

- 1 A function f is defined by $f(x) = ax^3 + bx^2 + cx + d$. The graph of $y = f(x)$ has turning points at $(-1, 4)$ and $\left(\frac{1}{2}, -\frac{11}{4}\right)$. Find the values of a, b, c , and d . [4]

1	$f(x) = ax^3 + bx^2 + cx + d$ passes through the points $(-1, 4)$ and $\left(\frac{1}{2}, -\frac{11}{4}\right)$. At $(-1, 4)$, $-a + b - c + d = 4$(1) At $\left(\frac{1}{2}, -\frac{11}{4}\right)$, $\frac{a}{8} + \frac{b}{4} + \frac{c}{2} + d = -\frac{11}{4}$(2) At the turning points, $f'(x) = 3ax^2 + 2bx + c$ $f'(-1) = 3a - 2b + c = 0$(3) $f'\left(\frac{1}{2}\right) = \frac{3a}{4} + b + c = 0$(4) From GC, $a = 4, b = 3, c = -6, d = -1$	Generally well done. <u>1. Wrong approach:</u> 1a. Some mistakes were in writing $f'(x) = (x+1)(x-\frac{1}{2}) = 3ax^2 + 2bx + c$ where the roots are correct but the coeff of a would be wrong. 1b. Those who wrote $f'(x) = k(x+1)(x-\frac{1}{2}) = 3ax^2 + 2bx + c$ obtained something which ended up being not very useful anyway. Either way, not a good approach. <u>2. Did not form equations/use GC</u> 2a. Few students couldn't form 4 equations, getting 2 only. 2b. After forming 4 equations, some did not recognise this is a system of 4 linear equations (SLE) that can be solved using a GC (Polysm2).
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- 2 A geometric series has first term a and common ratio r . The sum to infinity of the series is 16 and its fourth term is $\frac{3}{16}$.

- (a) Show that r satisfies the equation $256r^4 - 256r^3 + 3 = 0$. Given that $|r| < 0.5$, solve this equation and hence show that $a = 12$. [5]

2(a)	$\frac{a}{1-r} = 16$ --- (1) $ar^3 = \frac{3}{16}$ ---- (2) $(2) \div (1) : r^3(1-r) = \frac{\frac{3}{16}}{16} = \frac{3}{256}$ $256r^3(1-r) = 3$ $-256r^4 + 256r^3 - 3 = 0$ $256r^4 - 256r^3 + 3 = 0$ (shown) Using GC, $r = 0.25$ or 0.988 (rejected as $r < 0.5$)	Some students wrote $S_{\infty} = a \frac{(1-r^n)}{1-r} = 16$, without making sense that n, number of terms is infinite. Did not know that they could write $S_{\infty} = \frac{a}{1-r} = 16$ directly. Students need to show all answers for r and reject with reason. This is not a complex number question. So, complex solutions can be omitted. But if written, need to write/reject correctly:
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	Sub $r = 0.25$ into equation (1) or (2) e.g. $\frac{a}{1-0.25} = 16$ $a = 12$ (shown)	$r = -0.118 \pm 0.183i$ (rejected, complex root) Some students wrongly wrote $r = -0.118 + 0.183$ or $r = -0.118$, did not read complex number answers in GC correctly.
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- (b) Find the exact value of the sum to infinity of the even-numbered terms of the geometric series. [2]

2(b)	Sum to infinity of the even numbered terms $= \frac{12(0.25)}{1-(0.25)^2}$ $= 3.2$	Students who listed the even numbered terms appear to have a better chance of getting this right.
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- 3 It is given that

$$f(x) = \begin{cases} \frac{10}{9-4x^2} & \text{for } -1 \leq x < 1, \\ 2 & \text{for } 1 \leq x < 3, \end{cases}$$

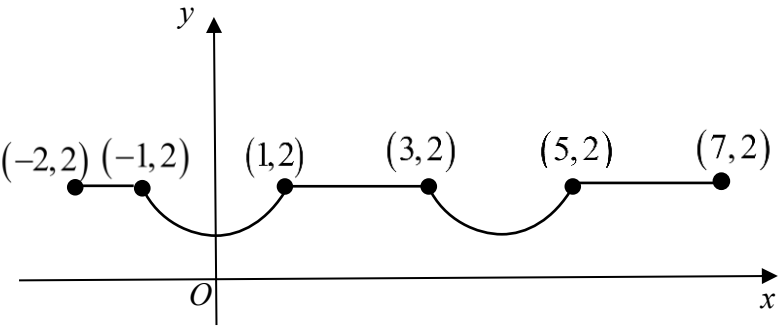
and that $f(x) = f(x+4)$ for all real values of x .

- (a) Evaluate $f(-49)$ [1]

3(a)	$f(-49) = f(-49 + 4 \times 12)$ $= f(-1)$ $= \frac{10}{9-4(-1)^2}$ $= 2$	- Evaluating $f(-49)$ means the answer should be a value rather than left in terms of the function f. - Note that $x = -49$ is outside the domain of the expressions given for $f(x)$ by the question. Hence you cannot just substitute and find $\frac{10}{9-4(-49)^2}$. $f(x) = f(x+4)$ means that the function is periodic with period 4. Use this to obtain $f(-49) = f(-49+4) = \dots = f(-49+4 \times 12) = f(-1)$ where -1 is within the domain of the given expression for $f(x)$. Hence evaluate $f(-1)$ to obtain the answer.
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(b) Sketch the graph of $y = f(x)$ for $-2 \leq x \leq 7$.

[3]

3(b)		Given: $f(x) = \begin{cases} \frac{10}{9-4x^2} & \text{for } -1 \leq x < 1, \\ 2 & \text{for } 1 \leq x < 3, \end{cases}$ $f(x) = f(x+4)$
	- Note that the graph of $y = f(x)$ is required for $-2 \leq x \leq 7$ and not just for $-1 \leq x < 3$. In this case, use the periodicity of the function to sketch the graph as required. - Since the function is periodic with period 4, 'copy and paste' the graph for $-1 \leq x < 3$ (which has 'width' 4) to $3 \leq x < 7$. Realise that the $f(x) = 2$ for $-2 \leq x < -1$ such that the graph would be a horizontal line here. - Note that the requirement for $-2 \leq x \leq 7$ means that $x = 7$ must be included for the sketch graph. Consider that $f(7) = f(7 - 4 \times 2) = f(-1) = 2$. Hence this endpoint must be closed and not open. - There are no open endpoints in the sketch graph. For instance, consider $x = 1$; although the expression $y = \frac{10}{9-4x^2}$ should not be used, realise that as $x \rightarrow 1$, $y \rightarrow 2$, and at $x = 1$ itself, $y = 2$ exactly since $x = 1$ is in the domain such that $f(x) = 2$. This means that there are no gaps or 'overlaps' of closed and open circles representing this and other such points where the graphs are 'joined'. - Be alert to the number of marks available. It would not be reasonable for 3 marks to be awarded where the coordinates of only the two i.e. left- and right-most endpoints are given. The coordinates of all 6 endpoints must be given for full credit. In particular, the y-coordinate(s) must not be excluded.	

(c) Find the exact value of $\int_4^6 f(x) dx$.

[3]

3(c)	$\begin{aligned} \int_4^6 f(x) dx &= 1(2) + \int_0^1 \frac{10}{9-4x^2} dx \\ &= 2 + \int_0^1 \frac{10}{(3)^2 - (2x)^2} dx \\ &= 2 + 10 \left[\left(\frac{1}{2(3)} \times \frac{1}{2} \right) \ln \left \frac{3+2x}{3-2x} \right \right]_0^1 \\ &= 2 + \frac{10}{12} (\ln 5 - \ln 1) \\ &= 2 + \frac{5}{6} \ln 5 \end{aligned}$	- You should use the part (b) sketch graph to visualise the definite integral as the area under the graph between $x = 4$ and $x = 6$. Realise that this area will be composite, including an area under the curve and the area of a rectangle. - Note that the limits for the area under the curve, specifically $x = 4$ and $x = 5$, are outside the domain of the expression for the curve by the question. This means that you cannot just evaluate $\int_4^5 \frac{10}{9-4x^2} dx$. The periodicity of the function (with period 4) leads to
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<u>Alternative</u> $\begin{aligned}\int_4^6 f(x) dx &= 1 \times 2 + \int_0^1 \frac{10}{9-4x^2} dx \\ &= 2 + \frac{10}{4} \int_0^1 \frac{1}{9/4 - x^2} dx \\ &= 2 + \frac{10}{4} \left[\left(\frac{1}{2(3/2)} \right) \ln \left \frac{3/2 + x}{3/2 - x} \right \right]_0^1 \\ &= 2 + \frac{10}{12} (\ln 5 - \ln 1) \\ &= 2 + \frac{5}{6} \ln 5\end{aligned}$	$\int_4^5 f(x) dx = \int_0^1 f(x) dx = \int_0^1 \frac{10}{9-4x^2} dx$ to be evaluated, using integration techniques since the question specified for an exact value. Realise that this can be integrated using the integral for $\frac{1}{a^2 - x^2}$ provided in MF27. However, note that $(2x)^2$ rather than just x^2 is involved. This means there must be another factor of $\frac{1}{2}$ to the integral and not just $\frac{1}{2a} = \frac{1}{2(3)}$. In addition, the expression that logarithm is taken of must be in modulus, since there is no a priori reason for it to be always non-negative. (NB: The integral provided in MF27 does not involve modulus since the domain is explicitly restricted therein.)
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Modification: Change $f(x)$ to the following and solve (d) and (e).

$$f(x) = \begin{cases} \frac{10}{9-4x^2} & \text{for } -1 \leq x < 1, \\ 3-x & \text{for } 1 \leq x < 3, \end{cases}$$

(d) Sketch the graph of $y = f(x)$ for $-2 \leq x < 7$.

3(d)	Label 8 pairs of coordinates (can also use dotted lines) with open circle at appropriate points
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(e) Find the exact value of $\int_4^6 f(x) \, dx$.

3(e)	$\begin{aligned}\int_4^6 f(x) \, dx &= \frac{1}{2}(1)(2+1) + \int_0^1 \frac{10}{9-4x^2} \, dx \\ &= \frac{3}{2} + \int_0^1 \frac{10}{(3)^2 - (2x)^2} \, dx \\ &= \frac{3}{2} + 10 \left[\left(\frac{1}{2(3)} \times \frac{1}{2} \right) \ln \left \frac{3+2x}{3-2x} \right \right]_0^1 \\ &= \frac{3}{2} + \frac{10}{12} (\ln 5 - \ln 1) \\ &= \frac{3}{2} + \frac{5}{6} \ln 5\end{aligned}$	Area of trapezium: $\frac{1}{2}(1)(1+2) \quad \text{or} \quad \int_1^2 3-x \, dx$
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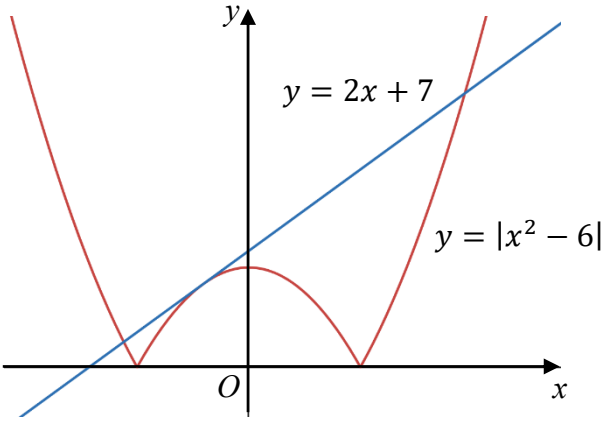
4 (a) Find the exact solutions of the equation $|x^2 - 6| = 2x + 7$. [4]

4(a)	$\begin{aligned}x^2 - 6 &= 2x + 7 \\ x^2 - 2x - 13 &= 0 \\ x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-13)}}{2(1)} \\ \text{or } (x-1)^2 - 14 &= 0 \\ x &= 1 \pm \sqrt{14}\end{aligned}$ $\begin{aligned}x^2 - 6 &= -(2x + 7) \\ x^2 + 2x + 1 &= 0 \\ (x+1)^2 &= 0 \\ x &= -1\end{aligned}$	Recall the definition of the modulus function, where $ x^2 - 6 = \begin{cases} x^2 - 6 & \text{for some values of } x \\ -(x^2 - 6) & \text{for another set of values of } x \end{cases}$ For the purpose of solving this equation, these respective values of x are not essential, since the roots of the equation would naturally lie within the respective sets. Hence, ‘removing’ the modulus from the equation leads to two quadratic equations to be solved. Since the question specified for the exact solutions, algebraic methods are expected, specifically use of the quadratic formula, by completing the square or by factorisation. - Another way to ‘remove’ the modulus is to square both sides of the equation i.e. $(x^2 - 6)^2 = (2x + 7)^2$. However, this leads to a quartic (degree 4) equation, which cannot be solved by use of a GC. The algebraic method in this case must involve factorisation of the quartic into the product of linear and quadratic factors, again leading to use of the quadratic formula or by completing the square to solve exactly the resulting quadratic equation. - Note that there is no restriction on the roots or values of x, since for instance the equation does not involve some logarithmic function. In particular, you should not be rejecting any of the solutions for being negative.
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- (b) Sketch, on the same axes, the graphs of $y = |x^2 - 6|$ and $y = 2x + 7$.

Hence solve exactly the inequality $|x^2 - 6| < 2x + 7$.

[3]

4(b)	 $1 - \sqrt{14} < x < -1$ or $-1 < x < 1 + \sqrt{14}$, or $1 - \sqrt{14} < x < 1 + \sqrt{14}$, $x \neq -1$	- Realise the close association between parts (a) and (b) of this question. Specifically, the algebraic solution of the equation has graphical representation which must be seen in your sketch graphs, and the exact solutions of the equation from part (a) will be the critical values for the exact solutions of the associated inequality for part (b). - Note that in part (a), there are two distinct real solutions, namely $x = 1 + \sqrt{14}$ and $x = 1 - \sqrt{14}$, and one repeated root, $x = -1$. This means that there must be three points at which the curve and the line intersect in your sketch graphs, where the repeated root at $x = -1$ must be represented graphically as a point of tangency. Do not merely 'copy and paste' the graph you see on your GC screen, which omits the point of intersection at $x = 1 + \sqrt{14}$ since it's outside the default zoom; change the window setting or zoom to visualise and include in your sketch all three points of intersection. - The exact solutions from part (a) are the critical values at which the line crosses or is tangent to the curve. Hence, interpret the given inequality graphically whereby its solution involves the values of x where the curve lies below the line. In particular, the inequality to be solved is a strict inequality i.e. we do not want the value of x at which both expressions are equal. Realise that the point at which the line is tangent to the curve represents a value of x where both expressions are actually equal, hence this value, $x = -1$, must be excluded from your solution. This can be written in two ways. (Note that the conjunction or word to be used for a union of two sets must be 'or'.) - The modulus graph is a quadratic curve, which should be drawn appropriately. In particular, since it is a curve, your sketch must not have straight lines, and since it is quadratic, the curve must not bend out or bend (back) towards the y-axis.
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- 5 The sum of the first n terms of a sequence $u_1, u_2, u_3, \dots$ is given by

$$\sum_{r=1}^n u_r = a - \frac{2a}{(n+2)!}, \text{ where } a \text{ is a positive constant.}$$

- (a) Find u_n in terms of a , expressing your answer as a single algebraic fraction. [2]

5(a)	$u_n = S_n - S_{n-1}$ $= \left(a - \frac{2a}{(n+2)!} \right) - \left(a - \frac{2a}{(n+1)!} \right)$ $= \frac{2a}{(n+1)!} - \frac{2a}{(n+2)!}$ $= \frac{2a(n+2)}{(n+2)!} - \frac{2a}{(n+2)!}$ $= \frac{2a(n+1)}{(n+2)!}$ Check for u_1: $\frac{2a(1+1)}{(1+2)!} = \frac{2a}{3} \text{ and } u_1 = a - \frac{2a}{(1+2)!} = \frac{2a}{3}$ For $n \geq 1$, $u_n = \frac{2a(n+1)}{(n+2)!}$.	Recall the formula $u_n = S_n - S_{n-1}$ and do not mix up with $u_{n+1} = S_{n+1} - S_n$. Note that $(n+2)! \neq (n+2)(n+1)(n)$. It should instead be $(n+2)! = (n+2)(n+1)\dots(2)(1)$. It is a good habit to check that $u_1 = S_1$ but not necessary if the question is only worth 2 marks.
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- (b) Explain why the series $\sum_{r=1}^{\infty} u_r$ converges. [1]

5(b)	$\sum_{r=1}^n u_r = a - \frac{2a}{(n+2)!}$ As $n \rightarrow \infty$, $\frac{1}{(n+2)!} \rightarrow 0$. $\therefore \sum_{r=1}^n u_r = a - \frac{2a}{(n+2)!} \rightarrow a$ Hence the series $\sum_{r=1}^{\infty} u_r$ converges.	Explanations must be clear, with the limit a written. It is also advisable to have a concluding statement to answer the question that the series converges.
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(c) Find $\sum_{r=7}^{\infty} u_r$ in terms of a .

[2]

5(c)	$\sum_{r=7}^{\infty} u_r = \sum_{r=1}^{\infty} u_r - \sum_{r=1}^6 u_r$ $= a - \left(a - \frac{2a}{8!} \right)$ $= \frac{a}{20160}$	$\sum_{r=7}^{\infty} u_r \neq \sum_{r=1}^{\infty} u_r - \sum_{r=1}^7 u_r$ Remember to simplify a fraction when the numerator and denominator have common factors.
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Modification: Find $\sum_{r=5}^{\infty} u_{r+1}$ in terms of a .

	Modification $\sum_{r=5}^{\infty} u_{r+1} = u_6 + u_7 + u_8 + \dots$ $= \sum_{r=1}^{\infty} u_r - \sum_{r=1}^5 u_r$ $= a - \left(a - \frac{2a}{7!} \right)$ $= \frac{a}{2520}$	Note that the general term is now u_{r+1} so write out the series to see the terms and express in terms of $\sum_{r=1}^? u_r$
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(d) Find the least value of n for which the sum of the first n terms of the sequence is within $a \times 10^{-8}$ of its sum to infinity. [3]

5(d)	$ S_n - S_{\infty} \leq a \times 10^{-8}$ $\left a - \frac{2a}{(n+2)!} - a \right \leq a \times 10^{-8}$ $\frac{2}{(n+2)!} \leq 10^{-8}$ <table><tr><td>n</td><td>$\frac{2}{(n+2)!}$</td></tr><tr><td>9</td><td>5.01×10^{-8}</td></tr><tr><td>10</td><td>4.18×10^{-9}</td></tr><tr><td>11</td><td>3.21×10^{-10}</td></tr></table> The least value of n is 10.	n	$\frac{2}{(n+2)!}$	9	5.01×10^{-8}	10	4.18×10^{-9}	11	3.21×10^{-10}	A modulus sign should be used unless it is obvious whether S_n or S_{∞} is larger. GC table working should be clearly shown. Do not simply write “By GC”. Incorrect to simply write $n = 10$
n	$\frac{2}{(n+2)!}$									
9	5.01×10^{-8}									
10	4.18×10^{-9}									
11	3.21×10^{-10}									

- 6 (a) Describe a sequence of transformations that transform the graph of $y = \ln x$ onto the graph of $y = \ln(ax + b)$, where a and b are positive constants. [2]

6 (a)	$y = \ln x \rightarrow y = \ln(x + b) \rightarrow y = \ln(ax + b)$ Step 1: Translation of b units in the negative x-direction Step 2: Scaling parallel to x-axis by factor $\frac{1}{a}$ OR $y = \ln x \rightarrow y = \ln(ax) \rightarrow y = \ln\left(a\left(x + \frac{b}{a}\right)\right)$ Step 1: Scaling parallel to x-axis by factor $\frac{1}{a}$ Step 2: Translation of $\frac{b}{a}$ units in the negative x-direction	Recall that transformations on x only replace the x variable itself. Write out the replacements step by step to get the correct order of transformations. Do memorise the proper phrasing for translation, scaling (and reflection). Answers which do not adhere to the standard phrasing will be penalised, even if it is just an extra or missing word.
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Modification: What if $a < 0$ and $b > 0$?

	Modification (where $a < 0$ and $b > 0$) $y = \ln x \rightarrow y = \ln(x + b) \rightarrow y = \ln(a x + b) \rightarrow y = \ln(- a x + b)$ Step 1: Translation of b units in the negative x-direction Step 2: Scaling parallel to x-axis by factor $\frac{1}{ a }$ Step 3: Reflection in the y-axis (Note: Steps 2 and 3 can be interchanged as the sequence for scaling and reflection does not matter.) OR $y = \ln x \rightarrow y = \ln(a x) \rightarrow y = \ln(- a x) \rightarrow y = \ln\left(- a \left(x - \frac{b}{ a }\right)\right)$ Step 1: Scaling parallel to x-axis by factor $\frac{1}{ a }$ Step 2: Reflection in the y-axis Step 3: Translation of $\frac{b}{ a }$ units in the positive x-direction (Note: Steps 1 and 2 can be interchanged as the sequence for scaling and reflection does not matter.)	Can use $-a$ instead of a in the solution Can use $-a$ instead of a in the solution
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- (b) The curve $y = f(x)$ cuts the axes at $(0, -3)$ and $(-1, 0)$. It is given that the equation of the asymptote is $x = 2$. State, if it is possible to do so, the equations of any asymptotes for the following curves and the coordinates of any points where the curves cut the axes.

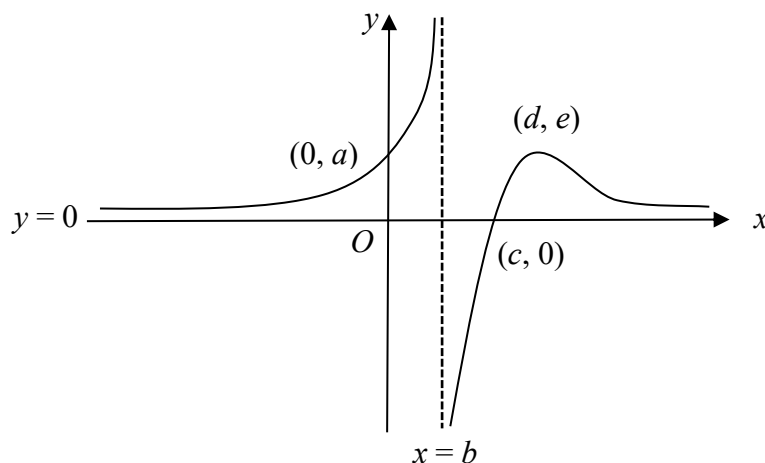
(i) $y = f(|x|)$, [2]

(ii) $y = \frac{1}{f(x)}$. [2]

(b)(i)	Cut axes at $(0, -3)$ Equations of asymptotes: $x = -2$, $x = 2$	Students are not required to sketch the graph. As instructed by the question, intercepts must be given in coordinates. Additional coordinates and asymptotes beyond the correct answer are likely to be penalised.
(b)(ii)	Cut axes at $(0, -\frac{1}{3})$, $(2, 0)$ Equation of asymptote: $x = -1$	

Modification

- (c) The diagram shows the curve $y = f(x)$ with a maximum point at (d, e) and two asymptotes $y = 0$ and $x = b$. The graph of $y = f(x)$ intersects the axes at the points $(0, a)$ and $(c, 0)$.

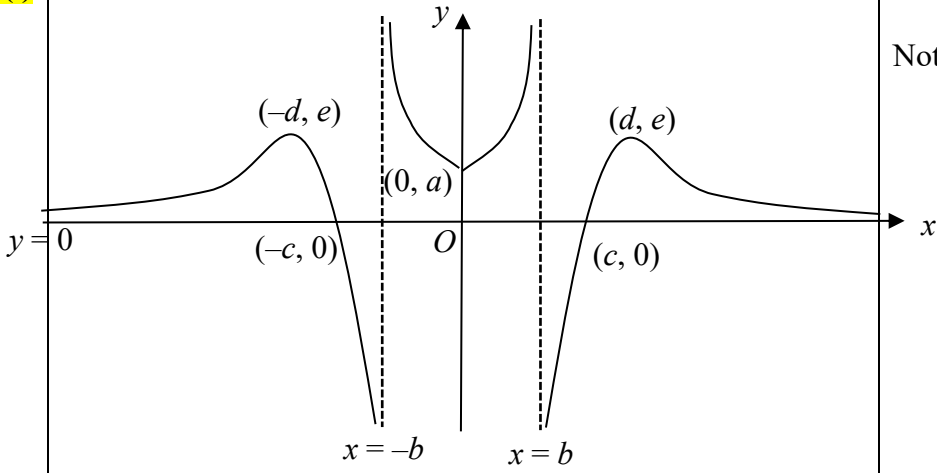
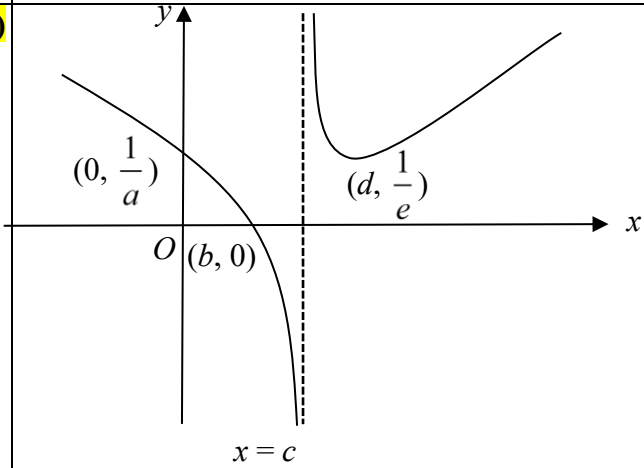
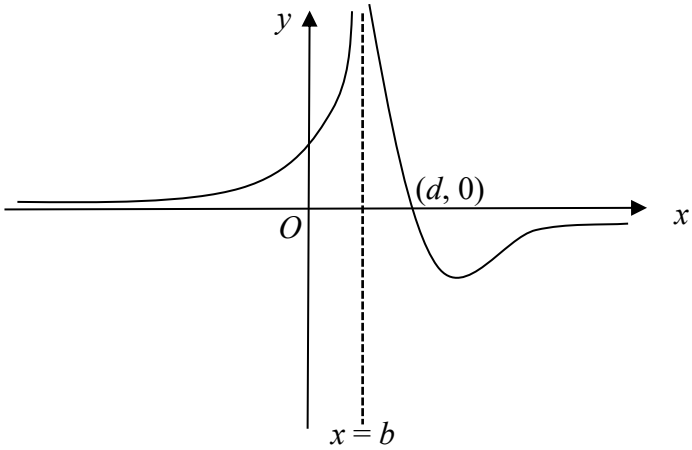


Sketch the following graphs on separate diagrams, labelling the equations of any asymptotes and the coordinates of any points where the graphs cross the axes and of any turning points.

(i) $y = f(|x|)$,

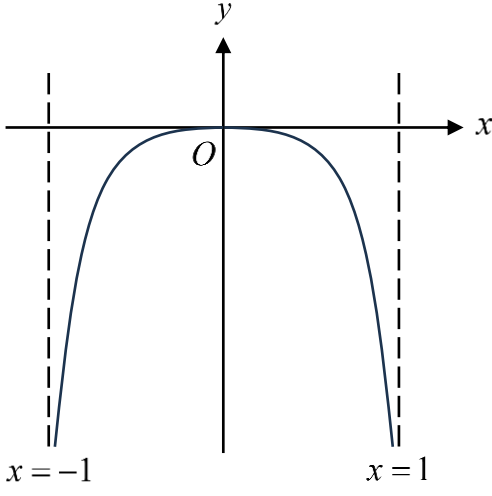
(ii) $y = \frac{1}{f(x)}$,

(iii) $y = f'(x)$.

(c)(i)	 The graph shows a function on a Cartesian coordinate system. The x-axis is labeled x and the y-axis is labeled y. The origin is marked O. The function has a sharp point at $(0, a)$ on the y-axis. It has two vertical asymptotes at $x = -b$ and $x = b$, indicated by dashed lines. The function has local maxima at $(-d, e)$ and (d, e), and crosses the x-axis at $(-c, 0)$ and $(c, 0)$. The line $y = 0$ is also labeled.	Note the sharp point at $(0, a)$
(c)(ii)	 The graph shows a function on a Cartesian coordinate system. The x-axis is labeled x and the y-axis is labeled y. The origin is marked O. The function has a vertical asymptote at $x = c$, indicated by a dashed line. It has a sharp point at $(d, \frac{1}{e})$. The function crosses the y-axis at $(0, \frac{1}{a})$ and the x-axis at $(b, 0)$.	
(c)(iii)	 The graph shows a function on a Cartesian coordinate system. The x-axis is labeled x and the y-axis is labeled y. The origin is marked O. The function has a vertical asymptote at $x = b$, indicated by a dashed line. It has a sharp point at $(d, 0)$ on the x-axis.	

- 7 A curve has parametric equations $x = \cos 2\theta$, $y = \ln(\sin 2\theta)$, $0 < \theta < \frac{\pi}{2}$.

(a) Sketch the curve, stating the equations of the two vertical asymptotes. [2]

7(a)		No endpoints labelled because when $\theta = 0$, $y = \ln(\sin 0)$ is undefined $\theta = \frac{\pi}{2}$, $y = \ln(\sin \pi)$ is undefined which actually results in the presence of the vertical asymptotes instead. Also ensure that the graph cuts the origin.
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(b) State the equation of the line of symmetry of the curve. [1]

(b)	$x = 0$	
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(c) Given that the gradient of the curve at a point P is between -1.5 and 1.5 , find the range of possible values of the x -coordinate of P . [4]

(c)	$x = \cos 2\theta$, $y = \ln(\sin 2\theta)$, $0 < \theta < \frac{\pi}{2}$ $\frac{dx}{d\theta} = -2 \sin 2\theta, \quad \frac{dy}{d\theta} = \frac{2 \cos 2\theta}{\sin 2\theta}$ $\frac{dy}{dx} = \frac{2 \cos 2\theta}{\sin 2\theta} \times \frac{1}{-2 \sin 2\theta} = -\frac{\cos 2\theta}{\sin^2 2\theta} \left(= -\frac{x}{1-x^2} \right)$ Let $-1.5 < -\frac{\cos 2\theta}{\sin^2 2\theta} < 1.5$ Using GC (graph), $0.3829497 < \theta < 1.1878466$ Using GC (graph), $-0.720759 < \cos 2\theta < 0.720759$ The range of possible values of the x-coordinate of P is $-0.721 < x < 0.721$ (3 s.f.).	Many students tried to solve $-1.5 < -\frac{\cos 2\theta}{\sin^2 2\theta} < 1.5$ algebraically, which is tedious and not required. The use of a graphing calculator will be useful here.
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8 A curve has equation $\frac{1}{x} - \frac{1}{y} = 3$, where $x < \frac{1}{3}$.

(a) Show that $\frac{dy}{dx} = \frac{y^2}{x^2}$. [2]

8(a)	$\frac{1}{x} - \frac{1}{y} = 3$ Differentiating wrt x, $-\frac{1}{x^2} + \frac{1}{y^2} \frac{dy}{dx} = 0$ $\frac{1}{y^2} \frac{dy}{dx} = \frac{1}{x^2}$ $\frac{dy}{dx} = \frac{y^2}{x^2}$	A handful of students tried to make y the subject before differentiating, which was unnecessary and complicates the question.
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(b) Using calculus, find the equation of the tangent to the curve at the point where $x = -1$, giving your answer in the form $y = mx + c$. [3]

(b)	When $x = -1$, $\frac{1}{-1} - \frac{1}{y} = 3$ $\frac{1}{y} = -4 \Rightarrow y = -\frac{1}{4}$ $\frac{dy}{dx} = \frac{\left(-\frac{1}{4}\right)^2}{(-1)^2} = \frac{1}{16}$ Equation of tangent : $y - \left(-\frac{1}{4}\right) = \frac{1}{16}(x - (-1))$ $y + \frac{1}{4} = \frac{1}{16}x + \frac{1}{16}$ $y = \frac{1}{16}x - \frac{3}{16}$	The use of $y - y_1 = m(x - x_1)$ is preferred over $y = mx + c$ as it makes the simplification steps easier.
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- (c) Given that y increases at a constant rate of 5 units per second, find the rate of change of x when $y = 1$. [3]

(c)	When $y = 1$, $\frac{1}{x} - \frac{1}{1} = 3 \Rightarrow x = \frac{1}{4}$ Given $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$ $= \frac{1}{\frac{y^2}{x^2}} \times 5$ $= \frac{5x^2}{y^2}$ At $x = \frac{1}{4}$ and $y = 1$, $\frac{dx}{dt} = \frac{5\left(\frac{1}{4}\right)^2}{1^2} = \frac{5}{16} \text{ units per second}$	Some students wrongly assumed that the value of $\frac{dy}{dx}$ is the same as (b). Students have to find the value of $\frac{dy}{dx}$ again by first obtaining the value of x when $y = 1$, and then substituting these values into $\frac{dy}{dx}$.
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- 9 (a) Without using a calculator, solve the equation $z^2 = 9 - 40i$, leaving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$. [5]

9(a)	Let $z = a + bi$ Then $(a + bi)^2 = 9 - 40i$. $a^2 + 2abi + (bi)^2 = 9 - 40i$ $a^2 - b^2 + 2abi = 9 - 40i$ Comparing the real and imaginary parts, $a^2 - b^2 = 9 \quad \text{--- (1)}$ $2ab = -40 \quad \text{--- (2)}$ From (2), $b = -\frac{20}{a} \quad \text{--- (3)}$ Substitute (3) into (1): $a^2 - \left(-\frac{20}{a}\right)^2 = 9$ $a^2 - \frac{400}{a^2} = 9$ $a^4 - 9a^2 - 400 = 0$ $c^2 - 9c - 400 = 0 \text{ (letting } c = a^2\text{)}$ $(c - 25)(c + 16) = 0$	Most students knew they must expand $(a + bi)^2$ and then compare real and imaginary parts. However, sign errors were often seen (e.g. obtaining $-2abi$) which would mess up the quadratic equation further down or the final answer. The rest of the students left this part blank, used the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (which led them nowhere), or wrongly applied the conjugate root theorem (even though not all the coefficients are real). Finally, as a reminder, the command "Without using a calculator" implies students should show all necessary working as credit will be given for such working.
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$c = 25$ or $c = -16$ $a^2 = 25$ or $a^2 = -16$ (rejected) $a = 5$ or $a = -5$ $b = -4$ or $b = 4$ $z = 5 - 4i$ or $z = -5 + 4i$.	
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- (b) It is given that the Maclaurin series for e^x can be generalised to expressions of the form e^{wx} for all complex numbers w and real numbers x .

- (i) Find the first six non-zero terms of the Maclaurin series for e^{ix} . [2]

(b) (i) $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$ $= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \dots$	Most students had no problem applying the Maclaurin series to e^{ix} . A few students mistakenly thought that $e^{ix} = e^i \times e^x$.
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- (ii) By considering the Maclaurin series for $\cos x$ and $\sin x$, verify that $e^{ix} = \cos x + i \sin x$. [1]

(b) (ii) $e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$ $= \cos x + i \sin x$	This is a straightforward verification. The better responses rearranged the terms in the series to show the identity clearly.
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- 10 It is given that $y = e^{\sin^{-1}(3x)}$, where $-\frac{1}{3} < x < \frac{1}{3}$.

- (a) Show that $(1 - 9x^2) \frac{d^2y}{dx^2} - 9x \frac{dy}{dx} - 9y = 0$. [3]

10(a)	EITHER $y = e^{\sin^{-1}(3x)}$ $\frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{1-9x^2}} e^{\sin^{-1}(3x)}$ $= \frac{3y}{\sqrt{1-9x^2}}$ $\sqrt{1-9x^2} \frac{dy}{dx} = 3y$	OR $y = e^{\sin^{-1}(3x)}$ $\ln y = \sin^{-1}(3x)$ Differentiating w.r.t. x , $\frac{1}{y} \frac{dy}{dx} = \frac{1}{\sqrt{1-9x^2}} \cdot 3$ $\sqrt{1-9x^2} \frac{dy}{dx} = 3y$	The most common error was omitting the factor 3 in the first derivative. Students should remind themselves to differentiate thoroughly using the chain rule . Once the first derivative was accurately obtained, students should look out for opportunities to simplify their expression such as replacing $e^{\sin^{-1}(3x)}$ by y . Also, it is better to 'cross-multiply' so that product rule (instead of quotient rule) can be used for further differentiation.
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	THEN EITHER Differentiating with respect to x, $\sqrt{1-9x^2} \frac{d^2y}{dx^2} + \frac{1}{2}(1-9x^2)^{-\frac{1}{2}}(-18x) \frac{dy}{dx} = 3 \frac{dy}{dx}$ Multiplying throughout by $\sqrt{1-9x^2}$, $(1-9x^2) \frac{d^2y}{dx^2} - 9x \frac{dy}{dx} = 3\sqrt{1-9x^2} \frac{dy}{dx}$ $= 3(3y)$ $(1-9x^2) \frac{d^2y}{dx^2} - 9x \frac{dy}{dx} - 9y = 0 \text{ (shown) --- (1)}$ OR Squaring both sides, $(1-9x^2) \left(\frac{dy}{dx} \right)^2 = 9y^2$ Differentiating with respect to x, $(1-9x^2) \cdot 2 \left(\frac{dy}{dx} \right) \frac{d^2y}{dx^2} - 18x \left(\frac{dy}{dx} \right)^2 = 18y \frac{dy}{dx}$ Dividing throughout by $2 \frac{dy}{dx}$, $(1-9x^2) \frac{d^2y}{dx^2} - 9x \left(\frac{dy}{dx} \right) = 9y$ $(1-9x^2) \frac{d^2y}{dx^2} - 9x \frac{dy}{dx} - 9y = 0 \text{ (shown)}$	
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- (b) By differentiating the above result, find the Maclaurin series for y , up to and including the term in x^3 . [4]

(b)	Differentiating with respect to x, $(1-9x^2) \frac{d^3y}{dx^3} - 18x \frac{d^2y}{dx^2} - 9x \frac{d^2y}{dx^2} - 9 \frac{dy}{dx} - 9 \frac{dy}{dx} = 0$ $(1-9x^2) \frac{d^3y}{dx^3} - 27x \frac{d^2y}{dx^2} - 18 \frac{dy}{dx} = 0 \text{ --- (2)}$ When $x = 0$, $y = 1$. $\frac{dy}{dx} = 3$ From (1): $\frac{d^2y}{dx^2} - 0 - 9 = 0 \Rightarrow \frac{d^2y}{dx^2} = 9$	This part was quite well done, with students who differentiated accurately in part (a) usually able to find the Maclaurin series. Still, errors such as writing $-9x \frac{d^2y}{dx^2} + 9 \frac{dy}{dx}$ (due to the omission of brackets) or $-27 \frac{d^2y}{dx^2}$ (somehow dropping the x from one line to the next) were seen from time to time.
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From (2): $\frac{d^3 y}{dx^3} - 0 - 18(3) = 0 \Rightarrow \frac{d^3 y}{dx^3} = 54$ $\therefore y = 1 + 3x + \frac{9}{2!}x^2 + \frac{54}{3!}x^3 + \dots$ $= 1 + 3x + \frac{9}{2}x^2 + 9x^3 + \dots$	For such standard questions, students must double-check that they made no slips.
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- (c) Using your result in part (b) and the standard series in the List of Formulae (MF27), obtain the Maclaurin series for $\sin^{-1}(3x)$, up to and including the term in x^3 . [3]

(c) $e^{\sin^{-1}(3x)} = 1 + 3x + \frac{9}{2}x^2 + 9x^3 + \dots$ $\sin^{-1}(3x) \approx \ln\left(1 + 3x + \frac{9}{2}x^2 + 9x^3\right)$ $\approx \left(3x + \frac{9}{2}x^2 + 9x^3\right) - \frac{1}{2}\left(3x + \frac{9}{2}x^2\right)^2 + \frac{1}{3}(3x)^3$ $\approx 3x + \frac{9}{2}x^2 + 9x^3 - \frac{1}{2}(9x^2 + 27x^3) + \frac{1}{3}(27x^3)$ $= 3x + \frac{9}{2}x^3$	The method shown is the most obvious way to use the result in part (b). Nevertheless, students were often unable to proceed properly. The first group to falter mistook $\ln\left(1 + 3x + \frac{9}{2}x^2 + 9x^3\right)$ to be equal to $\ln 1 + \ln(3x) + \ln\left(\frac{9}{2}x^2\right) + \ln(9x^3)$; others got closer to the answer but left out terms such as the $\frac{9}{2}x^2$ in $-\frac{1}{2}\left(3x + \frac{9}{2}x^2\right)^2$ or the $\frac{1}{3}(3x)^3$. This part rewards students who are careful and thorough in extracting out the terms – an important skill in expansions.
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- 11 The plane π_1 has equation $x + 2y + 2z = 14$. The line l_1 passes through the point A with coordinates $(1, 0, 2)$ and is parallel to $\mathbf{i} + \mathbf{j}$.

- (a) Find the acute angle between π_1 and l_1 . [2]

11 (a) Let the acute angle between π_1 and l_1 be θ. $\sin \theta = \frac{\left \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right }{\sqrt{9}\sqrt{2}} = \frac{3}{3\sqrt{2}}$ $\theta = 45^\circ$	Need to use direction vector of l_1 and normal vector of π_1 Include modulus since finding acute angle
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[illegible]

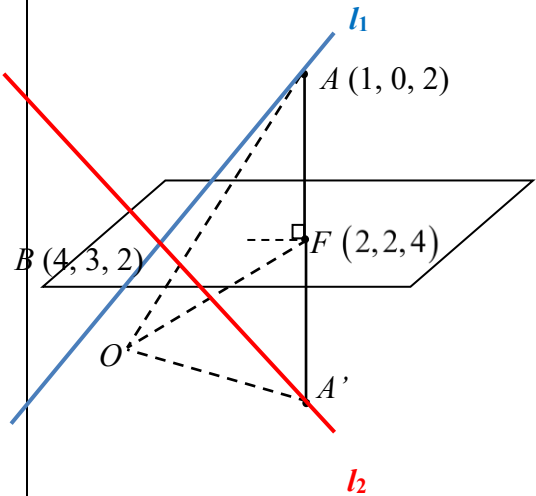
(c) Let F be the foot of perpendicular of A to the plane π_1. Equation of the line AF: $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$ Since F lies on the line, thus $\overrightarrow{OF} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ for some } \mu \in \mathbb{R}$ Since F also lies on the plane, $\overrightarrow{OF} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 14$	Recall that $\mathbf{r}$ represents the position vector of any point on the line. Since F lies on the line, $\overrightarrow{OF} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ for some } \mu \in \mathbb{R}$ It is very wrong to write $\overrightarrow{AF} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$
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$\left[\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 14$ $5 + 9\mu = 14$ $\mu = 1$ $\text{So } \overrightarrow{OF} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$	To keep it simple, you can just substitute equation of line AF $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mu \in \mathbb{R} \quad \text{into equation}$ $\text{of plane } \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 14$ A small number of students attempted to find the projection vector $\overrightarrow{AF}$ first but were usually not successful as they either couldn't remember the formula or were unclear about the direction of the vectors. Thus, it is recommended to use the standard method shown in the solution.
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The line l_2 is the reflection of l_1 in π_1 .

(d) Show that l_2 is parallel to $-\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

[2]

(d) Let A' be the reflection of A in the plane π_1. $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OA'}$ $\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA}$ $= 2 \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$ Let B be the point of intersection between π_1 and l_1. $\overrightarrow{BA'} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ Since l_2 is parallel to $\overrightarrow{BA'}$, l_2 is parallel to $-\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.	A simple diagram will help.  To find $\overrightarrow{OA'}$, use Ratio Theorem to relate the 3 collinear points F, A and A' with the ratio 1:1 A possible direction vector of l_2 is $\overrightarrow{BA'}$
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(e) The plane π_2 contains l_1 and l_2 . Find a cartesian equation of π_2 .

[3]

(e)	$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ $= 4$ Cartesian equation of π_2 is $2x - 2y + z = 4$	To find a normal vector of a plane, do cross product of 2 vectors which are parallel to the plane (but not parallel to each other) Before finding the cartesian equation, we need to find a vector equation in scalar product form using $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
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(f) Explain whether π_1 and π_2 are perpendicular.

[1]

(f)	$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 2 - 4 + 2 = 0$ Since the normal vectors of π_1 and π_2 are perpendicular, therefore π_1 and π_2 are also perpendicular. <u>Alternatively,</u> π_2 contains $\overline{AF}$, which is perpendicular to π_1. $\therefore \pi_1$ and π_2 are perpendicular	Important to conclude that normal vectors of π_1 and π_2 are perpendicular and not just show that dot product of the normal vectors = 0. Also, show clearly that the dot product is indeed 0.
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Vectors Extension Question 1

[CJC Prelim 9758/2023/02/Q4]

The plane Π_1 and the line l have equations

$$\Pi_1: \mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 3 \quad \text{and} \quad l: \frac{x+2}{3} = \frac{4-y}{2} = z-3$$

respectively.

- (a) Find the acute angle between Π_1 and l . [2]
- (b) Find the coordinates of the point of intersection between Π_1 and l . [3]
- (c) Find the perpendicular distance from $B(10, -4, 7)$ to Π_1 . [3]

The plane Π_2 contains l and is perpendicular to Π_1 .

- (d) Find a cartesian equation of Π_2 . [3]
- (e) Without using a calculator, find a vector equation of the line which lies in both Π_1 and Π_2 . [2]

(a)	Vector equation of the line l is $\mathbf{r} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ Method ①: Let the acute angle between Π_1 and l be θ. $\theta = \sin^{-1} \frac{\left \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right }{\left\ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\ \left\ \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right\ }$ $= \sin^{-1} \left \frac{3}{\sqrt{6}\sqrt{14}} \right $ $= 19.1^\circ \quad \text{or} \quad 0.333$
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Method ②:

Let α be the acute angle between the normal vector of plane Π_1 and line l .

$$\begin{aligned}\alpha &= \cos^{-1} \frac{\left| \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right|}{\left\| \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\| \left\| \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right\|}} \\ &= \cos^{-1} \left| \frac{3}{\sqrt{6}\sqrt{14}} \right| \\ &= 70.893^\circ \quad \text{or} \quad 1.2373\end{aligned}$$

$\therefore$ acute angle between the plane Π_1 and line l is

$$\begin{aligned}&= 90^\circ - 70.893^\circ \quad \text{or} \quad \frac{\pi}{2} - 1.2373 \\ &= 19.1^\circ \quad \text{or} \quad 0.333\end{aligned}$$

(b)

Substitute equation of line into the equation of plane:

$$\begin{aligned}\left[\begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} &= 3 \\ -3 + 3\lambda &= 3 \\ \lambda &= 2\end{aligned}$$

When $\lambda = 2$,

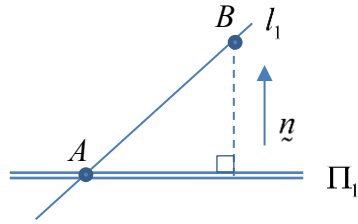
$$\vec{r} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$$

Coordinates of the point of intersection is (4, 0, 5).

(c)

Method ①:Perpendicular distance from B to Π_1

$$\begin{aligned}
 &= \frac{|\overrightarrow{AB} \cdot \vec{n}|}{|\vec{n}|} \\
 &= \frac{\left| \begin{bmatrix} 10 \\ -4 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right|}{\left| \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right|} \\
 &= \frac{\left| \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right|}{\sqrt{6}} \\
 &= \frac{|6|}{\sqrt{6}} \\
 &= \sqrt{6} \text{ units}
 \end{aligned}$$

**Method ② (not recommended)**Equation of line that is parallel to normal vector and passing through $(10, -4, 7)$ is:

$$\vec{r} = \begin{pmatrix} 10 \\ -4 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mu \in \mathbb{R}$$

At the point of intersection,

$$\begin{aligned}
 \left[\begin{pmatrix} 10 \\ -4 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} &= 3 \\
 9 + 6\mu &= 3 \\
 \mu &= -1
 \end{aligned}$$

When $\mu = -1$,

$$\vec{r} = \begin{pmatrix} 10 \\ -4 \\ 7 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \\ 8 \end{pmatrix}$$

	Perpendicular distance from B to Π_1 $= \left \begin{pmatrix} 8 \\ -5 \\ 8 \end{pmatrix} - \begin{pmatrix} 10 \\ -4 \\ 7 \end{pmatrix} \right $ $= \left \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right $ $= \sqrt{(-2)^2 + (-1)^2 + 1^2}$ $= \sqrt{6} \text{ units}$
(d)	A vector perpendicular to the plane Π_2 is $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$ Equation of plane Π_2 is $\vec{r} \cdot \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$ $x + 5y + 7z = 39$
(e)	Note that point $(4, 0, 5)$ lies on both planes Π_1 and Π_2. A vector parallel to the line of intersection of both planes is $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 12 \\ -15 \\ 9 \end{pmatrix} = 3 \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$ Vector equation of the line that lies in both planes is $\vec{r} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}$

Vectors Extension Question 2

ACJC Promo 9758/2022/Q10

The plane P_1 has the equation $-6x - 4y + 2z = 4$.

- (i) Find the vector equations of the planes such that the perpendicular distance from each plane to P_1 is 10 units. [2]

The plane P_2 has the equation $-x - y + 2z = k$, where k is a constant.

- (ii) Find the acute angle between P_1 and P_2 . [2]

- (iii) The planes P_1 and P_2 intersect in the line L . Show that a possible vector equation of L is

$$\mathbf{r} = \begin{pmatrix} 2k-2 \\ 2-3k \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}. \quad [3]$$

The plane P_3 has the equation $5x + \beta y + 5z = \mu$, where $\beta, \mu \in \mathbb{R}$.

- (iv) Given that the line L is contained in the plane P_3 , find β and μ , giving your answer in terms of k if necessary. [2]
- (v) Given instead that the line L does not intersect P_3 , what can be said about the values of β and μ ? [1]

(i)	Let equation of the required plane be $-6x - 4y + 2z = D$. $\frac{ D-4 }{\sqrt{6^2+4^2+2^2}} = 10$ $ D-4 = 10\sqrt{56}$ $D = 4 \pm 20\sqrt{14}$ $\mathbf{r} \cdot \begin{pmatrix} -6 \\ -4 \\ 2 \end{pmatrix} = 4 \pm 20\sqrt{14}$ $\mathbf{r} \cdot \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = 2 \pm 10\sqrt{14}$
(ii)	Let θ be the acute angle between P_1 and P_2 .

	$\theta = \cos^{-1} \frac{\begin{vmatrix} -6 \\ -4 \\ 2 \end{vmatrix} \cdot \begin{vmatrix} -1 \\ -1 \\ 2 \end{vmatrix}}{\begin{vmatrix} -6 \\ -4 \\ 2 \end{vmatrix} \begin{vmatrix} -1 \\ -1 \\ 2 \end{vmatrix}} = \cos^{-1} \frac{14}{\sqrt{56}\sqrt{6}} = 40.2^\circ$
(iii)	$\begin{pmatrix} -6 \\ -4 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 10 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$ Let $z = 0$, $-6x - 4y = 4 \text{ -----(1)}$ $-x - y = k \Rightarrow x = -k - y \text{ -----(2)}$ Subst. (2) into (1) $-6(-k - y) - 4y = 4$ $2y = 4 - 6k \Rightarrow y = 2 - 3k$ $\therefore x = -k - (2 - 3k) \Rightarrow x = 2k - 2$ $\therefore L: \mathbf{r} = \begin{pmatrix} 2k - 2 \\ 2 - 3k \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$
(iv)	$P_3: \mathbf{r} \cdot \begin{pmatrix} 5 \\ \beta \\ 5 \end{pmatrix} = \mu$ Since the line is contained in the plane, direction vector of L is perpendicular to the normal vector of P_3 $\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ \beta \\ 5 \end{pmatrix} = 0$ $-15 + 5\beta + 5 = 0$ $\beta = 2$ Since $(2k - 2, 2 - 3k, 0)$ on L lies on P_3 Subst. $\mathbf{r} = \begin{pmatrix} 2k - 2 \\ 2 - 3k \\ 0 \end{pmatrix}$ into $P_3: \mathbf{r} \cdot \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix} = \mu$ $\begin{pmatrix} 2k - 2 \\ 2 - 3k \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix} = \mu$ $\mu = 4k - 6$
(v)	$\beta = 2, \mu \neq 4k - 6$

- 12** Scientists are investigating the toxin levels in marine fish living in a particular habitat. The rate of change of the toxin level in a marine fish, u milligrams (mg), with respect to time, t days, is proportional to the difference between the toxin level in the habitat and the toxin level in the fish. It may be assumed that the toxin level in the habitat, a mg, is constant. Two models describe the fluctuation of the toxin level over time.

Model A is a continuous model using differential equations.

- (a)** Write down a differential equation for this model.

[1]

12 (a)	$\frac{du}{dt} = k(a - u)$	Not necessary to put modulus or negative sign. k can be any real number.
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- (b)** By solving the differential equation in part **(a)**, show that $u = a - Ae^{-kt}$, where A and k are constants.

[3]

(b)	$\frac{du}{dt} = k(a - u)$ $\int \frac{1}{a - u} du = \int k dt$ $-\ln a - u = kt + C, \text{ where } C \text{ is an arbitrary constant}$ $\ln a - u = -kt - C$ $a - u = \pm e^{-kt} e^{-C}$ $a - u = Ae^{-kt}, \text{ where } A = \pm e^{-C}$ $u = a - Ae^{-kt} \text{ (shown)}$	When separating variables, please make sure you have the integration sign with the respective du or dt. Need to have modulus for $a - u$ after integration because question did not give information about whether a or u is larger. Do not forget the negative sign too because of the coefficient of u. If it is clearer, can do it like this $\int \frac{1}{a - u} du = -\int \frac{-1}{a - u} du = -\ln a - u $ Remember $\pm$ when removing modulus. Remember to indicate what A is and make sure that your arbitrary constants tally with those in the question since it is a show question.
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The scientists narrow their investigation to a particular species of marine fish for which $a = 23$. Based on published data, when a fish of this species has an initial toxin level of 13 mg, its toxin level increases to 19.3 mg after 15 days.

- (c)** Find the toxin level in the fish after 30 days.

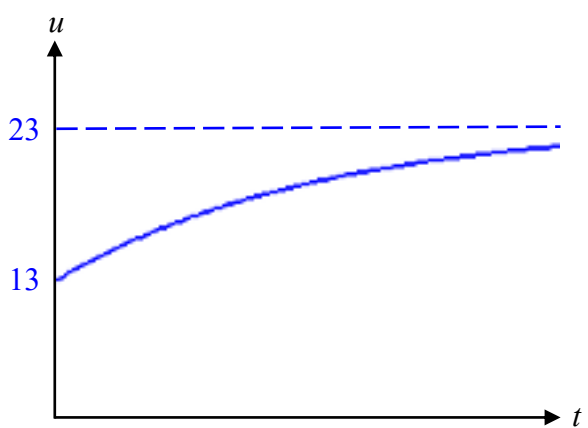
[4]

(c)	Given $a = 23$: $u = 23 - Ae^{-kt}$ When $t = 0$, $u = 13$ $13 = 23 - A \Rightarrow A = 10$ When $t = 15$, $u = 19.3$ $19.3 = 23 - 10e^{-15k}$ $e^{-15k} = 0.37$ $e^{-k} = (0.37)^{\frac{1}{15}} \quad \left(\text{or } k = -\frac{1}{15} \ln 0.37 \right)$	The general solution was shown in (b). In order to find the toxin level after 30 days, we will need to find the particular solution. Since (b) is a show question, you should utilise the shown general equation to solve (c). initial toxin level of 13 mg: $t = 0$, $u = 13$ toxin level increases to 19.3 mg after 15 days: $t = 15$, $u = 19.3$
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$u = 23 - 10(0.37)^{\frac{1}{15}t} \quad \left(\text{or } u = 23 - 10e^{\left(\frac{1}{15} \ln 0.37\right)t} \right)$ When $t = 30$, $u = 23 - 10e^{\left(\frac{1}{15} \ln 0.37\right)(30)}$ $= 21.631 \text{ (or 21.6 (3 s.f.))}$	Common error: Substituting $k = 0.66283$ instead of 0.066283 AND mixing up of the negative signs.
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(d) Sketch the graph of u against t .

[2]

(d)		Based on context, $t \geq 0$. Need to know based on the equation $u = 23 - 10e^{\left(\frac{1}{15} \ln 0.37\right)t}$ that there will be a horizontal asymptote. Need to label vertical intercepts and axes must be correctly labelled. Curve must tend towards asymptote and not touch it.
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Model B is a discrete model using the recurrence relation $u_{n+1} = b + 0.96u_n$, where u_n is the toxin level in the fish on the n th day and b is a constant. The starting term is u_0 . This recurrence relation has a general solution $u_n = C + (0.96)^n D$ for some constants C and D .

(e) Use the earlier published data on the toxin levels to find the values of C and D , giving your answers correct to 3 decimal places. [2]

(e)	Given $u_n = C + (0.96)^n D$: $u_0 = C + (0.96)^0 D \Rightarrow C + D = 13 \quad \text{--- (1)}$ $u_{15} = C + (0.96)^{15} D \Rightarrow C + (0.96)^{15} D = 19.3 \quad \text{--- (2)}$ Solving (1) and (2) $C = 26.758, D = -13.758 \text{ (3 d.p.)}$ [i.e. $u_n = 26.758 - 13.758(0.96)^n$]	Question mentioned: "Use the earlier published data on the toxin" so we are asked to use: - initial toxin level of 13 mg: $n = 0$, $u_0 = 13$ - toxin level increases to 19.3 mg after 15 days: $n = 15$, $u_{15} = 19.3$ It was mentioned in the question that the starting term is u_0 and therefore it should be the initial amount of 13mg. You will only need to use the general solution in this question, do not need to use the recurrence relation.
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		u_n is the toxin level in the fish on the nth day $\rightarrow$ Which means after 15 days, it will be u_{15} Utilise the GC to solve the simultaneous equations and ensure answers are left to required accuracy (3 d.p.). One common error was $0.96^0 = 0$. u_{30} is not part of the given conditions in the published data so it should not be used in this question.
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(f) Describe how the toxin level changes over time based on this model. [1]

(f)	The toxin level increases and converges/approaches/tends towards 26.758 mg.	Not sufficient to only state the convergence because it could have been decreasing/increasing before reaching the limit. Use correct terms to describe the convergence and state the value it is converging to.
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The fish will die if the toxin level in its body reaches 25 mg. A study reveals that many of the fish have died from the toxin in recent years.

(g) Explain which of the models, A or B, should be used in view of this study. [1]

(g)	Model B should be used as it shows that the toxin level approaches 26.758 mg which means that Model B can predict that the toxin level can reach 25mg. OR Model B should be used because Model A shows that the toxin level approaches 23 mg which means that Model A cannot predict that toxin level cannot reach 25mg.	Need to quote data value of 25mg from the study to show clearly the understanding of the question.
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