4.97841$ (5dp)	

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$$x_{n+1} = x_n - \frac{e^{-x_n} + \ln\left(\frac{x_n}{5}\right)}{\frac{-x_n + e^{x_n}}{x_n e^{x_n}}}$$

n	x_n
1	4.97841
2	4.96522
3	4.96524

Note that $f(4.9655) = 5.05 \times 10^{-5} > 0$

$f(4.9645) = -1.43 \times 10^{-4} < 0$

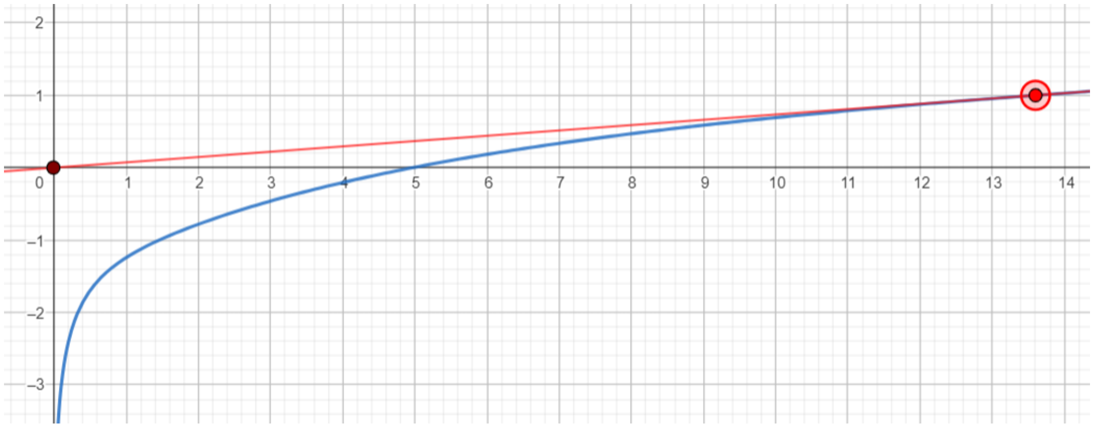
$\therefore$ the root is 4.965 correct to 3 decimal places.

</td><td>Newton-Rhapson
GC or manual iteration</td></tr>
<tr><td>(iv)</td><td>

Equation of tangent at $(k, f(k))$:

$$y - e^{-k} - \ln\left(\frac{k}{5}\right) = \left(-e^{-k} + \frac{1}{k}\right)(x-k)$$

$$y - e^{-k} - \ln\left(\frac{k}{5}\right) = -xe^{-k} + \frac{x}{k} + ke^{-k} - 1$$



For tangent to pass through $(0,0)$,

</td><td>Tangent

Origin
Or use of $x > 0$</td></tr>
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	$0-\mathrm{e}^{-k}-\ln\left(\frac{k}{5}\right)=k\mathrm{e}^{-k}-1$ $k\mathrm{e}^{-k}+\mathrm{e}^{-k}+\ln\left(\frac{k}{5}\right)-1=0$ Using GC, $k=13.591$ $=13.6$ (3sf)	

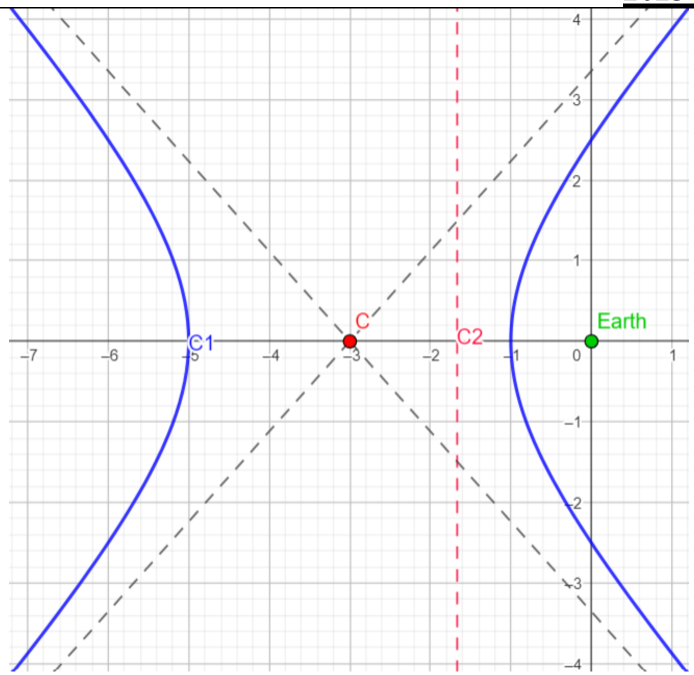
Question 8

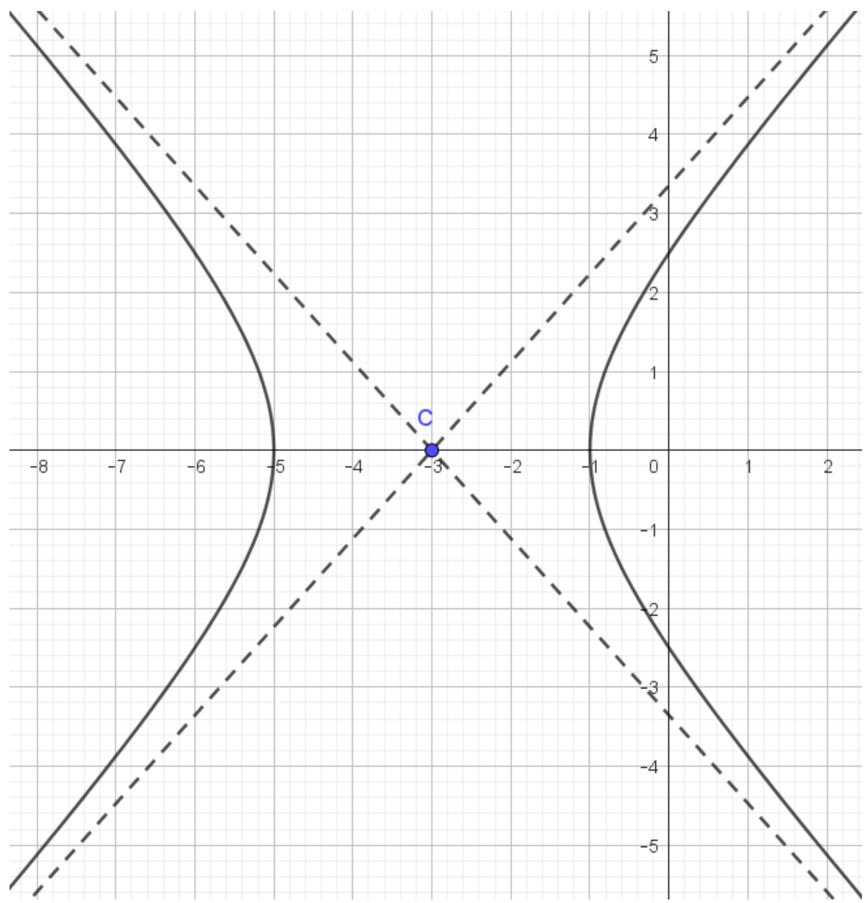
		Key Points
	(i) Since $G_n = F_n + M_n$, $F_{n+1} = 2G_n - 2M_n$ $\Rightarrow F_{n+1} = 2(F_n + M_n) - 2M_n$ $\therefore F_{n+1} = 2F_n \quad -----(1)$ General solution: $F_n = 2^n F_0 = 2^n(10)$ since $F_0 = 10$ $M_{n+2} - G_{n+1} = 2M_n - F_{n+1}$ $M_{n+2} - (M_{n+1} + F_{n+1}) = 2M_n - F_{n+1}$ $M_{n+2} - M_{n+1} - 2M_n = 0 \qquad$ using (1) Auxilliary equation: $m^2 - m - 2 = 0$ $(m-2)(m+1) = 0$ $m = -1$ or 2 General solution: $M_n = A(-1)^n + B(2)^n$ $n = 0, M_0 = A + B = 2$ $n = 1, M_1 = -A + 2B = 4$ Solving: $A = 0, B = 2$ $\therefore M_n = 2^{n+1}$ Since $G_n = F_n + M_n = 2^n(10) + 2^{n+1}$ $G_n = 6(2^{n+1})$.	Learning point: There are only 2 RRs but 3 "variables" and we are required to solve for F_n and M_n. We need to formulate a third RR to eliminate G_n based on the information given.

	(ii)	When $n = 3$, $F_3 = 2^3(10) = 80$ $M_3 = 2^4 = 16$ For the new cycle. $F_0 = 40$ and $M_0 = 4$ From earlier working: $F_n = 2^n F_0 = 2^n(40)$ General solution: $M_n = A(-1)^n + B(2)^n$ $n = 0, M_0 = A + B = 4$ $n = 1, M_1 = -A + 2B = 5$ Solving: $A = 1, B = 3$ $\therefore M_n = (-1)^n + 3(2^n)$ $G_n = 2^n(40) + (-1)^n + 3(2^n)$ At the end of the year, $n = 6$ $G_6 = 2753$	

Question 9

		Key Points
(a)	$r=\dfrac{5}{2-3\cos\theta}$ $2r-3r\cos\theta=5$ $2r=5+3x$ Square both sides $4\left(x^2+y^2\right)=25+30x+9x^2$ $5x^2+30x-4y^2+25=0$ $5\left(x^2+6x\right)-4y^2=-25$ $5\left(x+3\right)^2-4y^2=-25+9\left(5\right)$ $5\left(x+3\right)^2-4y^2=20$ $\dfrac{\left(x+3\right)^2}{4}-\dfrac{y^2}{5}=1$ Asymptotes $\dfrac{\left(x+3\right)^2}{4}-\dfrac{y^2}{5}=0$ $y=\pm\dfrac{\sqrt{5}}{2}\left(x+3\right)$	

		
(b)	From the above cartesian equation, the center of C1 is at (-3,0) $\therefore$ Satellite is at $(-3,0)$ in cartesian coordinates. When $y=0, x=-1$ or $x=-5$ Since the comet travels on the part of C_1 where $r\cos\theta \geq -1$, ie $x \geq -1$. Hence $x=-1$. the shortest distance from the satellite to the comet $= 2$ units	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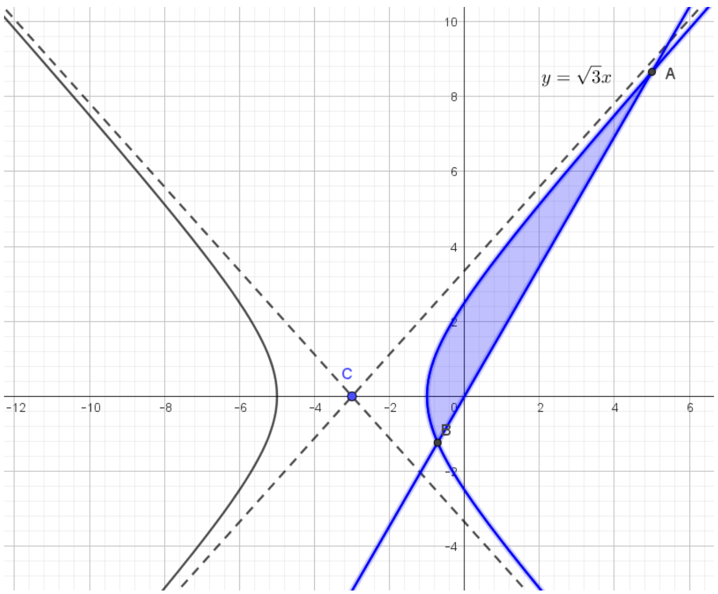
(c)	For $y=\sqrt{3}x$, The polar equation for $y=\sqrt{3}x$ is $\theta=\frac{\pi}{3}$ or $\theta=\frac{4\pi}{3}$ When $\theta=\frac{\pi}{3}$, using $r=\frac{5}{2-3\cos\theta}$, we have $r=10$ When $\theta=\frac{4\pi}{3}$, using $r=\frac{5}{2-3\cos\theta}$, we have $r=\frac{10}{7}$ Hence point of intersections is $\left(10,\frac{\pi}{3}\right),\left(\frac{10}{7},\frac{4\pi}{3}\right)$ 	Integral with correct limits $h=\frac{\pi}{4}$ Simpson's rule

	area $=\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} r^2 d\theta$ $=\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}}\left(\frac{5}{2-3\cos\theta}\right)^2 d\theta$ $\approx\frac{1}{2}\frac{\pi}{12}\left(\left(\frac{5}{2-3\cos\frac{\pi}{3}}\right)^2+4\left(\frac{5}{2-3\cos\frac{7\pi}{12}}\right)^2+2\left(\frac{5}{2-3\cos\frac{10\pi}{12}}\right)^2+4\left(\frac{5}{2-3\cos\frac{13\pi}{12}}\right)^2+\left(\frac{5}{2-3\cos\frac{4\pi}{3}}\right)^2\right)$ ≈ 15.910 $=15.9$ (3sf)	

Question 10

		Key Points
	(a) Given $\mathbf{Mx} = \lambda\mathbf{x}$ $\Rightarrow \mathbf{M}^{-1}\mathbf{Mx} = \mathbf{M}^{-1}(\lambda\mathbf{x})$ since $\det(\mathbf{M}) \neq 0$, $\mathbf{M}^{-1}$ exists. $\Rightarrow \quad \mathbf{x} = \lambda\mathbf{M}^{-1}\mathbf{x}$ $\Rightarrow \quad \mathbf{M}^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$ since $\lambda \neq 0$. $\therefore \mathbf{M}^{-1}$ has an eigenvalue $\frac{1}{\lambda}$ and corresponding eigenvector $\mathbf{x}$. (b) Given $\mathbf{M}_1\mathbf{x} = \mu\mathbf{x}$ $\mathbf{MM}_1\mathbf{x} = \mathbf{M}(\mu\mathbf{x})$ $= \mu\mathbf{Mx}$ $= \mu\lambda\mathbf{x}$ Thus, $\mathbf{MM}_1$ has eigenvalue $\mu\lambda$ and corresponding eigenvector **x.** (c) $\mathbf{A}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \mathbf{0}$ since $\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\}$ is a basis for the nullspace of $\mathbf{A}$. $\Rightarrow \mathbf{A}^T\mathbf{A}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \mathbf{A}^T\mathbf{0} = \mathbf{0} = 0\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$	

$$\left(3\mathbf{I}-\mathbf{A}^T\mathbf{A}\right)\begin{pmatrix}1\\2\\3\\4\end{pmatrix}=3\begin{pmatrix}1\\2\\3\\4\end{pmatrix}-\mathbf{A}^T\mathbf{A}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}=3\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

$$\left(3\mathbf{I}+\mathbf{A}^T\mathbf{A}\right)\begin{pmatrix}1\\2\\3\\4\end{pmatrix}=3\begin{pmatrix}1\\2\\3\\4\end{pmatrix}+\mathbf{A}^T\mathbf{A}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}=3\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

Using (a), $\left(3\mathbf{I}+\mathbf{A}^T\mathbf{A}\right)^{-1}$ have eigenvalue $\frac{1}{3}$ and eigenvector $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$.

Using (b), $\mathbf{B}$ has eigenvalue $3\left(\frac{1}{3}\right)=1$ and eigenvector $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$.

(d)

Suppose $\mathbf{A}^T\mathbf{A}$ has eigenvalue λ and associated eigenvector $\mathbf{v}$, then $\mathbf{A}^T\mathbf{A}\mathbf{v}=\lambda\mathbf{v}$

$\left(3\mathbf{I}-\mathbf{A}^T\mathbf{A}\right)\mathbf{v}=3\mathbf{v}-\lambda\mathbf{v}=(3-\lambda)\mathbf{v}$

$\left(3\mathbf{I}+\mathbf{A}^T\mathbf{A}\right)\mathbf{v}=3\mathbf{v}+\lambda\mathbf{v}=(3+\lambda)\mathbf{v}\Rightarrow\left(3\mathbf{I}+\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{v}=\frac{1}{(3+\lambda)}\mathbf{v}$ i.e. if $\lambda\neq-3$

$\therefore\left(3\mathbf{I}-\mathbf{A}^T\mathbf{A}\right)\left(3\mathbf{I}+\mathbf{A}^T\mathbf{A}\right)^{-1}$ has eigenvalue $\gamma=\frac{3-\lambda}{3+\lambda}$ and associated eigenvectors $\mathbf{v}$.

since $\lambda>0\Rightarrow\frac{3-\lambda}{3+\lambda}=-1+\frac{6}{3+\lambda}>-1$

on the otherhand, $\frac{3-\lambda}{3+\lambda}<\frac{3}{3+\lambda}<1$

Thus, $|\gamma|<1$.