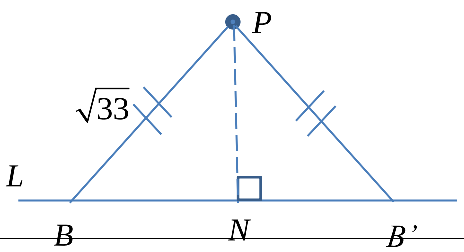
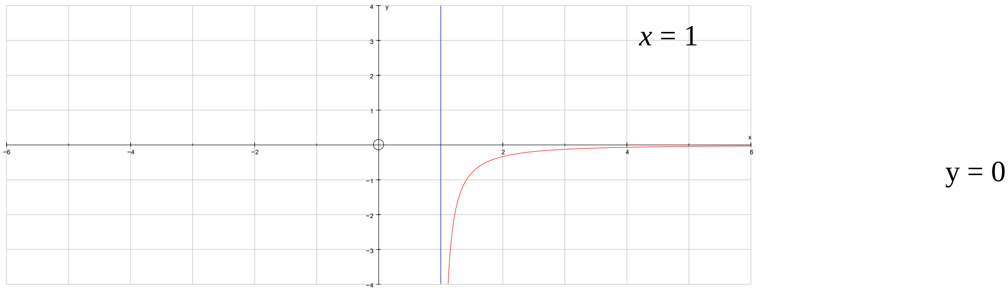
2015 H2 Maths Paper 2 (edited for new H2 Syllabus 9758)

Qn	Suggested Solutions
	Section A (Pure Math)
1(i)	Max height is when $\frac{\mathrm{d}h}{\mathrm{d}t}=0$. $\frac{1}{10}\sqrt{16-\frac{1}{2}h}=0$ $\frac{1}{2}h=16$ $h=32$ m
(ii)	Separating variables and integrating w.r.t. t, $-2\int \frac{1\times\left(-\frac{1}{2}\right)}{\sqrt{16-\frac{1}{2}h}}\,\mathrm{d}h=\int\frac{1}{10}\mathrm{d}t$ $-2\times 2\sqrt{16-\frac{1}{2}h}=\frac{1}{10}t+c$ $-4\sqrt{16-\frac{1}{2}h}=\frac{1}{10}t+c$ When t = 0, h = 0: $c=-16$ $t=-40\sqrt{16-\frac{1}{2}h}+160$ When the tree is 16m, $t=-40\sqrt{16-\frac{1}{2}(16)}+160$ $t=46.9$ years In general, $\int\frac{\mathrm{f}'(x)}{\sqrt{a+\mathrm{f}(x)}}\mathrm{d}x=\int \mathrm{f}'(x)[a+\mathrm{f}(x)]^{-\frac{1}{2}}\,\mathrm{d}x$ $=\frac{[a+\mathrm{f}(x)]^{\frac{1}{2}}}{\frac{1}{2}}+C$ $=2[a+\mathrm{f}(x)]^{\frac{1}{2}}+C$ In particular, $\int(a+bx)^n\,\mathrm{d}x=\frac{(a+bx)^{n+1}}{(b)(n+1)}+C$
2(i)	Let the angle between L and the x-axis be θ. $\cos\theta=\left\lvert\frac{\begin{pmatrix}2\\3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{49}\sqrt{1}}\right\rvert=\frac{2}{7}$ $\Rightarrow\ \theta=73.4°$ Acute angle between L and the x-axis = 73.4°
(ii)	$L:\ \mathbf{r}=\begin{pmatrix}1\\-2\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-6\end{pmatrix},\ \lambda\in\mathbb{R}$.  Let B be a point on L such that $\lvert\overrightarrow{PB}\rvert=\sqrt{33}$

	and let N be the point on L closest to P. $\overrightarrow{PB}=\overrightarrow{OB}-\overrightarrow{OP}$ $=\left[\begin{pmatrix}1\\-2\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\-6\end{pmatrix}\right]-\begin{pmatrix}2\\5\\-6\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $=\begin{pmatrix}-1+2\lambda\\-7+3\lambda\\2-6\lambda\end{pmatrix}$ $\left\|\overrightarrow{PB}\right\|=\sqrt{(-1+2\lambda)^2+(-7+3\lambda)^2+(2-6\lambda)^2}$ $=\sqrt{54-70\lambda+49\lambda^2}$ $\therefore\sqrt{54-70\lambda+49\lambda^2}=\sqrt{33}$ Square both sides and solve for λ, $54-70\lambda+49\lambda^2-33=0$ $49\lambda^2-70\lambda+21=0$ $7\lambda^2-10\lambda+3=0$ $(7\lambda-3)(\lambda-1)=0$ $\lambda=1$ or $\frac{3}{7}$ $\overrightarrow{OB}=\begin{pmatrix}1\\-2\\-4\end{pmatrix}+1\begin{pmatrix}2\\3\\-6\end{pmatrix}=\begin{pmatrix}3\\1\\-10\end{pmatrix}$ or $\overrightarrow{OB}=\begin{pmatrix}1\\-2\\-4\end{pmatrix}+\frac{3}{7}\begin{pmatrix}2\\3\\-6\end{pmatrix}=\frac{1}{7}\begin{pmatrix}13\\-5\\-46\end{pmatrix}$ The points are (3, 1, −10) and $\frac{1}{7}(13,-5,-46)$. Hence, as N is the midpoint of BB', $\overrightarrow{ON}=\frac{1}{2}\left[\begin{pmatrix}3\\1\\-10\end{pmatrix}+\frac{1}{7}\begin{pmatrix}13\\-5\\-46\end{pmatrix}\right]=\frac{1}{7}\begin{pmatrix}17\\1\\-58\end{pmatrix}$
(iii)	Normal to the new plane: $\begin{pmatrix}-1\\-7\\2\end{pmatrix}\times\begin{pmatrix}2\\3\\-6\end{pmatrix}=\begin{pmatrix}36\\-2\\11\end{pmatrix}$ $\mathbf{r}.\begin{pmatrix}36\\-2\\11\end{pmatrix}=\begin{pmatrix}1\\-2\\-4\end{pmatrix}\cdot\begin{pmatrix}36\\-2\\11\end{pmatrix}=-4$ Cartesian equation: $36x-2y+11z=-4$.

3(ai)

$\mathrm{f}(x) = \dfrac{1}{1-x^2}$, $x > 1$



Any horizontal line $y = k$ intersects the graph of $y = \mathrm{f}(x)$ at most once.

$\therefore$ f is a one-one function and so its inverse exists.

(ii)

$\mathrm{f}(x) = \dfrac{1}{1-x^2}$, $x > 1$

Let $y = \dfrac{1}{1-x^2}$

Make x the subject: $y(1-x^2) = 1$

$y - yx^2 = 1$

$yx^2 = y - 1$

$x = \pm\sqrt{\dfrac{y-1}{y}}$

As $x > 1$, $x = \sqrt{\dfrac{y-1}{y}} = \sqrt{1 - \dfrac{1}{y}}$

$\mathrm{f}^{-1}(x) \to \sqrt{1-\dfrac{1}{x}}$, $x \in \mathbb{R}, x < 0$

$D_{f^{-1}} = R_f = (-\infty, 0)$.

(b)

Let $y = \mathrm{g}(x) = \dfrac{2+x}{1-x^2}$.

$y(1-x^2) = 2 + x$

$yx^2 + x + (2-y) = 0 \quad ---(1)$

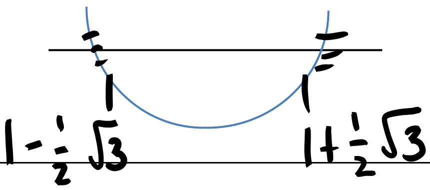
The range of g corresponds to the values of y for which (1) has real root x i.e. $D \geq 0$.

$1^2 - 4y(2-y) \geq 0$

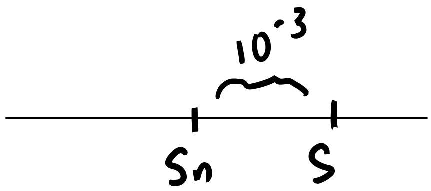
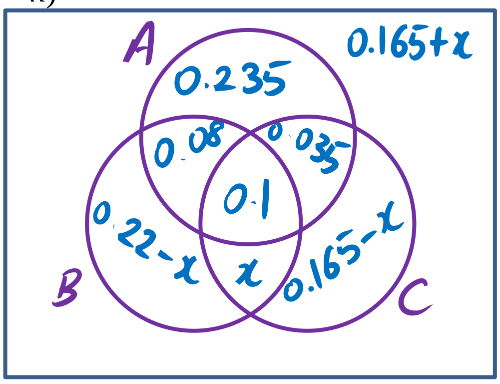
$1 - 8y + 4y^2 \geq 0$

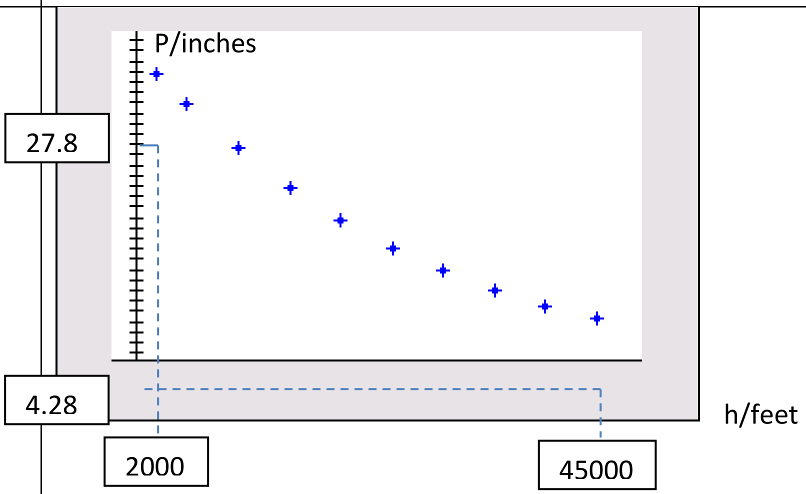
Solve $1 - 8y + 4y^2 = 0$:

$y = \dfrac{8 \pm \sqrt{64 - 4(4)}}{2(4)} = 1 \pm \dfrac{1}{2}\sqrt{3}$



	$\therefore \mathrm{R}_g = \left\{ y \in \mathbb{R} : y \le 1 - \frac{\sqrt{3}}{2} \text{ or } y \ge 1 + \frac{\sqrt{3}}{2} \right\}$
4(bi)	$\frac{2}{4r^2+8r+3} = \frac{A}{2r+1} + \frac{B}{2r+3}$ $2 = A(2r+3) + B(2r+1)$ Let $r = -\frac{3}{2}$: $2 = B\left(2\left(-\frac{3}{2}\right)+1\right) \Rightarrow B = -1$ Let $r = -\frac{1}{2}$: $2 = A\left(2\left(-\frac{1}{2}\right)+3\right) \Rightarrow A = 1$ $\therefore \frac{2}{4r^2+8r+3} = \frac{1}{2r+1} - \frac{1}{2r+3}$
(ii)	$S_n = \sum_{r=1}^{n} \frac{2}{4r^2+8r+3} = \sum_{r=1}^{n} \left(\frac{1}{2r+1} - \frac{1}{2r+3} \right)$ $\sum_{r=1}^{n} \frac{1}{2r+1} - \frac{1}{2r+3}$ $= \frac{1}{3} - \cancel{\frac{1}{5}}$ $+ \cancel{\frac{1}{5}} - \frac{1}{7}$ $\vdots$ $+ \cancel{\frac{1}{2(n-1)+1}} - \cancel{\frac{1}{2(n-1)+3}}$ $+ \cancel{\frac{1}{2n+1}} - \frac{1}{2n+3}$ $= \frac{1}{3} - \frac{1}{2n+3}$
(iii)	Let S be the sum to infinity. From (ii), as $n \to \infty$, $S_n \to \frac{1}{3}$. $\therefore S = \frac{1}{3}$ [Note: Since $S_n = \frac{1}{3} - \frac{1}{2n+3} < \frac{1}{3}$, $S_n < S$.] For S_n to be within 10^{-3} of the sum to infinity,

	$\left\|S-S_n\right\|<10^{-3}$ $\left\|\frac{1}{3}-\left(\frac{1}{3}-\frac{1}{2n+3}\right)\right\|<10^{-3}$ $\frac{1}{2n+3}<10^{-3}$ $2n+3>1000$ $n>498.5$ Smallest n = 499. 
	Section B (Statistics)
6(i)	Let X be the random variable number of red sweets in a small pack of 10. X~B(10, 0.25) P($X \geq 4$) =1 – P($X \leq 3$) = 0.224
(iii)	Y~B(100, 0.25) P($Y \geq 30$)=1 – P($X \leq 29$)≈ 0.14954 Let S be the random variable number of large packs that contain at least 30 red sweets out of 15 large packs. S~B(15, 0.14954) P($S \leq 3$) = 0.824
8	An unbiased estimate of population mean is $\bar{x}=0.8825$ An unbiased estimate of population variance is $s^2 \approx 0.074785^2 \approx 0.00559$
9(i)	P($B\|A$) = P(B) since A and B are independent = 0.4
(ii)	Let $\mathrm{P}(B \cap C)$ = 0.1+x. $\mathrm{P}(A \cup B \cup C)=\mathrm{P}(A)+(0.22-x)+x+(0.165-x)$ $=0.45+0.385-x$ $=0.835-x$ $\mathrm{P}(A' \cap B' \cap C')=1-(0.835-x)$ $=0.165+x \quad ---(1)$  When B and C are also independent, $\mathrm{P}(B \cap C)=\mathrm{P}(B) \times \mathrm{P}(C)$ $0.1+x=0.4 \times 0.3$ $x=0.02$ Sub into (1): $\mathrm{P}(A' \cap B' \cap C')=0.185$

(iii)	When B and C are not independent, Min $x = 0$ Max $x = 0.165$ (as $0.165 - x$ cannot be negative) Sub into (1): $\min \mathrm{P}(A' \cap B' \cap C') = 0.165 + 0 = 0.165$ $\max \mathrm{P}(A' \cap B' \cap C') = 0.165 + 0.165 = 0.33$
10 (i)	
(ii)	(a) $r = -0.9807$ (b) $r = -0.9748$ (c) $r = -0.9986$ (all to 4 d.p.)
(iii)	Most appropriate case is (c) since $\lvert r \rvert$ is closest to 1. Equation is $P = -0.147\sqrt{h} + 34.8$ (3 s.f.)
(iv)	(Changing from feet to metres is a scaling by factor $\frac{1}{3.28}$ parallel to the x-axis.) The equation changes from $P = \mathrm{f}(h)$ to $P = \mathrm{f}(3.28h)$: $P = -0.14687\sqrt{3.28h} + 34.789$ $P = -0.266\sqrt{h} + 34.8$ (3 s.f.)
11(i)	No. of different arrangements $= \dfrac{8!}{2!2!} = 10080$ since there are 2 As and 2 Bs
(ii)	No. of different arrangements = 10079
(iii)	No. of different arrangements = 6! = 720 <u>AA</u> <u>BB</u> C G E S
(iv)	No. of different arrangements = Total no. of ways – No. of ways with As or Bs together or both $= 10080 - \left[\dfrac{7!}{2!} + \dfrac{7!}{2!} - 6!\right] = 5760$ Note that: No of different arrangements with 2 As or 2 Bs together: AA B B C G E S $= \dfrac{7!}{2!}$.

	No of different arrangements with 2As and 2Bs together = 6!.
12(i)	Let X be the mass of an apple in grams and Y be mass of a pear in grams. $X \sim \mathrm{N}(300, 20^2)$ $\quad$ $Y \sim \mathrm{N}(200, 15^2)$ $X_1 + ... + X_5 \sim N(5\times300, 5\times20^2) = N(1500, 2000)$ $\mathrm{P}(X_1 + ... + X_5 > 1600) \approx 0.0127$.
(ii)	Let $A = X_1 + ... + X_5 \sim \mathrm{N}(5\times300, 5\times20^2) = \mathrm{N}(1500, 2000)$ and $B = Y_1 + \ldots + Y_8 \sim \mathrm{N}(8\times200, 8\times15^2) = \mathrm{N}(1600, 1800)$ $A - B \sim \mathrm{N}(1500-1600, 2000+1800) = \mathrm{N}(-100, 3800)$ $\mathrm{P}(A - B > 0) \approx 0.0524$.
(iii)	The total mass of 5 apples and 8 pears after preparation is the random variable $T = 0.85A + 0.9B \sim N(0.85\times1500 + 0.9\times1600, 0.85^2\times2000 + 0.9^2\times1800)$ $T \sim \mathrm{N}(2715, 2903)$ $\mathrm{P}(T < 2750) \approx 0.742$.