2026 J2 H2 FM
Recurrence Relations – Solutions

1	To improve the biodiversity in the forest of Anteeyew, a botanist plans to introduce a new species of wildflower to the forest. The wildflower produces seeds which will germinate at different times. It is known that of the seeds produced by each wildflower each week, on average, 3 seeds will germinate in one week's time, and 5 seeds will germinate in two weeks' time. Let W_n denote the number of wildflowers present in the forest of Anteeyew at the end of the n^{th} week since the introduction of the wildflowers to the forest.	
	(a) Show that $W_n = 4W_{n-1} + 5W_{n-2}$ for $n \geq 2$.	[1]
	Initially, the botanist plants 10 wildflowers in the forest of Anteeyew.	
	(b) Taking $W_0 = 10$, state the value of W_1. Hence, find the general formula for W_n for $n \geq 0$.	[4]
	(c) Give one possible reason for why the recurrence relation in part **(a)** may not be a suitable model for the population of wildflowers in the forest of Anteeyew.	[1]
	A new insect species is attracted to the wildflowers in the forest of Anteeyew. Let I_n denote the population of the insect at the end of the n^{th} week since the introduction of the wildflowers to the forest. With the presence of this new insect species, the populations of wildflowers and insects due to their interactions may be modelled by $$W_n = \begin{cases} 10 & , n = 0 \\ W_{n-1} + I_{n-1} & , n \geq 1 \end{cases} \quad \text{and} \quad I_n = \begin{cases} 5 & , n = 0 \\ I_{n-1} + 2W_{n-1} - 5 & , n \geq 1 \end{cases}.$$	
	(d) Obtain a second order recurrence relation for I_n and show that the insect population approaches $\left(a + b\sqrt{2}\right)\left(1 + \sqrt{2}\right)^n$ in the long run, where a and b are rational constants to be found.	[6]
Solution: 2020/NJC/1/10		
(a)	$W_n = W_{n-1} + 3W_{n-1} + 5W_{n-2}$ $= 4W_{n-1} + 5W_{n-2}, \quad n \geq 2.$	B1 AG
(b)	$W_1 = 40$. $W_n = 4W_{n-1} + 5W_{n-2}$ $W_n - 4W_{n-1} - 5W_{n-2} = 0$ Auxiliary equation: $m^2 - 4m - 5 = 0$. $\Rightarrow m = -1$ or $m = 5$. $\therefore W_n = A(-1)^n + B(5)^n$, where A and B are arbitrary constants. When $n = 0$, $10 = A + B$. When $n = 1$, $40 = -A + 5B$. $\Rightarrow A = \frac{5}{3}, B = \frac{25}{3}$. Therefore, the solution to the recurrence relation is $W_n = \frac{5}{3}(-1)^n + \frac{25}{3}(5)^n$, $n \geq 0$.	B1 M1 M1 A1
(c)	• The model does not take into account deaths of wildflowers.	

	• The given average values of 3 and 5 are just average values. In cases of sudden spikes or plunges in number of seeds germinating in the following weeks, the model may not be accurate any more. • Once the wildflowers exceed the boundary of the forest, any wildflowers growing outside the forest will not count towards W_n.	B1
(d)	For $n \geq 1$, $W_n = W_{n-1} + I_{n-1}$ $W_{n-1} = W_n - I_{n-1}$ Thus, $I_n = I_{n-1} + 2W_{n-1} - 5$ $= I_{n-1} + 2(W_n - I_{n-1}) - 5$ $= 2W_n - I_{n-1} - 5$ $W_n = \dfrac{I_n + I_{n-1} + 5}{2}$ For $n \geq 2$, from $W_n = W_{n-1} + I_{n-1}$, $\dfrac{I_n + I_{n-1} + 5}{2} = \dfrac{I_{n-1} + I_{n-2} + 5}{2} + I_{n-1}$ $I_n + I_{n-1} + 5 = I_{n-1} + I_{n-2} + 5 + 2I_{n-1}$ $I_n - 2I_{n-1} - I_{n-2} = 0$ Auxiliary equation: $m^2 - 2m - 1 = 0$. $m = \dfrac{2 \pm \sqrt{2^2 - 4(-1)}}{2} = 1 \pm \sqrt{2}$. General solution: $I_n = C\left(1+\sqrt{2}\right)^n + D\left(1-\sqrt{2}\right)^n$, $n \geq 2$, where C and D are arbitrary constants. Initial conditions: $I_0 = 5$ $I_1 = I_0 + 2W_0 - 5$ $= 5 + 2(10) - 5$ $= 20.$ Substituting into the general solution: $20 = C\left(1+\sqrt{2}\right) + D\left(1-\sqrt{2}\right)$ $5 = C + D$ $D = 5 - C$ and $= C\left(1+\sqrt{2}\right) + (5 - C)\left(1-\sqrt{2}\right)$ $= 2\sqrt{2}C + 5 - 5\sqrt{2}$ $C = \dfrac{15 + 5\sqrt{2}}{2\sqrt{2}} = \dfrac{10 + 15\sqrt{2}}{4}$ $D = 5 - \dfrac{10 + 15\sqrt{2}}{4} = \dfrac{10 - 15\sqrt{2}}{4}$ Therefore, for $n \geq 2$, $I_n = \dfrac{10 + 15\sqrt{2}}{4}\left(1+\sqrt{2}\right)^n + \dfrac{10 - 15\sqrt{2}}{4}\left(1-\sqrt{2}\right)^n$. Now, $\left\|1+\sqrt{2}\right\| > 1$ and $\left\|1-\sqrt{2}\right\| < 1$. Thus, in the long run, i.e., as $n \to \infty$,	M1 M1 M1 A1 A1 B1

	$\left(1-\sqrt{2}\right)^n \to 0$ and therefore $I_n \to \left(\frac{5}{2}+\frac{15}{4}\sqrt{2}\right)\left(1+\sqrt{2}\right)^n$.	AG

2. A sequence $u_0, u_1, u_2, \ldots$ is given by

$$u_{n+2} = 8u_{n+1} - 15u_n,\ u_0 = u_1 = 1.$$

(a) Show that $u_{n+2} - 3u_{n+1} = 5\left(u_{n+1} - 3u_n\right)$. [1]

(b) Let $v_n = u_{n+1} - 3u_n$. Write down a first-order recurrence relation for v_n, and hence express v_n in terms of *n*. [3]

(c) Let $w_n = u_{n+1} - 5u_n$. Write down a first-order recurrence relation for w_n, and hence show that $w_n = (-4)3^n$. [3]

(d) By finding and solving a pair of simultaneous linear equations in u_{n+1} and u_n, or otherwise, express u_n in terms of *n* for $n \geq 0$. [2]

Solution:

2	**ACJC\2017\Prelim\P1\Q5**	
(a)	$u_{n+2} = 8u_{n+1} - 15u_n$ $u_{n+2} - 3u_{n+1} = 5u_{n+1} - 15u_n$ $= 5\left(u_{n+1} - 3u_n\right)$	AG1
(b)	Since $v_n = u_{n+1} - 3u_n$, using (a): $v_{n+1} = 5v_n$ for $n \geq 0 \Rightarrow v_n$ is a Geometric Progression with common ratio 5 $v_n = v_0 5^{n-1}$ $= \left(u_1 - 3u_0\right)5^{n-1}$ $= (1-3)5^{n-1} = (-2)5^{n-1}$	M1 M1 A1
(c)	$u_{n+2} = 8u_{n+1} - 15u_n$ $u_{n+2} - 5u_{n+1} = 3u_{n+1} - 15u_n$ $= 3\left(u_{n+1} - 5u_n\right)$ Let $w_n = u_{n+1} - 5u_n$, then: $w_{n+1} = 3w_n$ for $n \geq 0 \Rightarrow w_n$ is a Geometric Progression with common ratio 3 $w_n = w_0 3^{n-1}$ $= \left(u_1 - 5u_0\right)3^{n-1}$ $= (1-5)3^{n-1} = (-4)3^{n-1}$	M1 M1 A1
(d)	$u_{n+1} - 3u_n = (-2)5^n$ $u_{n+1} - 5u_n = (-4)3^n$ By eliminating u_{n+1}: $u_n = \frac{1}{2}\left[(-2)5^n - (-4)3^n\right]$ $= 2\left(3^n\right) - 5^n$	M1 A1

3	The aggregate expenditure of Country X is given by the sum of household consumption and government expenditure in a year. Let Y_n and C_n be the aggregate expenditure and household consumption at the end of the n^{th} year respectively, where $n \geq 1$. Let G be the government expenditure which is assumed to be constant from year to year. In an economic model, the household consumption in each year is assumed to be a linear function of the aggregate expenditure of the preceding year. **(a)** Based on the information above, find a recurrence relation for the aggregate expenditure in the form $Y_{n+1} = \mathrm{f}(Y_n)$. Hence, show that $Y_{n+1} = (G+k)\left(\frac{m^n-1}{m-1}\right) + m^n Y_1$ for all positive integer n, where k and m are constants. [4] **(b)** If $0 < m < 1$, find the long term aggregate expenditure of the country, in terms of G, k and m. [2]

Solution:

3	(a) Given $Y_{n+1} = G + C_{n+1}$, $C_{n+1} = mY_n + k$, where k and m are constants. So, $Y_{n+1} = G + k + mY_n$ By repeated substitution, $Y_{n+1} = G + k + mY_n$ $= G + k + m(G + k + mY_{n-1})$ $= (1+m)(G+k) + m^2\left(G + k + mY_{n-2}\right)$ $= \ldots$ $= (1 + m + m^2 + \ldots + m^{n-1})(G+k) + m^n Y_1$ $= (G+k)\left(\frac{m^n-1}{m-1}\right) + m^n Y_1$ since $1 + m + m^2 + \ldots + m^{n-1}$ is a GP. (ii)If $0 < m < 1$, $m^n \to 0$ when $n \to \infty$. Hence, $Y_{n+1} \to \frac{G+k}{1-m}$.	M1 A1 M1: repeated substitution A1: prove to AG M1,A1

4 The terms in the sequence u_1, u_2, u_3 … satisfy the recurrence relation

$$u_n = 2\sqrt{2}\,\alpha\, u_{n-1} - 4\alpha^2\, u_{n-2},$$

where α is a positive constant.

(a) Find the general solution of this recurrence relation. [4]

(b) Find the set of values of α for which the sequence is convergent. [2]

	VJC\2018\Prelim\P1\Q2	
(a)	$u_n = \left(2\sqrt{2}\,\alpha\right)u_{n-1} - 4\alpha^2 u_{n-2}$ Auxiliary equation: $\lambda^2 - 2\sqrt{2}\alpha\lambda + 4\alpha^2 = 0$ $\lambda = \frac{2\sqrt{2}\alpha \pm \sqrt{8\alpha^2 - 4(4\alpha^2)}}{2(1)} = \sqrt{2}\alpha \pm \sqrt{2}\alpha i$	 M1 A1

	$r=\sqrt{2\alpha^2+2\alpha^2}=2\alpha$, $\theta=\tan^{-1}\left(\frac{\sqrt{2}\alpha}{\sqrt{2}\alpha}\right)=\frac{\pi}{4}$. $\therefore u_n=$ $(2\alpha)^n\left(A\cos\frac{n\pi}{4}+B\sin\frac{n\pi}{4}\right)$, where A and B are arbitrary constants.	M1 A1
(b)	Since $A\cos\frac{n\pi}{4}+B\sin\frac{n\pi}{4}$ is bounded, u_n is convergent provided $\lvert 2\alpha\rvert<1$. Since $\alpha>0$, we have $0<\alpha<\frac{1}{2}$.	M1 A1