

6A

VECTORS IN TWO AND THREE DIMENSIONS, SCALAR AND VECTOR PRODUCTS

SYLLABUS

- Basic properties of vectors in two- and three-dimensions
 - Addition and subtraction of vectors, multiplication of a vector by a scalar, and their geometrical interpretations
 - Position vectors, displacement vectors and direction vectors
 - Magnitude of a vector
 - Unit vectors
 - Distance between two points
 - Collinearity
 - Use of the ratio theorem in geometrical applications
- Scalar and vector products in vectors
 - Concepts of scalar product and vector product of vectors and their properties
 - Angle between two vectors
 - Geometrical meanings of $|\mathbf{a} \cdot \hat{\mathbf{n}}|$ and $|\mathbf{a} \times \hat{\mathbf{n}}|$, where $\hat{\mathbf{n}}$ is a unit vector

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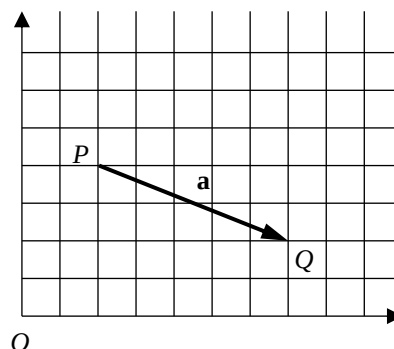
LECTURE 1

Lesson Outline

- Definitions and notation
- Simple operations that involve addition, subtraction and scalar multiple of vectors
- Fundamental results such as parallel vectors, collinear points, linear combinations of vectors and Ratio Theorem

1 DEFINITIONS AND NOTATION

1. A **scalar** is a quantity that is fully defined by **magnitude** alone.
Examples: length, area, volume, distance, speed, temperature, energy, charge.
2. A **vector** is a quantity which has both **magnitude** and **direction**.
Examples: displacement, velocity, acceleration, force, momentum
3. The vector **a** in the diagram goes from *P* to *Q*.
 - (i) The direction of the vector in the diagram is denoted by the arrowhead.
 - (ii) The vector can be denoted as $\overrightarrow{PQ}$, or simply as **a**.
In printed text, vectors are indicated by bold text.
In writing, we indicate vectors with a wavy line underneath the letter: $\underline{a}$.
 - (iii) The **position vector** of a point *P* is the vector with reference to a fixed point *O* (origin) i.e. position vector of *P* is vector $\overrightarrow{OP}$.
 - (iv) A **free vector** is a vector of which only the magnitude and direction are specified but not the position.
 - (v) A point moves from *P* to *Q*. The **displacement vector** $\overrightarrow{PQ}$ is a vector whose length is the shortest distance from the initial position *P* to the final position *Q* of the point.
4. The **magnitude** of $\overrightarrow{PQ}$ is the length of the line segment *PQ*, denoted as *PQ*, $|\overrightarrow{PQ}|$ or $|\mathbf{a}|$.
5. Two vectors **u** and **v** are **equal** if and only if they have the same magnitude and direction,
i.e. $\mathbf{u} = \mathbf{v} \Leftrightarrow |\mathbf{u}| = |\mathbf{v}|$ and **u** and **v** are parallel and in the same direction.



Practice: Draw another vector that is equal to **a** on the diagram.

6. Negative of a vector

If two vectors $\mathbf{u}$ and $\mathbf{v}$ have the same magnitude but opposite directions, then we say that $\mathbf{v} = -\mathbf{u}$, i.e., $-\mathbf{u}$ is a vector in a direction opposite to $\mathbf{u}$ and of the same magnitude as $\mathbf{u}$.

Practice: Draw $-\mathbf{a}$ on the diagram.

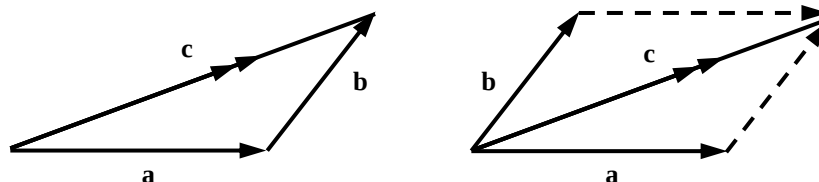
7. The **zero vector** or the **null vector** has zero magnitude. Geometrically, it is a point. It is denoted by $\mathbf{0}$.

2 SIMPLE OPERATIONS

2.1 Addition of Vectors

We can add vectors using either the Triangle Law or the Parallelogram Law.

Given vectors $\mathbf{a}$ and $\mathbf{b}$, their sum $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is defined as shown in the diagrams.



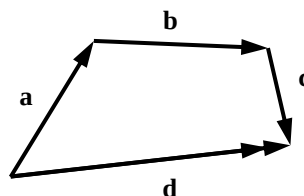
Note that, in general, $|\mathbf{c}| = |\mathbf{a} + \mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$.

Equality holds only in the special case where $\mathbf{a}$ and $\mathbf{b}$ are parallel and in the same direction.

In general, $|\mathbf{c}| \leq |\mathbf{a}| + |\mathbf{b}|$ (the triangle inequality)

To add three or more vectors, we can use the polygon law of addition.

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{d}$$

**Note**

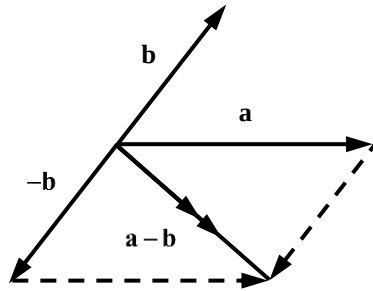
Vector addition is commutative and associative, i.e.,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \quad (\text{Commutative})$$

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \quad (\text{Associative})$$

2.2 Subtraction of Vectors

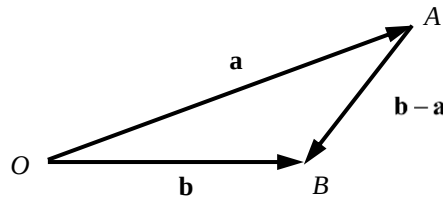
Given vectors $\mathbf{a}$ and $\mathbf{b}$, the vector $\mathbf{a} - \mathbf{b}$ is defined as $\mathbf{a} + (-\mathbf{b})$, as shown in the diagram.



Note

If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of points A and B respectively, we can write the vector $\overrightarrow{AB}$ in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$.

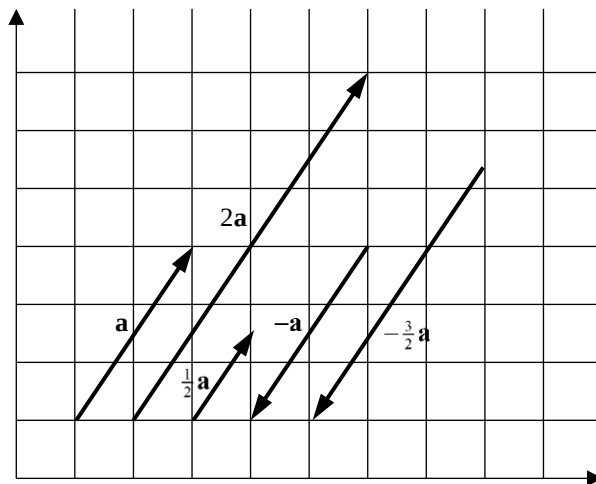
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}\end{aligned}$$



2.3 Scalar Multiple of a Vector

If λ is a positive real number, then $\lambda\mathbf{a}$ is a vector with magnitude $\lambda|\mathbf{a}|$ in the direction of $\mathbf{a}$.

If λ is a negative real number, then $\lambda\mathbf{a}$ is a vector with magnitude $|\lambda||\mathbf{a}|$ in the opposite direction to $\mathbf{a}$.



3 FUNDAMENTAL RESULTS

3.1 Parallel Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $\mathbf{a}$ and $\mathbf{b}$ are **parallel** if and only if

$$\mathbf{b} = k\mathbf{a} \text{ for some } k \in \mathbb{R}, k \neq 0.$$

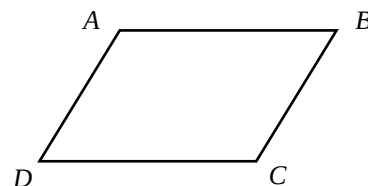
Note

- If $\mathbf{a}$ and $\mathbf{b}$ are parallel and in the same direction, then k is positive.
- If $\mathbf{a}$ and $\mathbf{b}$ are parallel and in opposite directions, then k is negative.
- If $\lambda\mathbf{a} = \mu\mathbf{b}$ and $\mathbf{a}$ is not parallel to $\mathbf{b}$, then it must be the case that $\lambda\mathbf{a} = \mu\mathbf{b} = \mathbf{0}$, so $\lambda = \mu = 0$.

Example 1

$ABCD$ is a quadrilateral and P is any point on AD . If $\overrightarrow{AP} + \overrightarrow{PB} + \overrightarrow{PD} = \overrightarrow{PC}$, prove that $ABCD$ is a parallelogram.

Solution

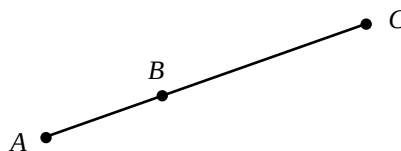


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3.2 Collinear Points

When 3 points A , B and C lie on the same straight line, we say that A , B and C are **collinear**.

From the diagram, since $\overrightarrow{AB} \parallel \overrightarrow{AC}$, i.e. $\overrightarrow{AB} = k\overrightarrow{AC}$ for some real value of k , and there is a **common point** A , therefore A , B , C are collinear.



A , B and C are **collinear**, if and only if
 $\overrightarrow{AB} = k\overrightarrow{AC}$ for some real value of k , $k \neq 0$,
 where A is a common point.

Example 2

If O , P , Q and R are four points such that $\overrightarrow{OP} = 10\mathbf{a}$, $\overrightarrow{OQ} = 5\mathbf{b}$ and $\overrightarrow{OR} = 4\mathbf{a} + 3\mathbf{b}$, show that points P , Q and R are collinear.

Solution

Do **not** write

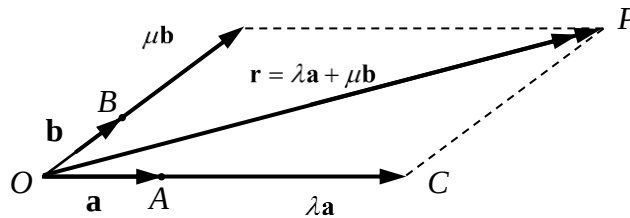
$\frac{\overrightarrow{PQ}}{\overrightarrow{QR}} = -\frac{5}{2}$, because
division of vectors is
not defined.

■ Note that $\frac{PQ}{RQ} = \frac{5}{2}$
and the ratio $PQ : RQ = 5 : 2$

3.3 Linear Combinations of Vectors

Given 2 non-parallel, non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, any vector $\mathbf{r}$ that is coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be expressed as a **linear combination** of $\mathbf{a}$ and $\mathbf{b}$, i.e.

$$\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b}, \text{ for some } \lambda, \mu \in \mathbb{R}.$$

Explanation

If $\overrightarrow{OP} = \mathbf{r}$, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, we can construct $\overrightarrow{OC}$ parallel to $\mathbf{a}$ and $\overrightarrow{CP}$ parallel to $\mathbf{b}$.

Then $\overrightarrow{OC} = \lambda\mathbf{a}$ and $\overrightarrow{CP} = \mu\mathbf{b}$ for some $\lambda, \mu \in \mathbb{R}$.

Since $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$, $\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b}$.

Example 3

If $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors and

$$\mathbf{r} = (3 - \lambda)\mathbf{a} + (3\mu - 1)\mathbf{b} = (\mu - 1)\mathbf{a} + (2\lambda + 1)\mathbf{b}$$

where λ and μ are real constants, find λ and μ and hence find the vector $\mathbf{r}$.

Solution

If $\lambda\mathbf{a} = \mu\mathbf{b}$ and $\mathbf{a}$ is not parallel to $\mathbf{b}$, then it must be the case that $\lambda\mathbf{a} = \mu\mathbf{b} = \mathbf{0}$ so $\lambda = \mu = 0$.

$$(3 - \lambda)\mathbf{a} + (3\mu - 1)\mathbf{b} = (\mu - 1)\mathbf{a} + (2\lambda + 1)\mathbf{b}$$

$$(4 - \lambda - \mu)\mathbf{a} = (2 + 2\lambda - 3\mu)\mathbf{b}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors,

$$4 - \lambda - \mu = 0 \quad \text{---(1)}$$

$$2 + 2\lambda - 3\mu = 0 \quad \text{---(2)}$$

What happens if $\mathbf{a}$ and $\mathbf{b}$ are parallel?

Hint: Use $\mathbf{a} = k\mathbf{b}$.

Solving (1) and (2), we get $\lambda = 2$, $\mu = 2$.

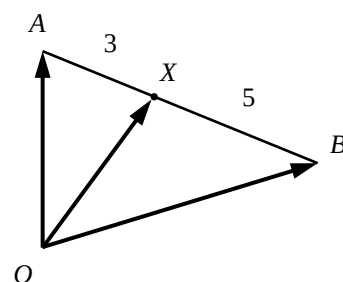
$$\therefore \mathbf{r} = \mathbf{a} + 5\mathbf{b}$$

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Example 4

In triangle OAB , the point X divides the side AB in the ratio 3:5. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OX} = \mathbf{r}$. Express $\mathbf{r}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

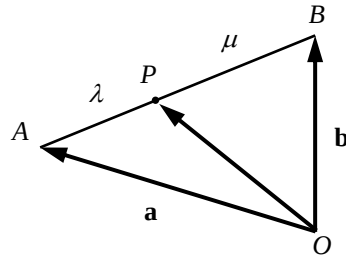
Solution



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This is an example of the **ratio theorem**.

3.4 The Ratio Theorem (Found in MF27)



Let P be a point that divides the line segment AB in the ratio $\lambda : \mu$. Then

$$\overrightarrow{OP} = \frac{\mu \mathbf{a} + \lambda \mathbf{b}}{\lambda + \mu}.$$

Proof

Using the triangle law of addition,

$$\begin{aligned} \overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= \overrightarrow{OA} + \left(\frac{\lambda}{\lambda + \mu} \right) \overrightarrow{AB} \\ &= \overrightarrow{OA} + \left(\frac{\lambda}{\lambda + \mu} \right) (\overrightarrow{OB} - \overrightarrow{OA}) \\ &= \left(\frac{\mu}{\lambda + \mu} \right) \overrightarrow{OA} + \left(\frac{\lambda}{\lambda + \mu} \right) \overrightarrow{OB} \\ &= \frac{\mu \mathbf{a} + \lambda \mathbf{b}}{\lambda + \mu} \end{aligned}$$

Note that $\overrightarrow{OP}$ is coplanar with $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Note

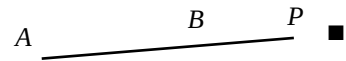
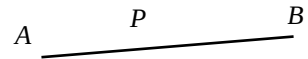
- If P is the mid-point of AB , then $AP : PB = 1 : 1$, $\therefore \overrightarrow{OP} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Example 5

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Find $\overrightarrow{OP}$ if

- (i) P is a point on the line segment AB such that $\frac{AP}{PB} = \frac{3}{2}$.
- (ii) P is a point on the line AB such that $\overrightarrow{AB} = 3\overrightarrow{AP}$.
- (iii) P is a point on the line AB produced such that $AB : BP = 2 : 1$.

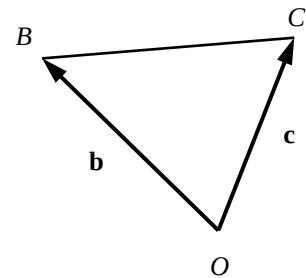
Solution

**Example 6**

The position vectors of points B and C referred to point O are $\mathbf{b}$ and $\mathbf{c}$ respectively.

- (i) Find an expression for the position vector of L , the point on BC such that $BL : LC = 2 : 1$.
- (ii) The point A with position vector $\mathbf{a}$ is such that the origin is the mid-point of AL , prove that $3\mathbf{a} + \mathbf{b} + 2\mathbf{c} = \mathbf{0}$.
- (iii) M is a point on CA such that $CM : MA = 3 : 2$. Show that BOM is a straight line.

Solution



LECTURE 2

Lesson Outline

- Two and three dimensional vectors in cartesian form
- Unit vectors

4 TWO AND THREE DIMENSIONAL VECTORS IN CARTESIAN FORM

4.1 Vectors in Two Dimensions

To go from point P to point Q , we move 2 units in the positive x -direction and 4 units in the positive y -direction.

We can write the vector $\overrightarrow{PQ}$ as $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

This is known as **column vector** notation.

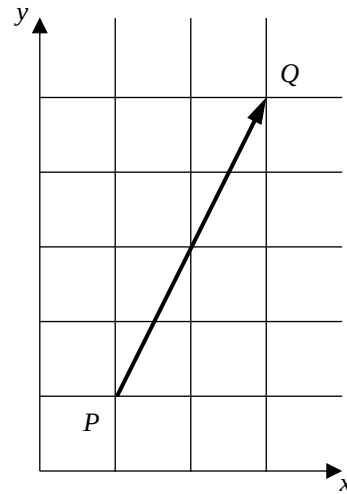
We can also define **standard unit vectors**, which are vectors of magnitude 1 along the x - and y -axes.

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can then write $\overrightarrow{PQ}$ as $2\mathbf{i} + 4\mathbf{j}$.

By Pythagoras' Theorem, magnitude of $\overrightarrow{PQ}$ is $|\overrightarrow{PQ}| = \sqrt{2^2 + 4^2} = \sqrt{20}$.

In general, for the vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} = a\mathbf{i} + b\mathbf{j}$, we have $|\mathbf{v}| = \sqrt{a^2 + b^2}$.

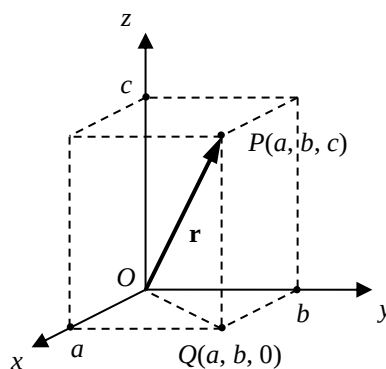


4.2 Vectors in Three Dimensions

In three dimensions, we add a z -axis perpendicular to both the x - and y -axes.

The **standard unit vectors** are vectors of magnitude 1 along the x , y and z axes, written as

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$



Let P be a point with coordinates (a, b, c) .

Then the **position vector** of P is $\overrightarrow{OP} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

The **magnitude** of $\overrightarrow{OP}$ is $OP = |\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$.

4.3 Operations on Vectors

We have the following properties:

Two dimensions	Three dimensions
Equality of vectors	
$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ if and only if $a_1 = a_2$ and $b_1 = b_2$.	$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ if and only if $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$
Addition of vectors	
$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \pm \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 \pm a_2 \\ b_1 \pm b_2 \end{pmatrix}$	$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \pm \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 \pm a_2 \\ b_1 \pm b_2 \\ c_1 \pm c_2 \end{pmatrix}$
Multiplication of vector by a scalar	
$\lambda \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$ for all $\lambda \in \mathbb{R}$	$\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$ for all $\lambda \in \mathbb{R}$
The following hold for all $\lambda, \mu \in \mathbb{R}$ and vectors $\mathbf{u}, \mathbf{v}$: $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$ $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$ $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$ $ \lambda\mathbf{v} = \lambda \mathbf{v} $	
$ \mathbf{u} + \mathbf{v} \leq \mathbf{u} + \mathbf{v} ,$ with equality only if $\mathbf{u}$ and $\mathbf{v}$ are parallel and in the same direction.	

Extra Practice (Ratio Theorem with vectors in ijk notation)

Annex B: Practice Question 1

4.4 Unit Vectors

Definition

A **unit vector** is a vector of unit length, i.e. with magnitude 1.

Consider the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.

We want to find a vector $\hat{\mathbf{v}}$, which is in the same direction as $\mathbf{v}$, but has a magnitude of 1.

We have $\hat{\mathbf{v}} = \lambda \mathbf{v}$ for some $\lambda \in \mathbb{R}^+$, therefore

$$|\hat{\mathbf{v}}| = |\lambda \mathbf{v}| = 1$$

$$\Rightarrow \lambda |\mathbf{v}| = 1 \text{ (since } \lambda > 0 \text{)}$$

$$\Rightarrow \lambda = \frac{1}{|\mathbf{v}|} = \frac{1}{\sqrt{3^2 + 4^2}} = \frac{1}{5}$$

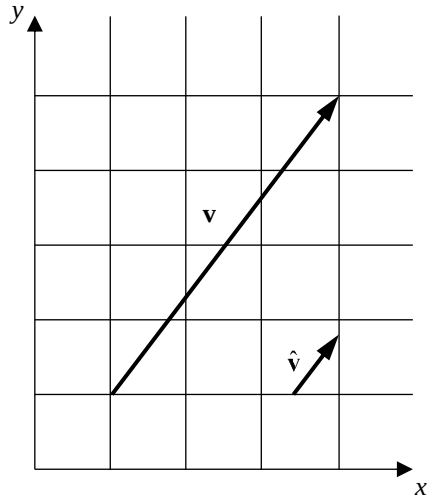
Therefore the vector we are looking for is

$$\hat{\mathbf{v}} = \lambda \mathbf{v} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

$\hat{\mathbf{v}}$ is known as the **unit vector** in the direction of $\mathbf{v}$.

In general, if the vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then the unit vector of $\mathbf{r}$, denoted by $\hat{\mathbf{r}}$, is such that

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$



Example 7

Given that $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j}$, find a unit vector $\mathbf{c}$ in the direction of $\mathbf{a} - \mathbf{b}$.

Solution

■

Example 8

Find **a** if it is parallel to $\begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}$ and equal in magnitude to $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Solution

■

LECTURE 3

Lesson Outline

- Definition of scalar product and its properties
- Angle between two vectors
- Projection of a vector in a given direction

5 SCALAR (DOT) PRODUCT

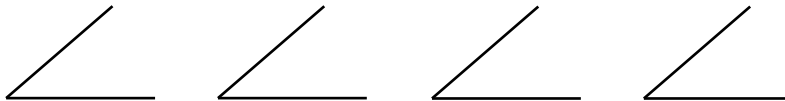
5.1 Definition

The **scalar (dot) product** of two non-zero vectors **a** and **b**, written as **a.b** is a scalar defined as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta, \text{ where } \theta \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b}.$$

Note

Angle between two vectors is the angle between the arrowheads either **both pointing inward** or **outward**.



5.2 Properties of Scalar (Dot) Product

- (i) **a.b** is a scalar quantity
- (ii) **a.b = b.a** (commutative property)
- (iii) **a.a = |a|²**
In particular, **i.i = j.j = k.k = 1**
- (iv) **λ(a.b) = (λa).b = a.(λb)**, where $\lambda \in \mathbb{R}$
- (v) **a.(b + c) = a.b + a.c** (distributive property)
- (vi) Non-zero vectors **a** and **b** are perpendicular if and only if **a.b = 0**.
In particular, **i.j = j.k = k.i = 0**.
- (vii) If **a = a₁i + a₂j + a₃k** and **b = b₁i + b₂j + b₃k**, then

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- (viii) If $\mathbf{a} \cdot \mathbf{b} > 0$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is _____
 If $\mathbf{a} \cdot \mathbf{b} < 0$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is _____

5.3 Angle between Two Vectors

The angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ can be computed using scalar (dot) product.

Note that if $\mathbf{a}$ and $\mathbf{b}$
 are perpendicular,
 then $\mathbf{a} \cdot \mathbf{b} = 0$

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

Example 9

The position vectors of A and B relative to origin O are $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

- (i) Find the scalar product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
 (ii) Find the angle between $\overrightarrow{AB}$ and $\mathbf{j}$.

Solution

■

Example 10

A vector $\mathbf{v}$ is given as $a\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ and makes an angle of 45° with the positive x -axis. Find the exact value of a .

Solution

■

Example 11

Let O be the centre of a circle and A and B be two distinct points on the circumference. Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$. Let A' be a point such that $\overrightarrow{OA'} = -\mathbf{a}$. By considering $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$, show that $\angle ABA' = 90^\circ$.

Solution

$$\begin{aligned} (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - |\mathbf{b}|^2 \\ &= 0 \end{aligned}$$

since $|\mathbf{a}| = |\mathbf{b}|$ (radius of circle).

Also

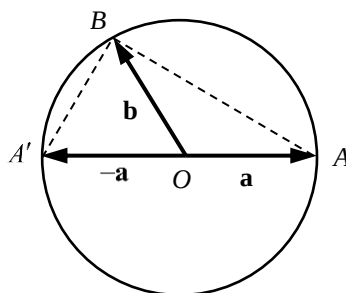
$$\overrightarrow{BA'} = \overrightarrow{BO} + \overrightarrow{OA'} = -\mathbf{b} + (-\mathbf{a}) = -(\mathbf{b} + \mathbf{a})$$

and $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$.

$$\text{Hence, } \cos \angle ABA' = \frac{\overrightarrow{BA'} \cdot \overrightarrow{BA}}{|\overrightarrow{BA'}| |\overrightarrow{BA}|} = \frac{-(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})}{|\mathbf{a} + \mathbf{b}| |\mathbf{a} - \mathbf{b}|} = 0$$

$$\Rightarrow \angle ABA' = 90^\circ \text{ (shown)}$$

■

**Example 12**

A vector $\mathbf{a}$ is such that $\mathbf{a} = 2\mathbf{b} + \mathbf{c}$, where $|\mathbf{b}| = 2$, $|\mathbf{c}| = 3$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is 60° . Using $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, find the magnitude of $\mathbf{a}$.

Solution

■

Example 13

(i) By expanding $(\mathbf{b}-\mathbf{c}).(\mathbf{b}-\mathbf{c})$, simplify $|\mathbf{b}|^2 + |\mathbf{c}|^2 - (\mathbf{b}-\mathbf{c}).(\mathbf{b}-\mathbf{c})$.

(ii) By taking $\mathbf{b} = \overrightarrow{AC}$ and $\mathbf{c} = \overrightarrow{AB}$, deduce the cosine formula for triangle ABC . [N2001/II/15]

Solution

How are the dot product and magnitude of vectors related?

$$(i) \quad (\mathbf{b}-\mathbf{c}).(\mathbf{b}-\mathbf{c}) = \mathbf{b}.\mathbf{b} - \mathbf{b}.\mathbf{c} - \mathbf{c}.\mathbf{b} + \mathbf{c}.\mathbf{c} = |\mathbf{b}|^2 - 2\mathbf{b}.\mathbf{c} + |\mathbf{c}|^2$$

$$\begin{aligned} \therefore |\mathbf{b}|^2 + |\mathbf{c}|^2 - (\mathbf{b}-\mathbf{c}).(\mathbf{b}-\mathbf{c}) \\ &= |\mathbf{b}|^2 + |\mathbf{c}|^2 - (|\mathbf{b}|^2 - 2\mathbf{b}.\mathbf{c} + |\mathbf{c}|^2) \\ &= |\mathbf{b}|^2 + |\mathbf{c}|^2 - |\mathbf{b}|^2 + 2\mathbf{b}.\mathbf{c} - |\mathbf{c}|^2 \\ &= 2\mathbf{b}.\mathbf{c} \end{aligned}$$

(ii) Taking $\mathbf{b} = \overrightarrow{AC}$ and $\mathbf{c} = \overrightarrow{AB}$,

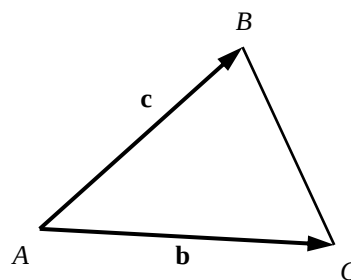
$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\mathbf{c} + \mathbf{b}$$

$$|\overrightarrow{BC}|^2 = \overrightarrow{BC}.\overrightarrow{BC}$$

$$= (\mathbf{b}-\mathbf{c}).(\mathbf{b}-\mathbf{c})$$

$$= |\mathbf{b}|^2 - 2\mathbf{b}.\mathbf{c} + |\mathbf{c}|^2$$

$$= |\overrightarrow{AC}|^2 - 2|\overrightarrow{AC}||\overrightarrow{AB}|\cos \angle BAC + |\overrightarrow{AB}|^2 \quad (\text{Deduced})$$

**Extra Practice**

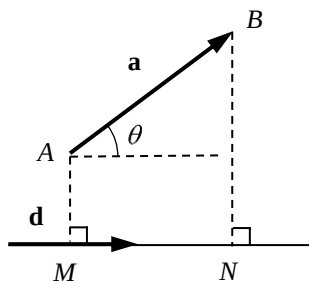
Annex B: Practice Question 2

5.4 Projection of a Vector in a Given Direction

Suppose $\mathbf{a} = \overrightarrow{AB}$ is a vector and l is a line. Let $\mathbf{d}$ be a vector parallel to the line l .

Suppose we draw a line from A perpendicular to l to intersect l at point M . M is called the **foot of the perpendicular** from A to l .

Likewise, let N be the foot of the perpendicular from B to l . $\overrightarrow{MN}$ is called the **projection** of $\overrightarrow{AB}$ onto l (or the projection of $\overrightarrow{AB}$ on $\mathbf{d}$).



If the angle between $\mathbf{a}$ and $\mathbf{d}$ is acute,

length of projection of $\overrightarrow{AB}$ on $\mathbf{d}$

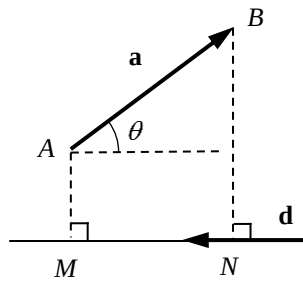
$$= MN = AB \cos \theta$$

$$= |\mathbf{a}| \frac{\mathbf{a} \cdot \mathbf{d}}{|\mathbf{a}| |\mathbf{d}|}$$

$$= \frac{\mathbf{a} \cdot \mathbf{d}}{|\mathbf{d}|}$$

$$= \mathbf{a} \cdot \frac{\mathbf{d}}{|\mathbf{d}|}$$

$$= \mathbf{a} \cdot \hat{\mathbf{d}}$$



If the angle between $\mathbf{a}$ and $\mathbf{d}$ is obtuse,

length of projection of $\overrightarrow{AB}$ on $\mathbf{d}$

$$= MN = AB \cos \theta$$

$$= -AB \cos(180^\circ - \theta)$$

$$= -|\mathbf{a}| \frac{\mathbf{a} \cdot \mathbf{d}}{|\mathbf{a}| |\mathbf{d}|}$$

$$= -\frac{\mathbf{a} \cdot \mathbf{d}}{|\mathbf{d}|}$$

$$= -\mathbf{a} \cdot \frac{\mathbf{d}}{|\mathbf{d}|}$$

$$= -\mathbf{a} \cdot \hat{\mathbf{d}}$$

Since angle between $\mathbf{a}$ and $\mathbf{d}$ is obtuse,
 $\mathbf{a} \cdot \hat{\mathbf{d}}$ would be negative, hence
 $-\mathbf{a} \cdot \hat{\mathbf{d}}$ is positive.

We can summarise both cases together by stating that the

length of projection of $\mathbf{a}$ on $\mathbf{d}$ is $\mathbf{a} \cdot \hat{\mathbf{d}}$.
--

This is the **geometrical meaning** of $|\mathbf{a} \cdot \hat{\mathbf{d}}|$.

Recall the unit vector, $\hat{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|}$. Since $\hat{\mathbf{d}}$ is a vector of magnitude 1, then

Why is it $(\mathbf{a} \cdot \hat{\mathbf{d}})\hat{\mathbf{d}}$
and not $|\mathbf{a} \cdot \hat{\mathbf{d}}|\hat{\mathbf{d}}$?

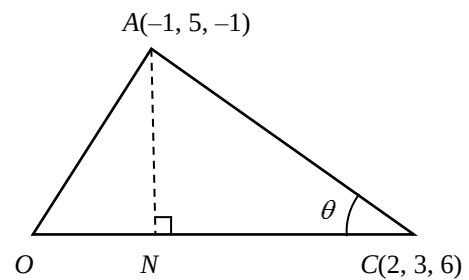
$\lambda \hat{\mathbf{d}}$ is a vector of magnitude λ in the same direction as $\hat{\mathbf{d}}$ (and hence in the same direction as $\mathbf{d}$), i.e. $\overrightarrow{MN} = (\mathbf{a} \cdot \hat{\mathbf{d}})\hat{\mathbf{d}}$ where $\mathbf{d}$ is a vector in the direction of $\overrightarrow{MN}$.

The **vector component** of $\mathbf{a}$ parallel to $\mathbf{d}$, i.e. the projection of $\mathbf{a}$ on $\mathbf{d}$, is $(\mathbf{a} \cdot \hat{\mathbf{d}})\hat{\mathbf{d}}$.

Example 14

A and C are points with coordinates $(-1, 5, -1)$ and $(2, 3, 6)$ respectively. Find the length of the projection of $\overrightarrow{AC}$ on $\overrightarrow{OC}$. Hence, find the position vector of the foot of the perpendicular from A to OC .

Solution



■

Extra Practice

Annex B: Practice Question 3

LECTURE 4

Lesson Outline

- Definition of vector product and its properties
- Perpendicular distance from a point to a line
- Other applications of vector product

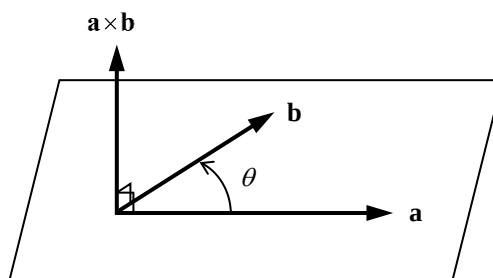
6 VECTOR (CROSS) PRODUCT

6.1 Definition

The **vector (cross) product** of two non-zero vectors **a** and **b**, written as **$\mathbf{a} \times \mathbf{b}$** , is a vector defined as

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin \theta) \hat{\mathbf{n}}$$

where θ is the angle between **a** and **b** and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both **a** and **b**, in the direction of a right-hand screw turned from **a** to **b**.



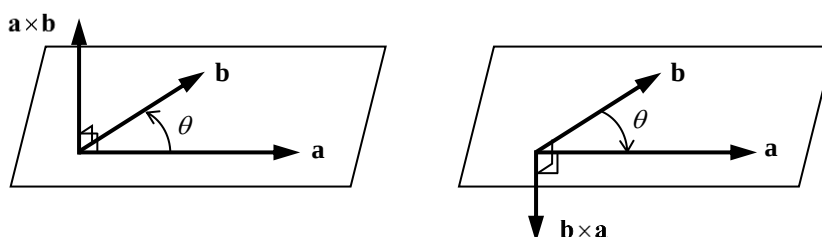
Magnitude: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin \theta$ where θ is the angle between **a** and **b**

Direction: **$\mathbf{a} \times \mathbf{b}$** is perpendicular to both **a** and **b**, in the direction of a right-hand screw turned from **a** to **b**.

6.2 Properties of Vector (Cross) Product

- $\mathbf{a} \times \mathbf{b}$** is a vector perpendicular to both **a** and **b**.
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$** , i.e. vector product is **not** commutative.

Given vectors **a** and **b**, we use the right hand screw rule to determine the direction of the perpendicular vectors as follows.



The perpendicular vectors are in opposite directions, hence $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

(iii) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. In particular, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

More generally, if $\mathbf{a} \parallel \mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

(iv) $\lambda(\mathbf{a} \times \mathbf{b}) = \lambda\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \lambda\mathbf{b}$, where $\lambda \in \mathbb{R}$.

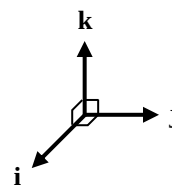
(v) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive property)

(vi) If $\mathbf{a} \perp \mathbf{b}$, then $\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin 90^\circ)\hat{\mathbf{n}} = |\mathbf{a}||\mathbf{b}|\hat{\mathbf{n}}$.

In particular, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

(vii) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \quad (\text{Found in MF27})$$



Example 15

Find a vector perpendicular to both $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - 2\mathbf{j}$.

Solution



■

Note

- To check the correctness of the vector product, we use scalar product as $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4 - 1 - 3 = 0 \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 2 - 2 + 0 = 0$$

Example 16

Three non-zero vectors, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, are such that $4\mathbf{a} \times \mathbf{b} = \mathbf{a} \times 3\mathbf{c}$. Show that $4\mathbf{b} - 3\mathbf{c} = k\mathbf{a}$ where k is a scalar.

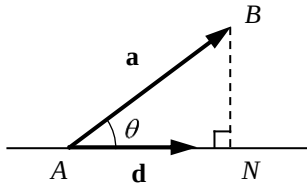
Solution

■

6.3 Perpendicular Distance from a Point to a Line

Suppose $\mathbf{a} = \overrightarrow{AB}$ is a vector and l is a line through A . Let $\mathbf{d}$ be a vector parallel to the line l , and N be the foot of the perpendicular from B to l .

We wish to find the perpendicular distance from B to the line l , that is, the distance BN .



If the angle between $\mathbf{a}$ and $\mathbf{d}$ is acute,

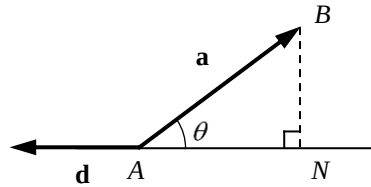
$$BN = AB \sin \theta$$

$$= |\mathbf{a}| \frac{|\mathbf{a} \times \mathbf{d}|}{|\mathbf{a}||\mathbf{d}|}$$

$$= \frac{|\mathbf{a} \times \mathbf{d}|}{|\mathbf{d}|}$$

$$= \left| \mathbf{a} \times \frac{\mathbf{d}}{|\mathbf{d}|} \right|$$

$$= |\mathbf{a} \times \hat{\mathbf{d}}|$$



If the angle between $\mathbf{a}$ and $\mathbf{d}$ is obtuse,

$$BN = AB \sin \theta = AB \sin(180^\circ - \theta)$$

$$= |\mathbf{a}| \frac{|\mathbf{a} \times \mathbf{d}|}{|\mathbf{a}||\mathbf{d}|}$$

$$= \frac{|\mathbf{a} \times \mathbf{d}|}{|\mathbf{d}|}$$

$$= \left| \mathbf{a} \times \frac{\mathbf{d}}{|\mathbf{d}|} \right|$$

$$= |\mathbf{a} \times \hat{\mathbf{d}}|$$

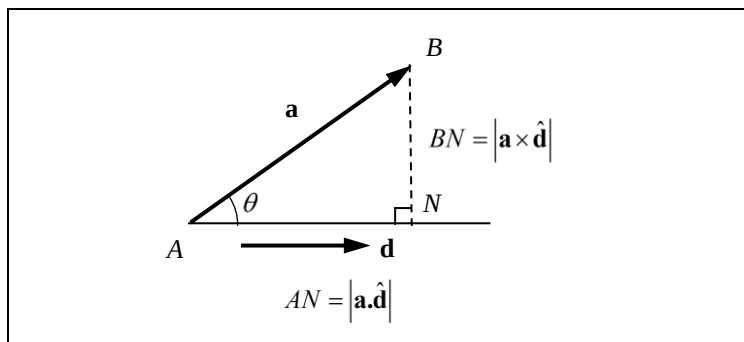
In both cases, we find that the

perpendicular distance from B to the line l is $|\mathbf{a} \times \hat{\mathbf{d}}|$

where $\mathbf{a} = \overrightarrow{AB}$ and l is the line through A parallel to $\mathbf{d}$.

This is the **geometrical meaning** of $|\mathbf{a} \times \hat{\mathbf{d}}|$.

Combining with the result on projection length, we have



Example 17

The points A and C have coordinates $(-1, 5, -1)$ and $(2, 3, 6)$ respectively. Find the shortest distance from point A to the line OC .

Solution

■

Extra Practice

Annex B: Practice Question 4

6.4 Other Applications of the Vector Product

Let $OACB$ be a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

Then the area of $OACB$ is $|\mathbf{a} \times \mathbf{b}|$.

Proof

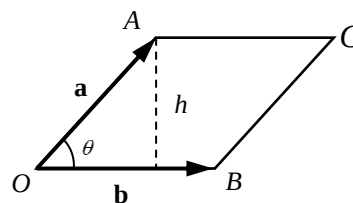
Since $h = OA \sin \theta$,

Area of parallelogram $= (OB)(h)$

$$= |\mathbf{b}|h$$

$$= |\mathbf{a}||\mathbf{b}|\sin \theta$$

$$= |\mathbf{a} \times \mathbf{b}|$$



■

Area of the parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as its adjacent sides is $|\mathbf{a} \times \mathbf{b}|$.

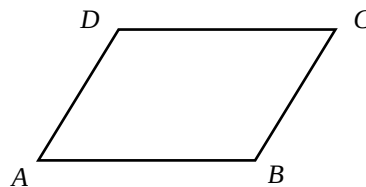
This is the **geometrical meaning** of $\boldsymbol{\theta}$.

$$\text{Hence area of } \triangle OAB = \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin \theta = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

Example 18

The parallelogram $ABCD$ has three of its vertices A , B and C at the points $(1, 2, -1)$, $(1, 3, 2)$ and $(-1, 3, -1)$. Find the area of the parallelogram.

Solution



■

Example 19

Show that the area of a triangle ABC is $\frac{1}{2}|\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are position vectors of A , B and C .

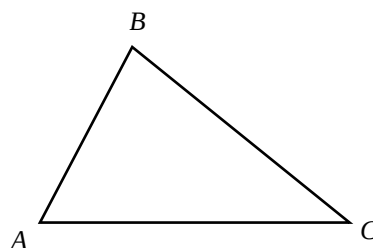
Hence, find the area of $\triangle ABC$ given that the position vectors of A , B and C relative to the origin O are $-3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$, $-\mathbf{i} - 5\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ respectively.

Solution

$$\text{Area of } \triangle ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

(or any two pairs of adjacent sides)

$$\begin{aligned} &= \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})| \\ &= \frac{1}{2}|\mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{a}| \\ &= \frac{1}{2}|\mathbf{b} \times \mathbf{c} - (-\mathbf{c} \times \mathbf{a}) - (-\mathbf{a} \times \mathbf{b})| \\ &= \frac{1}{2}|\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}| \end{aligned}$$



$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -3 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 20 \\ -13 \\ -4 \end{pmatrix}$$

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ -5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \\ -2 \end{pmatrix}$$

$$\mathbf{c} \times \mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} -3 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -16 \\ 11 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \therefore \text{Area of } \triangle ABC &= \frac{1}{2} \left| \begin{pmatrix} 20 \\ -13 \\ -4 \end{pmatrix} + \begin{pmatrix} 10 \\ -8 \\ -2 \end{pmatrix} + \begin{pmatrix} -16 \\ 11 \\ 2 \end{pmatrix} \right| \\ &= \frac{1}{2} \left| \begin{pmatrix} 14 \\ -10 \\ -4 \end{pmatrix} \right| = \sqrt{78} \text{ units}^2 \end{aligned}$$

■

Extra Practice

Annex B: Practice Question 5

Example 20 (Applications of Vectors in Physics)

The work done, W , by a force, $\mathbf{F}$, measured in Newtons, on an object that has moved by a displacement $\mathbf{s}$, measured in metres, is given by

$W = \mathbf{F} \cdot \mathbf{s}$, where W is measured in joules.

A force given by $\mathbf{F} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ acts on a particle and moves it a distance of 12 metres in the direction of the vector $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find $\mathbf{s}$ and hence find W .

*Solution*Displacement $\mathbf{s}$

$$\begin{aligned} \therefore W = \mathbf{F} \cdot \mathbf{s} &= \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix} \\ &= 32 + 16 - 12 \\ &= 36 \text{ J} \end{aligned}$$

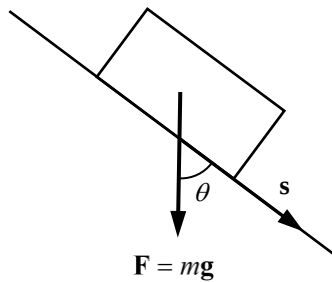
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ANNEX A

Further Exploration: Applications of Vectors in Physics

The scalar and vector products were originally defined and used by physicists such as William Clifford, Oliver Heaviside and Josiah Gibbs to help deal with problems in the study of electromagnetism.

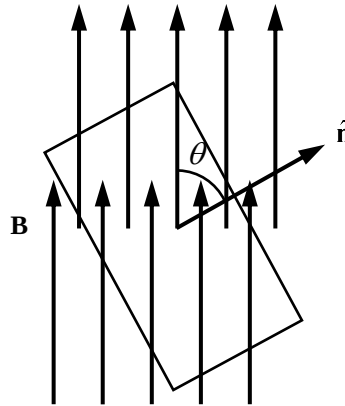
They turn out to have many applications in various branches of physics as well.



The work done by a force $\mathbf{F}$ on an object that has moved by a displacement $\mathbf{s}$ is given by

$$W = \mathbf{F} \cdot \mathbf{s}$$

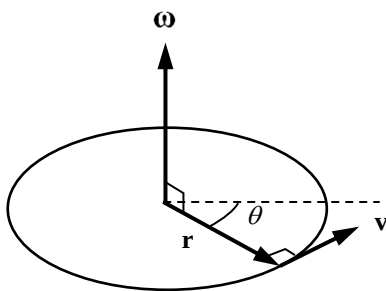
(Refer to Example 20 on page 26)



The magnetic flux of a field $\mathbf{B}$ through an area A is given by

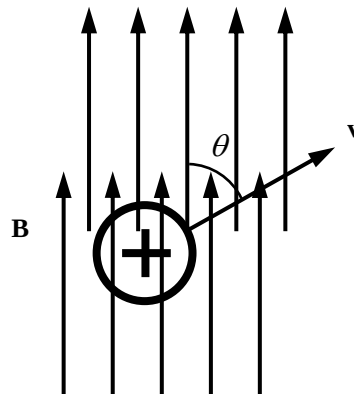
$$\Phi = (\mathbf{B} \cdot \hat{\mathbf{n}}) A$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the area.



The angular velocity of an object with position vector $\mathbf{r}$ and velocity $\mathbf{v}$ is given by

$$\boldsymbol{\omega} = \frac{1}{|\mathbf{r}|^2} (\mathbf{r} \times \mathbf{v}).$$



The force on a charged particle with charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}).$$

ANNEX B**Practice Questions on Vectors I**

- 1 Let $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OB} = -3\mathbf{i} + 2\mathbf{k}$. Find $\overrightarrow{OP}$ if
- P is a point on the line segment AB such that $\frac{AP}{PB} = \frac{1}{4}$.
 - P is a point on the line AB such that $\overrightarrow{BA} = 3\overrightarrow{BP}$.
 - P is a point on the line AB produced such that $AB : BP = 3 : 2$.
- 2 Find the angle between the following pairs of vectors.
- $\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
 - $\begin{pmatrix} -1 \\ -1 \\ \sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
 - $\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
- 3 Given that $\mathbf{a} = 4\mathbf{i} + 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ find the length of the projection and the vector projection of
- $\mathbf{a}$ along $\mathbf{c}$,
 - $\mathbf{b}$ along $\mathbf{a}$,
 - $\mathbf{c}$ along $\mathbf{b}$.
- 4 Given the points $A(1, 2, -1)$, $B(3, 4, 0)$, $C(1, 5, -2)$ and that O is the origin, find the shortest distance
- from point C to the line AB ,
 - from point O to the line AC ,
 - from point B to the line OA .
- 5
- The vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ represent two sides of a triangle. Find the area of the triangle.
 - The triangle ABC has its vertices at the points $A(0, 0, 1)$, $B(1, 0, 1)$, $C(2, 1, 3)$. Find the area of the triangle ABC .
 - A parallelogram $OABC$ has one vertex O at the origin and the vertices A and B at the points $(0, 1, 3)$ and $(0, 2, 5)$. Find the area of $OABC$.

Answers

- 1 (a) $\mathbf{i} + \frac{12}{5}\mathbf{j} - \frac{6}{5}\mathbf{k}$ (b) $-\frac{4}{3}\mathbf{i} + \mathbf{j} + \frac{2}{3}\mathbf{k}$ (c) $-\frac{19}{3}\mathbf{i} - 2\mathbf{j} + \frac{14}{3}\mathbf{k}$
- 2 (a) 20.4° (b) 135° (c) 41.4°
- 3 (i) $\frac{2}{3}$, $-\frac{2}{9}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ (ii) $\frac{18}{5}$, $\frac{18}{25}(4\mathbf{i} + 3\mathbf{k})$ (iii) $\frac{1}{\sqrt{14}}$, $-\frac{1}{14}(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$
- 4 (i) $\frac{\sqrt{65}}{3}$ (ii) $\sqrt{\frac{11}{10}}$ (iii) $\sqrt{\frac{29}{6}}$
- 5 (a) $\frac{1}{2}\sqrt{35}$ (b) $\frac{1}{2}\sqrt{5}$ (c) 1