# Raffles Institution
# H2 Mathematics (9758)
# Solution for 2022 A-Level Paper 2

## Section A: Pure Mathematics

### Question 1

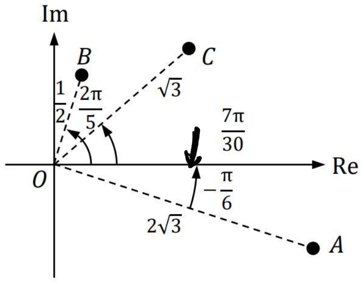
| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| | $u=\sqrt{x+2}$<br>$u^2=x+2$<br>$2u\frac{\mathrm{d}u}{\mathrm{d}x}=1$<br>$\int \frac{x}{\sqrt{x+2}}\ \mathrm{d}x = \int \frac{u^2-2}{u}\ 2u\ \mathrm{d}u$<br>$=2\int\left(u^2-2\right)\ \mathrm{d}u$<br>$=2\left(\frac{1}{3}u^3-2u\right)+C$<br>$=\frac{2}{3}(x+2)^{\frac{3}{2}}-4\sqrt{x+2}+C$ | Remember to change back to *x*. |

### Question 2

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(a)** | Volume $=72k=0.9h\times1000$<br>$\therefore k=\frac{900h}{72}=12.5h$ | Be careful of the units!<br>$72\,\text{sec}\times k$ litres/sec $=72k$ litres<br>$0.9\,\text{m}^2\times h\ \text{m}=0.9h\,\text{m}^3=0.9h\times1000$ litres |
| **(b)** | Let $V$ litres be the volume of water inside the container at time $t$.<br>$\frac{\mathrm{d}V}{\mathrm{d}t}=kt$<br>$V=\int kt\ \mathrm{d}t$<br>$=\frac{1}{2}kt^2+C$<br>$=\frac{25}{4}ht^2+C$, using (a)<br>$V=0$ when $t=0\Rightarrow C=0$<br>$\therefore V=\frac{25}{4}ht^2$<br>When full, $V=900h\Rightarrow 900h=\frac{25}{4}ht^2$<br>$t^2=144$<br>Since $t>0,\ t=12s$ | |

| | | |
|---|---|---|
| **(c)** | $\frac{\mathrm{d}V}{\mathrm{d}t}=kt+25=12.5ht+25$<br>$V=\int 12.5ht+25\ \mathrm{d}t$<br>$V=6.25ht^2+25t+D$<br>$V=0$ when $t=0\Rightarrow D=0$<br>$\therefore V=6.25ht^2+25t$<br>$t=10, V=900h\Rightarrow 900h=625h+250$<br>$\Rightarrow h=\frac{250}{275}=\frac{10}{11}\text{m}$ | |

**Question 3**

| **No.** | **Suggested Solution** | **Remarks for Student** |
|---|---|---|
| **(a)** | $z_1=3-\mathrm{i}\sqrt{3}=\sqrt{12}\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$<br>$z_2=\frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{2\pi}{5}}$<br>$z_3=z_1z_2=\frac{\sqrt{12}}{2}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{5}-\frac{\pi}{6}\right)}=\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{7\pi}{30}}$<br>$\lvert z_3\rvert=\sqrt{3},\ \arg(z_3)=\frac{7\pi}{30}$ | Need to specify $\lvert z_3\rvert$ and $\arg(z_3)$ explicitly. Answer left as $\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{7\pi}{30}}$ is not acceptable. |
| **(b)** | Let *A*, *B* and *C* represent complex numbers *z*₁, *z*₂ and *z*₃ respectively<br><br> | |
| **(c)** | $z_3^n=\left(\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{7\pi}{30}}\right)^n=\left(\sqrt{3}\right)^n\mathrm{e}^{\mathrm{i}\frac{7n\pi}{30}}$<br>$\frac{7n\pi}{30}=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\pm\frac{7\pi}{2},\pm\frac{9\pi}{2}$<br>$7n=15,45,75,105,\ldots$<br>smallest positive integer $n=15$<br>$\lvert z_3^{15}\rvert=\left(\sqrt{3}\right)^{15}=3^7\sqrt{3}=2187\sqrt{3}$<br>$\arg\left(z_3^{15}\right)=\frac{105\pi}{30}-4\pi=\frac{7\pi}{2}-4\pi=-\frac{\pi}{2}$ | |

**Question 4**

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)</td><td>$\frac{1}{9r^2+3r-2}=\frac{1}{(3r+2)(3r-1)}=\frac{1}{3(3r-1)}-\frac{1}{3(3r+2)}$</td><td></td></tr>
<tr><td>(b)</td><td>

$$\sum_{r=m}^{3m}\left(\frac{1}{9r^2+3r-2}\right)=\frac{1}{3}\sum_{r=m}^{3m}\left(\frac{1}{3r-1}-\frac{1}{3r+2}\right)$$

$$=\frac{1}{3}\left[\frac{1}{3m-1}-\cancel{\frac{1}{3m+2}}\right.$$

$$+\cancel{\frac{1}{3m+2}}-\cancel{\frac{1}{3m+5}}$$

$$+\cancel{\frac{1}{3m+5}}-\cancel{\frac{1}{3m+8}}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$+\cancel{\frac{1}{9m-4}}-\cancel{\frac{1}{9m-1}}$$

$$\left.+\cancel{\frac{1}{9m-1}}-\frac{1}{9m+2}\right]$$

$$=\frac{1}{3}\left(\frac{1}{3m-1}-\frac{1}{9m+2}\right)$$

$$=\frac{2m+1}{(3m-1)(9m+2)}$$

</td><td>You are required to combine and simplify the answer as a single fraction.</td></tr>
<tr><td>(c)</td><td>

$$\sum_{r=1}^{n}\left(\frac{1}{9r^2+3r-2}\right)=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3n+2}\right)=\frac{1}{6}-\frac{1}{3(3n+2)}$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{9r^2+3r-2}\right)=\lim_{n\to\infty}\left(\frac{1}{6}-\frac{1}{3(3n+2)}\right)=\frac{1}{6}$$

</td><td></td></tr>
<tr><td>(d)</td><td>

$$\left|\sum_{r=1}^{n}\left(\frac{1}{9r^2+3r-2}\right)-\sum_{r=1}^{\infty}\left(\frac{1}{9r^2+3r-2}\right)\right|<0.004$$

$$\left|\frac{1}{6}-\frac{1}{3(3n+2)}-\frac{1}{6}\right|<0.004$$

$$\frac{1}{3(3n+2)}<0.004\Rightarrow n>27.111$$

Least $n=28$

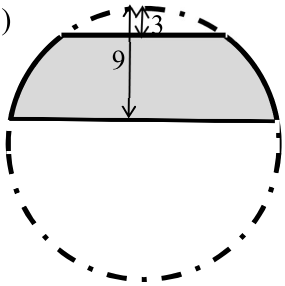
**Alternatively,**

<table>
<tr><td>$n$</td><td>$\left|\sum_{r=1}^{n}\left(\frac{1}{9r^2+3r-2}\right)-\sum_{r=1}^{\infty}\left(\frac{1}{9r^2+3r-2}\right)\right|$</td></tr>
<tr><td>27</td><td>$0.004016>0.004$</td></tr>
<tr><td>28</td><td>$0.003876<0.004$</td></tr>
</table>

Least $n$ is 28.

</td><td>Note the correct use of inequalities leading to the answer 28 which has to be an integer.</td></tr>
</table>

**Question 5**

| **No.** | **Suggested Solution** | **Remarks for Student** |
|---|---|---|
| **(a)** | Volume of spherical cap is<br>$\pi\int_{r-h}^{r} x^2\ \mathrm{d}y$<br>$=\pi\int_{r-h}^{r}\left(r^2-y^2\right)\ \mathrm{d}y$<br>$=\pi\left[r^2y-\frac{1}{3}y^3\right]_{r-h}^{r}$<br>$=\pi\left[r^3-\frac{1}{3}r^3-r^2\left(r-h\right)+\frac{1}{3}\left(r-h\right)^3\right]$<br>$=\pi\left[\frac{2}{3}r^3-r^3+r^2h+\frac{1}{3}r^3-r^2h+rh^2-\frac{1}{3}h^3\right]$<br>$=\pi\left[rh^2-\frac{1}{3}h^3\right]$<br>$=\frac{1}{3}\pi h^2\left(3r-h\right)$ | |
| **(b)** | $3402\pi=\frac{4}{3}\pi\left(15\right)^3-\frac{1}{3}\pi p^2\left(3\left(15\right)-p\right)-\frac{1}{3}\pi\left(3p\right)^2\left(3\left(15\right)-3p\right)$<br>$3402=4500-15p^2+\frac{1}{3}p^3-135p^2+9p^3$<br>$28p^3-450p^2+3294=0$<br>$p=3$ or $p=-2.5158$ or $p=15.587$<br>(reject since $p>0$) (reject since $p<15$)<br>$\therefore p=3$ | |
| **(c)** | Volume of 2nd ornament is<br>$\frac{1}{3}\pi\left(3p\right)^2\left(3r-3p\right)-\frac{1}{3}\pi p^2\left(3r-p\right)$<br>(with $p=3, r=15$)<br>$=846\pi$<br><br> | |

**Section B: Probability and Statistics**

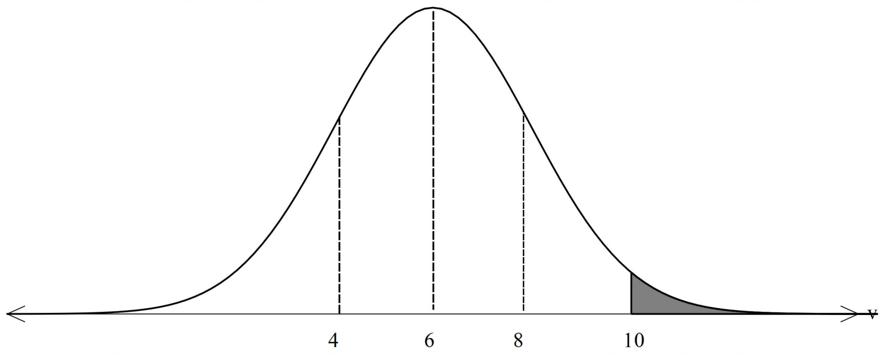
**Question 6**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(a)** | $\mathrm{P}(\text{A wins})$<br>$= \mathrm{P}(\text{A wins on throw 3})$<br>$+\mathrm{P}(\text{A wins on throw 5})$<br>$+\mathrm{P}(\text{A wins on throw 7})$<br>$+\ldots$<br>$= 1\times\frac{5}{6}\times\frac{1}{6}+1\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+1\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}+\ldots$<br>$= \frac{5}{36}\left(1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\left(\frac{5}{6}\right)^6+\ldots\right)$<br>$= \frac{5}{36}\frac{1}{1-\left(\frac{5}{6}\right)^2}$<br>$= \frac{5}{11}$ | Recognise that this is a geometric progression and the it is an infinite series<br>The required answer is found by using the sum to infinity formula of a GP. So, knowledge of Pure Math topic is required.<br>Exact answer is required. |
| **(b)** | $\mathrm{P}(\text{B wins on her second throw} \mid \text{B wins})$<br>$= \frac{\mathrm{P}(\text{B wins on throw 4})}{1-\mathrm{P}(\text{A wins})}$<br>$= \frac{1\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}}{1-\frac{5}{11}}$<br>$= \frac{275}{1296}$ | Exact answer is required. |

**Question 7**

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td><b>(a)</b></td><td>Sample, as there were 75 employees in Staffing and the 53 respondents is a subset of the 75 employees.</td><td></td></tr>
<tr><td><b>(b)</b></td><td>A random sample should be chosen.<br>This is to reduce bias.</td><td></td></tr>
<tr><td><b>(c)</b></td><td>
<table>
<tr><th>A<br>(7)</th><th>P<br>(6)</th><th>M<br>(4)</th><th>S<br>(3)</th><th>No.</th></tr>
<tr><td>5</td><td>1</td><td>1</td><td>1</td><td>$^7C_5 \times {}^6C_1 \times {}^4C_1 \times {}^3C_1 = 1512$</td></tr>
<tr><td>4</td><td>2</td><td>1</td><td>1</td><td>$^7C_4 \times {}^6C_2 \times {}^4C_1 \times {}^3C_1 = 6300$</td></tr>
<tr><td>4</td><td>1</td><td>2</td><td>1</td><td>$^7C_4 \times {}^6C_1 \times {}^4C_2 \times {}^3C_1 = 3780$</td></tr>
<tr><td>4</td><td>1</td><td>1</td><td>2</td><td>$^7C_4 \times {}^6C_1 \times {}^4C_1 \times {}^3C_2 = 2520$</td></tr>
<tr><td>3</td><td>2</td><td>2</td><td>1</td><td>$^7C_3 \times {}^6C_2 \times {}^4C_2 \times {}^3C_1 = 9450$</td></tr>
<tr><td>3</td><td>2</td><td>1</td><td>2</td><td>$^7C_3 \times {}^6C_2 \times {}^4C_1 \times {}^3C_2 = 6300$</td></tr>
<tr><td>3</td><td>1</td><td>2</td><td>2</td><td>$^7C_3 \times {}^6C_1 \times {}^4C_2 \times {}^3C_2 = 3780$</td></tr>
</table>
Total: 33642</td><td>Listing systematically is key in answering this question. Yes, there are 7 cases.</td></tr>
</table>

**Question 8**

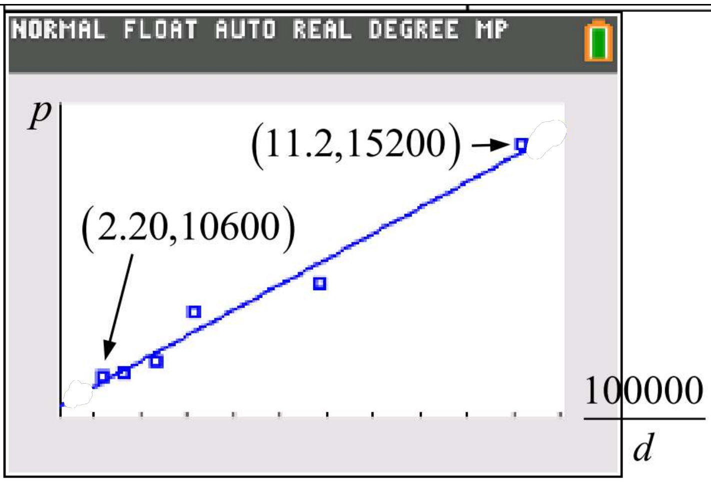
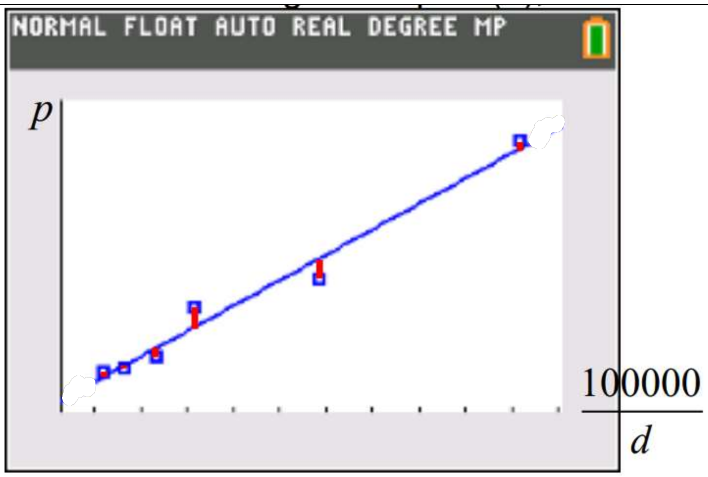
| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| (a) | $aX - bY \sim \mathrm{N}\left(ap - bs, a^2q^2 + b^2t^2\right)$ | |
| (b)<br>(i)<br>& (ii) | $V \sim \mathrm{N}\left(6, 2^2\right)$<br><br> <br>$\mathrm{P}\left(V > 10\right) = 0.0228$ | Label $\mu = 6$<br>And also the 4 and 8 |
| (c) | $\mathrm{E}\left(W\right) = 1.2\mathrm{Var}\left(W\right)$<br>$8p = 1.2\left[8p\left(1-p\right)\right]$<br>$p = 0$ (not meaningful) or $p = \frac{1}{6}$<br>$W \sim \mathrm{B}\left(8, \frac{1}{6}\right)$<br>$\mathrm{P}\left(W < 2\right) = \mathrm{P}\left(W \le 1\right) = 0.605$ | |

**Question 9**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(a)** | Note that it takes 5 moves from S to one of A,B,C,D,E or F.<br>Let $X$ denote number of left moves after first move. $X \sim \mathrm{B}(4, p)$<br>(first move has probability ½ to move left or right)<br>$\mathrm{P}(\text{counter arrives at B})$<br>$= \mathrm{P}(\text{first move is L and } X = 3) + \mathrm{P}(\text{first move is R and } X = 4)$<br>$= \frac{1}{2}\mathrm{P}(X = 3) + \frac{1}{2}\mathrm{P}(X = 4)$<br>$= \frac{1}{2} \times {}^4C_3 p^3 q + \frac{1}{2} \times {}^4C_4 p^4$<br>$= 2p^3 q + \frac{1}{2}p^4$ | Note the probability is ½ for first move from S to L or R. |
| **(b)** | 5 routes in total<br>Both taking Right →Left→Left→ Left→ Left has probability<br>$\left(\frac{1}{2}p^4\right)^2 = \frac{1}{4}p^8$<br>There are 4 routes where Both take Left on first move follow by another same 3 left and one right moves subsequently has probability<br>$4 \times \left(\frac{1}{2}p^3 q\right)^2 = p^6 q^2$<br>Sum of above<br>$= \frac{1}{4}p^8 + p^6 q^2$<br>$= \frac{1}{4}p^8 + p^6(1-p)^2$<br>$= \frac{1}{4}p^8 + p^6(1-2p+p^2)$<br>$= \frac{1}{4}p^6(p^2 + 4 - 8p + 4p^2)$<br>$= \frac{1}{4}p^6(5p^2 - 8p + 4)$<br>Required probability<br>$= \dfrac{\frac{1}{4}p^6(5p^2-8p+4)}{\left(2p^3 q + \frac{1}{2}p^4\right)^2} = \dfrac{\frac{1}{4}p^6(5p^2-8p+4)}{\frac{1}{4}p^6(4q+p)^2} = \dfrac{5p^2-8p+4}{(4-3p)^2}$ | Answer must be simplified and in terms of $p$ only |

| | | |
|---|---|---|
| **(c)** | $\mathrm{P}(\text{counter arrives at C})$<br>$=\frac{1}{2}\mathrm{P}(X=2)+\frac{1}{2}\mathrm{P}(X=3)$<br>$=\frac{1}{2}\times {}^4C_2p^2q^2+\frac{1}{2}\times {}^4C_3p^3q$<br>$=3p^2q^2+2p^3q$<br>$3p^2q^2+2p^3q=2p^3q+\frac{1}{2}p^4$<br>$3p^2q^2-\frac{1}{2}p^4=0$<br>$3p^2(1-p)^2-\frac{1}{2}p^4=0$<br>Since $p\neq 0,\ p=0.710$ | Can use GC, no need to find exact value of $p$ which is $\frac{6-\sqrt{6}}{5}$ |

**Question 10**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(a)** | *d* and *p* may not be linearly correlated as –0.78 is not very near to –1. The scatter diagram also suggests a non-linear negative relationship. | |
| **(b)** | No. Scaling of the values, including change of units, do not change the relationship. | |
| **(c)** | As this data becomes a outlier, since the special car has different features from the other 6 cars, it will disrupt the relationship to the rest of the data as they were taken from cars of similar version | Remember to elaborate in context. Merely stating that it is an outlier is NOT enough |
| **(d)** | <br> | |
| **(e)** | <br> <br>Squaring the distances so that the sum will not be zero or become negative, as the distances could be positive (above the line) or negative (below the line).<br>This is referred to "method of least squares" as we are trying to fit a line such that the sum of the square of distances is the smallest. | |

| | | |
|---|---|---|
| **(f)** | $p = 9398.492473 + 505.730253\left(\frac{100000}{d}\right) \approx 9400 + \frac{50600000}{d}$<br>$r = 0.9869962799 \approx 0.987$ | |
| **(g)** | $P$ = 19403.04743 = 19400 (3 s.f.)<br>Not expecting it to be reliable as 5055 is not in the given range of 8954 to 45452 | |

### Question 11

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(a)** | Let $X$ denote time in sec taken by Zhou to swim 100m, and $T$ be the time in sec taken by Tan to swim 100m.<br>$X \sim \mathrm{N}\left(80, 2^2\right),\ T \sim \mathrm{N}\left(79, 3^2\right)$<br>$X - T \sim \mathrm{N}\left(1, 13\right)$<br>$\mathrm{P}\left(X < T\right) = \mathrm{P}\left(X - T < 0\right)$<br>$= 0.391$ | Note that winning in this context means take shorter time |
| **(b)** | Respective unbiased estimates of population mean and variance are:<br>$\overline{x} = 79.21$<br>$s^2 = 14.72334483 \approx 14.7$ | Note that 79.21 is the exact answer |
| **(c)** | Null hypothesis H$_0$: $\mu = 80$<br>Alternative hypothesis H$_1$: $\mu < 80$<br>$\mu$ is the Zhou's population mean time<br>Under H$_0$,<br>$\overline{X} \sim \mathrm{N}\left(80, \frac{14.72334483}{30}\right)$ approximately<br>by Central Limit Theorem since 30 is large<br>$p - \text{value} = \mathrm{P}\left(\overline{X} < 79.21\right) = 0.12972836 > 0.05$<br>We do not reject H$_0$ and there is insufficient evidence at 5% level of significance to conclude that Zhou's times have reduced. | |
| **(d)** | He should use a 2-tail test as he is interested in testing whether his mean time has changed which could be greater or less than the original mean time. | |
| **(e)** | Assumption 1: his time still follows normal distribution<br>Assumption 2: As he did not calculate an unbiased estimate of the population variance from the data, he has to assume that population variance remains at $3^2$. | The statistical distribution of the time $T$ under this new diet is **unknown**. As the **sample size is small** (6), we **cannot** use CLT to say that $\overline{T}$ is approximately normal.<br>So, in order for us to use hypothesis testing on $\overline{T}$ to be normally distributed, then $T$ has to be normally distributed. |