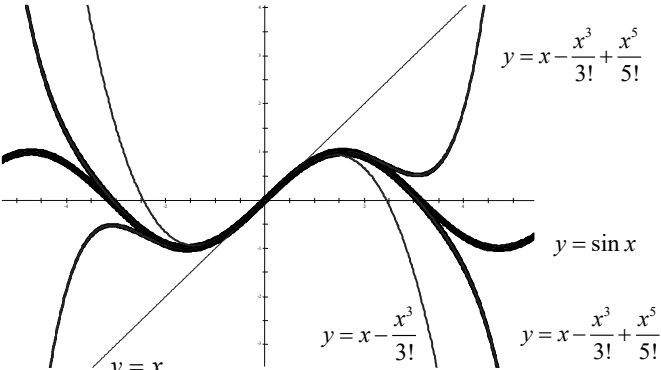


## SUMMARY: Maclaurin Series

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| <p><b>Maclaurin Series (in MF27)</b></p> $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$ <p>Note: <math>f^{(n)}(x)</math> denotes the <math>n</math>th derivative of <math>f(x)</math>.</p>                                                                                                                                                                                                                                                                                                                                    | <p><b>Implicit differentiation</b> is usually required when doing repeated differentiation of an equation involving the derivatives.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| <p><b>Standard Expansions (in MF27)</b></p> $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots,  x  < 1$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (-1 < x \leq 1)$                                                                                         | <ul style="list-style-type: none"> <li>The standard expansions can be applied when <math>x</math> is replaced by <math>kx</math> or <math>x^k</math>, where <math>k</math> is a constant.</li> <li>For expressions like <math>x+k</math>, it needs to be simplified before applying the standard expansion.</li> </ul> <p><b>Examples</b></p> $e^{x+3} = e^x e^3 = e^3 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \right)$ $\ln(2+x) = \ln \left[ 2 \left( 1 + \frac{x}{2} \right) \right] = \ln 2 + \ln \left( 1 + \frac{x}{2} \right)$ $= \ln 2 + \frac{x}{2} - \frac{1}{2} \left( \frac{x}{2} \right)^2 + \frac{1}{3} \left( \frac{x}{2} \right)^3 - \frac{1}{4} \left( \frac{x}{2} \right)^4 + \dots$ <ul style="list-style-type: none"> <li>Take note of the range of validity for each of the above expansion.</li> </ul> |
| <p><b>Convergence &amp; Approximation</b></p> <p>In general, there are <b>two ways to improve the approximation</b> of a function using Maclaurin series</p> <ul style="list-style-type: none"> <li>Comparing two Maclaurin series of a function consisting of different number of terms, the Maclaurin series with the <b>greater number of terms</b> gives a better approximation to the function.</li> <li>For a given Maclaurin series of a function, the approximation to the function gets better when the <b><math>x</math> value is closer to 0</b>.</li> </ul> |  <p>The graph illustrates the convergence of the Maclaurin series for <math>y = \sin x</math>. It shows the function <math>y = \sin x</math> and its approximations using the first, third, fifth, and seventh terms of the series. The approximations are shown as curves that get closer to the sine function as more terms are added. The x-axis ranges from -2 to 2, and the y-axis ranges from -1 to 1.</p>                                                                                                                                                                                                                                                                                                                                                                 |
| <p><b>Binomial Series (in MF27)</b></p> $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \dots,$ <p>where <math> x  &lt; 1</math> and <math>n</math> is any rational number.</p>                                                                                                                                                                                                                                                                                                                                                   | <ul style="list-style-type: none"> <li>This is an <b>infinite</b> series and it is valid for <math> x  &lt; 1</math> (i.e. <math>-1 &lt; x &lt; 1</math>).</li> <li>For the expansion of <math>(a+x)^n</math>, it is advisable to change it to the form <math>(1+y)^n</math> as follows: <ul style="list-style-type: none"> <li>(i) If <math>x</math> is <b>small</b>, the expansion is in <b>ascending powers of <math>x</math></b>: <math>\square \square</math></li> </ul> <math display="block">(a+x)^n = \left[ a \left( 1 + \frac{x}{a} \right) \right]^n = a^n \left( 1 + \frac{x}{a} \right)^n.</math> <p>The expansion is valid for <math>\left  \frac{x}{a} \right  &lt; 1</math></p> </li> </ul>                                                                                                                                                          |

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| <p><b>Special Binomial Series to remember:</b></p> <p>1. <math>(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots + (-1)^r x^r + \dots</math><br/> <math> x  &lt; 1 \Rightarrow -1 &lt; x &lt; 1</math></p> <p>2. <math>(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots + x^r + \dots</math><br/> <math> -x  &lt; 1 \Rightarrow -1 &lt; x &lt; 1</math></p> <p>We may apply these special series when finding the binomial expansions of <math>(1 \pm kx)^n</math> where <math>n = -1</math>.</p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      | <p>(ii) If <math>x</math> is <b>large</b>, then <math>\frac{a}{x}</math> is <b>small</b>, and the expansion is in <b>descending powers</b> of <math>x</math>:</p> $(x+a)^n = \left[ x \left( 1 + \frac{a}{x} \right) \right]^n = x^n \left( 1 + \frac{a}{x} \right)^n$ <p>The expansion is valid for <math>\left  \frac{a}{x} \right  &lt; 1</math>.</p> <ul style="list-style-type: none"> <li>When using binomial series for <b>approximation</b>: <ul style="list-style-type: none"> <li>(i) if the value of <math>x</math> is given, then just <b>substitute</b> it into the expansion.</li> <li>(ii) if not, then explore and look for a <b>value within the range of validity</b>.</li> </ul> </li> <li>If the denominator can be <b>factorised</b>, then change to <b>Partial Fractions</b> first.</li> </ul> <p><b>Example</b></p> $\begin{aligned} \frac{7x-5}{(2-x)(1+x)^2} &= \frac{1}{2-x} + \frac{1}{1+x} - \frac{4}{(1+x)^2} \\ &= (2-x)^{-1} + (1+x)^{-1} - 4(1+x)^{-2} \\ &= 2^{-1} \left( 1 - \frac{x}{2} \right)^{-1} + (1+x)^{-1} - 4(1+x)^{-2} \\ &\rightarrow \begin{cases} = \frac{1}{2} \left( 1 + \frac{x}{2} + \left( \frac{x}{2} \right)^2 + \left( \frac{x}{2} \right)^3 + \dots \right) \\ + (1 - x + x^2 - x^3 + \dots) \\ - 4 \left( 1 - 2x + \frac{(-2)(-3)}{2!} x^2 + \dots \right) = \dots \end{cases} \end{aligned}$ |
| <p><b>Small Angle Approximations</b></p> <p>When <math>x</math> is in <b>radians</b> and <math> x </math> is small (i.e. <math>x^3</math> &amp; higher powers of <math>x</math> can be neglected),</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <math>\sin x \approx x \quad \cos x \approx 1 - \frac{1}{2}x^2 \quad \tan x \approx x</math> </div> <p>In general, if <math>x</math> is small and <math>kx</math> is small, where <math>k</math> is a constant, we have</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <math>\sin kx \approx kx \quad \cos kx \approx 1 - \frac{1}{2}(kx)^2 \quad \tan kx \approx kx</math> </div> <p>Knowledge of geometrical results related to circles and triangles (e.g. arc length, area of sector, sine rule, cosine rule etc) are required.</p> <p><b>Sine Rule:</b> <math>\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}</math></p> <p><b>Cosine Rule:</b> <math>a^2 = b^2 + c^2 - 2bc \cos A</math></p> | <p><b>Example</b></p> <p>Given that <math>\theta</math> is a sufficiently small angle, show that</p> $\begin{aligned} \frac{\sin\left(\frac{\pi}{6} + \theta\right)}{\cos 2\theta} &\approx \frac{1}{2} + \frac{\sqrt{3}}{2}\theta + \frac{3}{4}\theta^2. \\ \frac{\sin\left(\frac{\pi}{6} + \theta\right)}{\cos 2\theta} &= \frac{\sin \frac{\pi}{6} \cos \theta + \cos \frac{\pi}{6} \sin \theta}{\cos 2\theta} = \frac{\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta}{\cos 2\theta} \\ &\approx \frac{\frac{1}{2} \left( 1 - \frac{1}{2}\theta^2 \right) + \frac{\sqrt{3}}{2} \theta}{1 - \frac{1}{2}(2\theta)^2} \\ &= \left( \frac{1}{2} + \frac{\sqrt{3}}{2}\theta - \frac{1}{4}\theta^2 \right) (1 - 2\theta^2)^{-1} \\ &\approx \left( \frac{1}{2} + \frac{\sqrt{3}}{2}\theta - \frac{1}{4}\theta^2 \right) (1 + 2\theta^2 + (2\theta^2)^2 + \dots) \\ &\approx \frac{1}{2} + \frac{\sqrt{3}}{2}\theta + \frac{3}{4}\theta^2 \end{aligned}$                                                                                                                                                                                                                                                                                                                                                                                         |