RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Additional Practice Questions for
Chapter 7B: Arithmetic Progression and Geometric Progression

1 **JJC/Prelim 9233/2006/01/Q1**
The sum of the first 100 terms of an arithmetic progression is 15050; the first, third and eleventh terms of this progression are three consecutive terms of a geometric progression. Find the first term, a and the non-zero common difference, d, of the arithmetic progression. [5]

Solutions

$$S_{100} = \frac{100}{2}\left(2a + 99d\right) = 15050$$
$$2a + 99d = 301 \qquad \text{---------(1)}$$
Also, $\dfrac{u_{11}}{u_3} = \dfrac{u_3}{u_1} \Rightarrow {u_3}^2 = u_{11}u_1$
$$\Rightarrow \left(a + 2d\right)^2 = a\left(a + 10d\right)$$
$$\Rightarrow 4d^2 = 6ad \text{ -------- (2)}$$
Since $d \neq 0 \therefore d = \dfrac{3}{2}a$

Substitute into (1), $2a + 99\left(\dfrac{3}{2}a\right) = 301 \Rightarrow a = 2, \quad d = 3.$

2 Jack starts working in a company with an annual salary of \$16 000 in the first year. He will receive an annual salary increase of 4% each year. Assuming that he works in the company till his retirement, find

(i) the amount he will earn in his 25th year, [2]
(ii) the total amount that he will earn over the 25-year period. [2]
(iii) the minimum number of years he has to work for his total earnings to exceed \$1 000 000. [4]

Solutions

(i)	$u_{25} = (16000)(1.04)^{25-1} = 41012.87$ (nearest cent) He will earn \$41 012.87 in his 25th year.
(ii)	$S_{25} = \dfrac{16000(1.04^{25} - 1)}{1.04 - 1} = 666334.53$ (nearest cent) He earns \$666 334.53 over the 25-year period.

(iii)	$\dfrac{16000(1.04^n-1)}{1.04-1}>1000000$ $1.04^n-1>2.5$ $1.04^n > 3.5$ $\ln 1.04^n > \ln 3.5$ $n\ln 1.04 > \ln 3.5$ $n>\dfrac{\ln 3.5}{\ln 1.04}$ $n > 31.9$ Minimum number of years he has to work = 32

3 VJC Promo 9758/2020/Q7

An infinite geometric series has first term a and common ratio r, where $r>0$. The third term is 36 and the sum to infinity is 243.

(i) Find the value of a and r. [3]

An arithmetic series has first term 1 and common difference d. The sum of the first 6 terms of the arithmetic series is equal to the sum of the first 3 terms of the geometric series.

(ii) Find the value of d. [3]

(iii) Find the least value of n for which the nth term of the arithmetic series is more than the sum of the first $2n$ terms of the geometric series. **[3]**

Solutions

(i)	Let u_n be the n^{th} term of the geometric series. $u_n = ar^{n-1}$ $u_3 = 36$ $ar^2 = 36 ----------(1)$ $S_\infty = 243$ $\dfrac{a}{1-r} = 243$ $a = 243(1-r) -------(2)$ Substitute (2) into (1), $243(1-r)r^2 = 36$ $243r^2 - 243r^3 - 36 = 0$ From GC, $r = -\dfrac{1}{3}$ (reject $\because r>0$) or $\dfrac{2}{3}$ $\therefore$ Common ratio, $r = \dfrac{2}{3}$.

	Substitute $r=\frac{2}{3}$ into (2), $a=243\left(1-\frac{2}{3}\right)=81$ $\therefore$ First term, $a=81$.
(ii)	$\frac{6}{2}\left[2(1)+(6-1)d\right]=\frac{81\left[1-\left(\frac{2}{3}\right)^3\right]}{1-\frac{2}{3}}$ $3(2+5d)=171$ $d=11$
(iii)	$1+(n-1)(11)>\frac{81\left[1-\left(\frac{2}{3}\right)^{2n}\right]}{1-\frac{2}{3}}$ $1+11n-11>243\left[1-\left(\frac{2}{3}\right)^{2n}\right]$ $243\left[1-\left(\frac{2}{3}\right)^{2n}\right]-11n+10<0$ Let $\mathrm{f}(n)=243\left[1-\left(\frac{2}{3}\right)^{2n}\right]-11n+10$ <table><tr><th>n</th><th>$\mathrm{f}(n)$</th></tr><tr><td>22</td><td>11 > 0</td></tr><tr><td>23</td><td>–0.000002 < 0</td></tr></table> The least value of n is 23.

4 YIJC Prelim 9758/2023/01/Q5

(a) An infinite geometric progression has first term a and common ratio r, where a and r are non-zero. The sum of all the terms after the nth term of the progression is equal to twice the nth term. Show that the sum to infinity of the progression is three times the first term. [3]

(b) The positive integers, starting at 1, are grouped into sets, as follows.

$$\{1\}, \{2,3\}, \{4,5,6\}, \ldots$$

(i) Find, in terms of r, the first integer and the last integer in the rth set. [3]

(ii) Prove that the sum of the integers in the rth set is $\frac{1}{2}r\left(1+r^2\right)$. [2]

Solutions

(a)	Sum of the all the terms after the nth term $= S_\infty - S_n = \dfrac{a}{1-r} - \dfrac{a\left(1-r^n\right)}{1-r}$ $= \dfrac{ar^n}{1-r}$ Given $S_\infty - S_n = 2u_n$, therefore $\dfrac{ar^n}{1-r} = 2ar^{n-1}$ $r = 2(1-r)$ $r = \dfrac{2}{3}$ Hence $S_\infty = \dfrac{a}{1-r} = \dfrac{a}{1-\frac{2}{3}} = 3a$ (Shown)
(b)(i)	Total number of integers in the first (r–1)th brackets is $1+2+3+\ldots+(r-1) = \dfrac{r-1}{2}\left(1+(r-1)\right) = \dfrac{r\left(r-1\right)}{2}$ Hence, first integer in the rth bracket $= \dfrac{r\left(r-1\right)}{2} + 1 = \dfrac{r^2-r+2}{2}$ Last integer in the rth bracket

	$=\dfrac{r^2-r+2}{2}+(r-1)$ $=\dfrac{r^2-r+2+2r-2}{2}$ $=\dfrac{r^2+r}{2}$ <u>Alternative method</u>: Last integer in the rth bracket = First integer in the (r+1)th bracket minus 1 $=\left[\dfrac{(r+1)(r)}{2}+1\right]-1=\dfrac{r^2+r}{2}$
(b)(ii)	There are r integers in the rth bracket. First integer in the rth bracket = $\dfrac{r^2-r+2}{2}$ Last integer in the rth bracket $=\dfrac{r^2+r}{2}$ Sum of all the integers in the rth bracket $=\dfrac{r}{2}\left(\dfrac{r^2-r+2}{2}+\dfrac{r^2+r}{2}\right)=\dfrac{r}{2}\left(\dfrac{2r^2+2}{2}\right)$ $=\dfrac{1}{2}r\left(1+r^2\right)$ (Shown)

5 **9758/2017/02/Q2**

An arithmetic progression has first term 3. The sum of the first 13 terms of the progression is 156.

(i) Find the common difference. [2]

A geometric progression has first term 3 and common ratio r. The sum of the first 13 terms of the progression is 156.

(ii) Show that $r^{13}-52r+51=0$. Show that the common ratio cannot be 1 even though $r=1$ is a root of this equation. Find the possible values of the common ratio. [4]

(iii) It is given that the common ratio of the geometric progression is positive, and that the nth term of this geometric progression is more than 100 times the nth term of the arithmetic progression. Write down an inequality, and hence find the smallest possible value of n. [3]

Solutions

(i)	Let a and d be the first term and common difference of the AP. Given that $a=3$ and $S_{13}=156$, $\frac{13}{2}\left[2(3)+(13-1)d\right]=156$ $d=1.5$
(ii)	Let a and r be the first term and common ratio of the GP. Given that $a=3$ and $S'_{13}=156$, $\frac{3\left(1-r^{13}\right)}{1-r}=156$ $1-r^{13}=52(1-r)$ $r^{13}-52r+51=0$ (shown) When $r=1$, $r^{13}-52r+51=0$. $\therefore$ $r=1$ is a root of the equation. However, if $r=1$, all the terms of the GP will be 3 and $S_{13}=3(13)=39\neq 156$. $\therefore$ $r\neq 1$. Using GC, $r=-1.45107$ or $r=1.21002$ $r=-1.45$ (3 sf) or $r=1.21$ (3 sf)
(iii)	Since $r>0$, $r=1.21002$ nth term of GP $>$ $100\times(n\text{th term of AP})$ $ar^{n-1}>100\left(a+(n-1)d\right)$ $3\left(1.21002^{n-1}\right)>100\left(3+1.5(n-1)\right)$ $3\left(1.21002^{n-1}\right)-150-150n>0$ Let $y=3\left(1.21002^{n-1}\right)-150-150n$ From G.C., <table><tr><td>n</td><td>y</td></tr><tr><td>41</td><td>-150.7 < 0</td></tr><tr><td>42</td><td>990.73 > 0</td></tr></table> $\therefore$ the smallest value of n is 42.

6 SAJC Promo 9758/2020/Q6

The sum of the first n terms of a series, G, is given by $S_n = c\left(2 - 2^{1-n}\right)$, where c is a positive constant.

(i) Show that the n^{th} term of the series G, $T_n = c2^{1-n}$. [2]

(ii) Hence, prove that the series G is a geometric series. [2]

(iii) It is given that the sixth term of G is $\frac{3}{8}$. Show that $T_1 = 12$. [2]

(iv) The first term and common difference of another arithmetic series A are the same as the first term and the common ratio of G respectively. Find the least value of n such that the sum of the first n terms of the arithmetic series A is more than one-quarter the sum of the first n terms of the series G by at least 250. [3]

Solutions

(i)	Given $S_n = c\left(2-2^{1-n}\right)$, For $n \geq 2$, $T_n = S_n - S_{n-1}$ $= c\left(2-2^{1-n}\right) - c\left(2-2^{2-n}\right)$ $= c\left(2^{2-n} - 2^{1-n}\right)$ $= c2^{1-n}(2-1)$ $= c2^{1-n}$ $T_1 = S_1 = c\left(2-2^{1-1}\right) = c = c2^{1-1}$ $\therefore T_n = c2^{1-n}$ for $n \geq 1$
(ii)	$\dfrac{T_n}{T_{n-1}} = \dfrac{c2^{1-n}}{c2^{1-(n-1)}} = \dfrac{c2^{1-n}}{c2^{2-n}} = 2^{1-n-(2-n)} = \dfrac{1}{2}$ Since ratio between any two consecutive terms is a constant, $\frac{1}{2}$, independent of n, the given series is a geometric series. (shown)
(iii)	Given $T_6 = \dfrac{3}{8}$, $c2^{1-6} = \dfrac{3}{8}$ $c = 12$ $T_1 = 12 \times 2^{1-1} = 12$ (shown)
(iv)	$r = d = \dfrac{1}{2}$, where d is the common difference of A

	$\frac{n}{2}\left[2(12)+(n-1)\left(\frac{1}{2}\right)\right] \geq \frac{1}{4}\left[12(2-2^{1-n})\right]+250$ $\frac{n}{2}\left[2(12)+(n-1)\left(\frac{1}{2}\right)\right]-\frac{1}{4}\left[12(2-2^{1-n})\right] \geq 250$ By G.C, the least value of n is 17.

n	$\frac{n}{2}\left[2(12)+(n-1)\left(\frac{1}{2}\right)\right]-\frac{1}{4}\left[12(2-2^{1-n})\right]$
16	$246 < 250$
17	$266 > 250$

7 MI Promo 9758/2020/PU1/Q9

Two athletes are training for a long distance race. During a training session, they run in 10-minute intervals and the distances they run in each 10-minute interval are recorded. Athlete A runs 2000 metres in the first 10-minute interval. On each subsequent 10-minute interval, she runs 45 metres less than in the previous interval.

(i) What distance is run on the 8th 10-minute interval? [1]

When Athlete A runs less than 1200 metres in a 10-minute interval, she will stop running after completing that interval.

(ii) Find the number of 10-minute intervals that Athlete A runs. [3]

(iii) Hence find the total distance run by Athlete A. [2]

Athlete B runs 2500 metres in the first 10-minute interval. On each subsequent 10-minute interval, she runs $\frac{11}{12}$ of the distance run on the previous interval.

(iv) Determine which athlete runs a longer distance on the 8th 10-minute interval. [2]

(iv) How many complete 10-minute intervals does it take for Athlete B to first exceed 95% of the theoretical maximum distance? [4]

Solutions

(i)	$T_8 = 2000+(8-1)(-45)=1685\text{m}$ Athlete A ran 1685m in the 8th 10 minute interval.

(ii)	$2000+(n-1)(-45)<1200$ **<u>Method 1: Table of Values</u>** Solve $2000+(n-1)(-45)<1200$ <table><tr><th>n</th><th>$2000+(n-1)(-45)$</th></tr><tr><td>18</td><td>$1235>1200$</td></tr><tr><td>19</td><td>$1190<1200$</td></tr></table> $\therefore$ Athlete A runs 19 10-minute intervals. **<u>Method 2: Algebraic</u>** $2000+(n-1)(-45)<1200$ $n>18.778$ $\therefore$ Athlete A runs 19 10-minute intervals.
(iii)	Total Distance $=\frac{19}{2}\left(2(2000)+(19-1)(-45)\right)=30305$ m
(iv)	For Athlete B, $T_8=2500\left(\frac{11}{12}\right)^{8-1}$ $=1359.63$ m Since Athlete A runs 1685 m on the 8th interval, Athlete A runs a longer distance on the 8th interval.
(v)	$S_\infty=\frac{2500}{1-\frac{11}{12}}=30000$ $S_n>0.95S_\infty$ $\frac{2500\left(1-\left(\frac{11}{12}\right)^n\right)}{1-\frac{11}{12}}>0.95\times 30000$ $30000\left(1-\left(\frac{11}{12}\right)^n\right)>28500$ **Method 1a: Using Table of Values to solve** $30000\left(1-\left(\frac{11}{12}\right)^n\right)>28500$ <table><tr><th>n</th><th>$30000\left(1-\left(\frac{11}{12}\right)^n\right)$</th></tr><tr><td>34</td><td>$28443<28500$</td></tr><tr><td>35</td><td>$28573>28500$</td></tr></table>

	Athlete B takes 35 complete 10 minutes intervals. **Method 2:** $1-\left(\frac{11}{12}\right)^n > 0.95$ $\left(\frac{11}{12}\right)^n < 0.05$ $n\ln\left(\frac{11}{12}\right) < \ln(0.05)$ $n > \frac{\ln(0.05)}{\ln\left(\frac{11}{12}\right)}$ $n > 34.429$ Athlete B takes 35 complete 10 minutes intervals.

8 TJC Promo 9758/2020/Q12

A bank offers both an ordinary account and a savings account.
On 1 January 2020, Perlin puts \$100 into an ordinary account. On the first day of each subsequent month, she saves \$5 more than in the previous month, so that she saves \$105 on 1 February 2020, \$110 on 1 March 2020, and so on. This account pays no interest.

(i) On what date will the amount in Perlin's account first exceed \$5,000? [5]

On 1 January 2020, Pauline puts \$100 into a savings account, and on the first day of each subsequent month she puts another \$$x$ into the account. The interest rate is 0.5% per month, so that on the last day of each month the amount in the account on that day is increased by 0.5%.

(ii) Show that the expression for the amount in Pauline's account on the last day of the nth month (where January 2020 is the 1st month, February 2020 is the 2nd month, and so on) is given by

$$1.005^n(200x+100)-201x\,. \quad [3]$$

(a) If $x=150$, find the number of complete months for the total amount in Pauline's account to first exceed \$5,000. [2]

(b) Pauline wishes for the amount in her account to be at least \$5,000 on 31 October 2021 in order to purchase a first-class flight to Japan for her year-end vacation. What is the smallest integer value of x would achieve this? [2]

Solutions

(i)

	Amount in account at beginning of month
1 Jan 2020	\$100
1 Feb 2020	$\$(100+105)$
1 Mar 2020	$\$(100+105+110)$

In n months, total amount in the saving account $S_n = 100+105+110+\ldots+$ (nth term)

$$= \frac{n}{2}\left[2(100)+5(n-1)\right]$$

$$= 100n + \frac{5}{2}n(n-1)$$

OR

$$S_n = 100n + 5\left[1+2+3+\ldots+(n-1)\right]$$

$$= 100n + \frac{5(n-1)}{2}[1+(n-1)]$$

$$= 100n + \frac{5}{2}n(n-1)$$

$$100n + \frac{5}{2}n(n-1) > 5000$$

Let $\mathrm{f}(n) = 100n + \frac{5}{2}n(n-1)$

From GC,

n	$\mathrm{f}(n)$
29	$4930 < 5000$
30	$5175 > 5000$

$\Rightarrow$ Least $n = 30$

Date is 1 June 2022.

OR

$$100n + \frac{5}{2}n(n-1) > 5000$$

$\Rightarrow \quad n^2 + 39n - 2000 > 0$

$\Rightarrow \quad n > 29.3 \quad$ or $\quad n < -68.3$

(Reject, since $n \geq 0$)

$\therefore$ Least $n = 30$

(ii)

	Beginning of month	End of month
1 Jan 2020	100	1.005(100)

	1 Feb 2020	$1.005(100)+x$	$1.005\left[1.005(100)+x\right]$ $=1.005^2(100)+1.005x$
	1 Mar 2020	$1.005^2(100)+1.005x+x$	$1.005\left[1.005^2(100)+1.005x+x\right]$ $=1.005^3(100)+1.005^2x+1.005x$

Total amount in Pauline's account at the end of nth month

$$=1.005^n(100)+1.005^{n-1}x+1.005^{n-2}x+...+1.005x$$

$$=1.005^n(100)+x\left[\frac{1.005(1.005^{n-1}-1)}{1.005-1}\right]$$

$$=1.005^n(100)+201x(1.005^{n-1}-1)$$

$$=1.005^n(100)+200x(1.005^n)-201x$$

$$=1.005^n(200x+100)-201x \quad \text{(shown)}$$

(ii) (a) When $x=150$, $1.005^n(30100)-30150>5000$

Let g(n) = $1.005^n(30100)-30150$

n	g(n)
31	4982.9 < 5000
32	5158.6 > 5000

From GC, least integer n = 32

OR $n>\dfrac{\lg\left(\frac{35150}{30100}\right)}{\lg 1.005}=31.1$

$\therefore$ Least $n=32$

It takes 32 complete months for Pauline to first save over $5,000.

(b) From 1 Jan 2020 to 31 Oct 2021, there are 22 months.

$$1.005^{22}(200x+100)-201x\geq 5000$$

$$x\geq\frac{4888.403}{22.19}=220.2$$

OR

Let h(n) = $1.005^{22}(200x+100)-201x$

N	h(n)
220	4994.4<5000
221	5016.6>5000

Using GC, smallest integer n = 221

9 EJC Promo 9758/2020/Q11 (modified)

(i) On 1 January 2020, Ms Eu took a study loan of $20,000 from BOPS bank to fund her part-time studies. Based on the loan agreement, Ms Eu will make a yearly repayment of $$x$ on 30 December of each year, starting from 30 December 2020. Furthermore, an interest rate of 4% per year is applied on the amount outstanding as at 31 December of each year, starting from 31 December 2020.

(a) Show that the amount outstanding at the end of the n^{th} year is

$$1.04^n(20000) - kx\left(1.04^n - 1\right),$$

where k is a constant to be determined. [3]

(b) Determine the value of x if Ms Eu intends to repay the loan fully at the end of 2024. [3]

(ii) While embarking on her part-time studies, Ms Eu started a new job and received her first monthly salary of $3,300 on 1 July 2020. She saved $1,600 from her first pay and deposited it into an account that paid no interest. Ms Eu intends to increase the amount saved from her salary by $50 per month, until her account balance exceeds $35,000. She will then transfer the full amount in this account into a fixed deposit in BOPS Bank immediately. She will keep the money for 12 months in the fixed deposit which earns a simple interest of 1.5% per year.

(a) Find the month and year in which Ms Eu will withdraw the fixed deposit. [3]

(b) Determine the value of the fixed deposit that Ms Eu will withdraw. [1]

(iii) After withdrawing the fixed deposit, Ms Eu has two options:

Option 1: Deposit $30,000 in a fixed deposit that earns a simple interest of 1.5% per year, assuming that the amount in the fixed deposit account remains at $30,000 each year.

Option 2: Invest $30,000 in a fund that will yield a compound interest of 1.2% per year, assuming that no withdrawal was made.

Showing your working clearly, explain which option would be more suitable for Ms Eu if she wants to grow the value of this $30,000 in the long term. [2]

Solutions

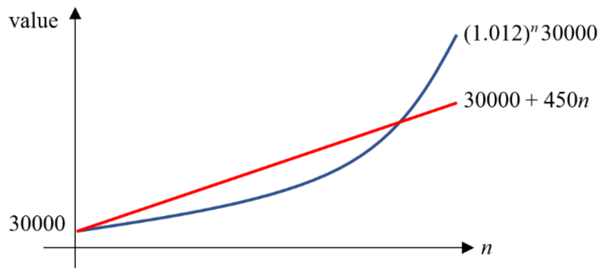
(i)(a)

Year	Amount outstanding on **31 Dec**
1	$1.04(20000-x)=1.04(20000)-1.04x$
2	$1.04\left[1.04(20000)-1.04x-x\right]$ $=1.04^2(20000)-1.04^2x-1.04x$
3	$1.04\left[1.04^2(20000)-1.04^2x-1.04x-x\right]$ $=1.04^3(20000)-1.04^3x-1.04^2x-1.04x$

	<table><tr><td>$\vdots$</td><td></td></tr><tr><td>n</td><td>$1.04^n(20000)-1.04^n x-1.04^{n-1}x-...-1.04x$</td></tr></table>Amount outstanding at the end of the n^{th} year $=1.04^n(20000)-1.04^n x-1.04^{n-1}x-...-1.04x$ $=1.04^n(20000)-\dfrac{1.04x\left(1.04^n-1\right)}{1.04-1}$ $=1.04^n(20000)-26x\left(1.04^n-1\right)$ where $k=26$.
(i)(b)	From year 2020 to 2024, we have $n=5$ $1.04^5(20000)-26x\left(1.04^5-1\right)=0$ $x=\dfrac{1.04^5\left(20000\right)}{26\left(1.04^5-1\right)}=4319.75$
(ii)(a)	$\dfrac{n}{2}\left(2(1600)+50(n-1)\right)>35000$ Let f(n) = $\dfrac{n}{2}\left(2(1600)+50(n-1)\right)$ <table><tr><td>n</td><td>f(n)</td></tr><tr><td>17</td><td>34000 < 35000</td></tr><tr><td>18</td><td>36450 > 35000</td></tr></table>$\therefore n\geq 18$ Ms Eu will transfer her savings in December 2021 and withdraw the fixed deposit in December 2022.
	Explanatory notes: It takes 18 months for the savings to reach \$36450 (to exceed \$35000). Ms Eu started her savings account (that pays no interest) in July 2020 and kept up her monthly deposits for 18 months; she would have exceeded \$35000 in Dec 2021. As given, Ms Eu *immediately* transferred the full amount into a fixed deposit. That is, the fixed deposit of \$36450 *started* in Dec 2021. Since the fixed deposit is for 12 months, Ms Eu will withdraw from this account in Dec 2022.
(ii)(b)	Amount placed in fixed deposit = \$36450 Amount available to withdraw $=\$36450\times 1.015$ $=\$36996.75$
(iii)	Let n be the number of years \$30000 was invested. Value of fixed deposit in the long run

$= 30000 + n\left[\frac{1.5}{100} \times 30000\right] = 30000 + 450n$

Value of money in the fund in the long run $= (1.012)^n \, 30000$



As shown in the graph, in the long term, the value of money in the fund will be greater than the value of the fixed deposit. Ms Eu should choose option 2.

OR

For $(1.012)^n \, 30000 > 30000 + 450n$ when $n \geq 37.05$

Since it takes at least 38 years for the amount in Option 2 to catch up to Option 1, which is quite long, Ms Eu might find Option 1 more suitable as it will offer her more flexibility in how she can grow the $30,000 in the long term.

10 YIJC Promo 9758/2020/Q13

A mask production factory produces 5000 reusable masks in the first week. In each subsequent week, the number of reusable masks produced is 200 more than the previous week.

(i) Find the number of reusable masks produced in the 10th week. [2]

(ii) Find the minimum number of weeks required by the factory to produce a total number of at least 100 000 reusable masks. [3]

Due to a certain flu pandemic, the demand for reusable masks has increased drastically. If the demand for reusable masks in the preceding week is M, it would become $100 + aM$ this week.

(iii) Given that the demand for reusable masks in the first week is 500, show that the demand for reusable masks in the Nth week can be written as $\frac{100}{a-1}\left(5a^N - 4a^{N-1} - 1\right)$. [3]

It is now given that $a = 1.15$.

To meet the growing demand of reusable masks, the factory adjusts its production capacity such that the number of reusable masks produced from the 10th week onwards is k more than the previous week.

(iv) Find the least value of k required by the factory in order to meet the total demand for the first 25 weeks. [4]

Solutions

(i)

AP : first term = 5000, common difference = 200

$$u_{10} = 5000 + (10-1)200$$
$$= 6800$$

Number of reusable masks produced in the 10th week is 6800.

(ii)

$$S_n \geq 100000$$
$$\frac{n}{2}\left[2(5000)+(n-1)200\right] \geq 100000$$
$$5000n + 100n^2 - 100n \geq 100000$$
$$n^2 + 49n - 1000 \geq 0$$

Method 1

$n \leq -64.5$ or $n \geq 15.5$

Hence, minimum number of weeks is 16.

Method 2

Using GC,

n	$n^2+49n-1000$
15	-40 (<0)
16	40 (>0)

Hence, minimum number of weeks is 16.

(iii)

Week	Demand
1	500
2	$100 + 500a$
3	$100+(100+500a)a$ $=100+100a+500a^2$
4	$100+(100+100a+500a^2)a$ $=100+100a+100a^2+500a^3$
$\vdots$	$\vdots$
N	$100+100a+\cdots+100a^{N-2}+500a^{N-1}$

The first $N-1$ terms follows a GP with first term = 100 and common ratio = a.

Demand for reusable masks in week N

	$=100\left(\dfrac{a^{N-1}-1}{a-1}\right)+500a^{N-1}$ $=\dfrac{100}{a-1}\left(a^{N-1}-1+5a^{N-1}(a-1)\right)$ $=\dfrac{100}{a-1}\left(5a^{N}-4a^{N-1}-1\right)$ (shown)
(iv)	Total number of reusable masks produced in the first 25 weeks =Total number of reusable masks produced in the first 8 weeks + Total number of reusable masks produced in the next 17 weeks $=\dfrac{8}{2}\left[2(5000)+(8-1)200\right]+\dfrac{17}{2}\left[2(6800-200)+(17-1)k\right]$ $=45600+112200+136k$ $=157800+136k$ Total demand of reusable masks in the first 25 weeks $=\sum_{N=1}^{25}\left[\dfrac{100}{1.15-1}\left(5\left(1.15^{N}\right)-4\left(1.15^{N-1}\right)-1\right)\right]$ $=231591.8537$ For $157800+136k\geq 231591.8537$ $k\geq 542.59$ Hence, least value of $k=543$

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A famous artist wants to create an artwork that uses squares only. On the first day, he draws the perimeter of a square with length x m as shown in Figure 1. On the second day, he divides the square into 9 equal squares and draws the perimeter of the middle square as shown in Figure 2. On the third day, he divides the remaining 8 empty squares from Day 2 into 9 equal squares, and draws the perimeters of the middle squares as shown in Figure 3. This procedure of "drawing the perimeters of the middle squares" is then repeated for subsequent days (see Figure 4 for his finished effort on Day 4).

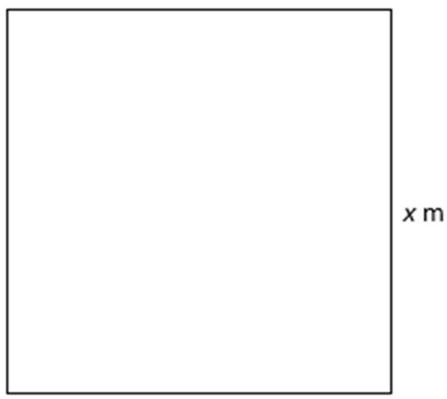

Figure 1 (Day 1)

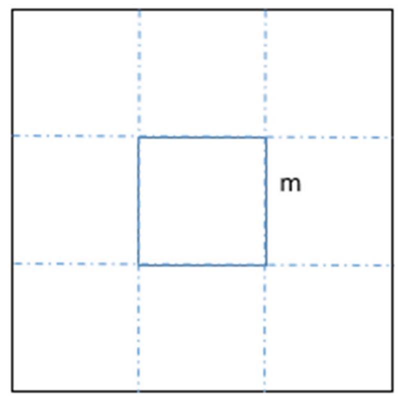

Figure 2 (Day 2)

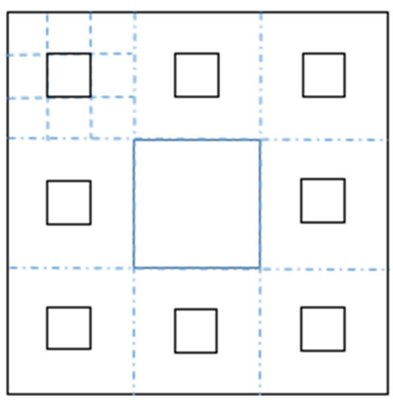

Figure 3 (Day 3)

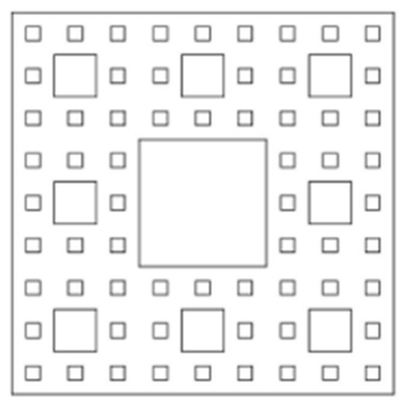

Figure 4 (Day 4)

(i) Write down the <u>number of new squares</u> the artist draws on Day 3 and Day 4. Hence write down the <u>number of new squares</u> the artist draws on Day N ($N \geq 2$). [2]

(ii) Write down the <u>length of one side of a new square</u> the artist draws on Day 3 and Day 4.
Hence write down the <u>length of one side of a new square</u> the artist draws on Day N ($N \geq 2$). [2]

(iii) Show that the <u>total perimeter of new squares</u> the artist draws on Day N ($N \geq 2$) is $\dfrac{x}{2}\left(\dfrac{8}{3}\right)^{N-1}$. [1]

(iv) Hence find the <u>total perimeter of all the squares</u> the artist would have drawn by the end of the N-th day ($N \geq 2$). [3]

(v) A company wants to support the artist. It decides to pledge \$10 for the drawing of the first square. For each subsequent square the artist draws, the company will pledge \$1 more than what was pledged for the previous square. (For the second square drawn, the artist will receive \$11. For the third square drawn, he will receive \$12, and so on.) The artist decides to stop drawing after he completes all the squares on Day 6.
A director of the company hears about the pledge, and he decides to personally pay for the 1st, 4th, 7th,10th, … square that the artist draws. Find the amount of money the director will personally pay. [4]

Solutions

(i)	Day 3 – 8 new squares Day 4 – 64 $=8^2$ new squares Day N – 8^{N-2} new squares
(ii)	Day 3 – $\left(\frac{1}{3}\right)^2 x$ Day 4 – $\left(\frac{1}{3}\right)^3 x$ Day N – $\left(\frac{1}{3}\right)^{N-1} x$
(iii)	Total perimeter of new squares drawn on Day N is $(8)^{N-2}\left(\frac{1}{3}\right)^{N-1}(4x)=\frac{4}{8}\left(\frac{8}{3}\right)^{N-1}x=\frac{1}{2}\left(\frac{8}{3}\right)^{N-1}x=\frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$
(iv)	Total perimeter of all the squares the artist draws by the end of the Nth day is $4x+\frac{x}{2}\left(\frac{8}{3}\right)^{2-1}+\frac{x}{2}\left(\frac{8}{3}\right)^{3-1}+\frac{x}{2}\left(\frac{8}{3}\right)^{4-1}+...+\frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$ $=4x+\frac{x}{2}\left(\frac{8}{3}\right)^{1}+\frac{x}{2}\left(\frac{8}{3}\right)^{2}+\frac{x}{2}\left(\frac{8}{3}\right)^{3}+...+\frac{x}{2}\left(\frac{8}{3}\right)^{N-1}$ $=4x+\dfrac{\frac{x}{2}\left(\frac{8}{3}\right)^{1}\left(\left(\frac{8}{3}\right)^{N-1}-1\right)}{\left(\left(\frac{8}{3}\right)-1\right)}$

	$= 4x + \dfrac{\frac{4x}{3}\left(\left(\frac{8}{3}\right)^{N-1} - 1\right)}{\frac{5}{3}}$ $= 4x + \dfrac{4x}{5}\left(\left(\dfrac{8}{3}\right)^{N-1} - 1\right)$ $= \dfrac{4x}{5}\left(\dfrac{8}{3}\right)^{N-1} + \dfrac{16}{3}x$
(v)	Total number of squares drawn by the end of Day 6 is $1+8^0+8^1+8^2+8^3+8^4=4682$ The director will pay for the 1st, 4th, 7th, 10th, …, 4681st square, a total of 1561 squares. He will be paying \$10 for the 1st square, \$13 for the 4th square, \$16 for the 7th square, and so on. $\dfrac{1561}{2}\left[2(10)+(1561-1)3\right]$ $= \dfrac{1561}{2}\left[20+1560\times 3\right]$ $= 3668350$ The amount of money the director will personally pay is \$3,668,350.

THE END