STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	3	-	7	1	0	3

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2023

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

PAPER 3

Thursday

24 August 2023

1 hr

1130 – 1230 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.
- Number of printed pages = **4.**

FOR MARKER USE ONLY:

Q1	Q2	TOTAL
		/55

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 26]

In this question you will explore the graph of hyperbolas and investigate how complex numbers are linked to hyperbolas.

Let z be a complex number such that

$$|z-1|-|z+1|=\pm c,$$

where $z \in \mathbb{C}$ and $0<c<2$.

(a) By expressing z in the form $x+\mathrm{i}y$, where $x, y \in \mathbb{R}$,

(i) show that $|z-1|=\sqrt{(x-1)^2+y^2}$, [2]

(ii) write down a similar expression for $|z+1|$. [1]

(b) (i) Show that equation $|z-1|-|z+1|=\pm c$ can be expressed as

$$4x+c^2=\pm 2c\sqrt{(x+1)^2+y^2}.$$ [3]

(ii) Show that the equation in (b)(i) can be further simplified to

$$4\left(4-c^2\right)x^2-4c^2y^2=c^2\left(4-c^2\right)$$ [2]

(c) (i) Express your equation in (b)(ii) in the form,

$$\frac{x^2}{h^2}-\frac{y^2}{k^2}=1.$$

leaving h and k in terms of c. [2]

(ii) Let $c=\sqrt{2}$. Show that the equation reduces to

$$x^2-y^2=\frac{1}{2}.$$ [1]

(This question continues on the following page)

(Question 1 continued)

A curve with the equation in the form $\frac{x^2}{h^2}-\frac{y^2}{k^2}=1$, where $h,k\in\mathbb{R}$, is called a hyperbola.

Now consider the equation of the hyperbola, $x^2-y^2=\frac{1}{2}$.

(d) (i) Show that the equation can be expressed as $y=\pm x\sqrt{1-\frac{1}{2x^2}}$. [2]

(ii) Sketch the graph of $x^2-y^2=\frac{1}{2}$, stating clearly the coordinates of any axial intercepts and the equation of each oblique asymptote. [4]

(e) Let F_1 and F_2 represent the points $(1,0)$ and $(-1,0)$ respectively, and $\mathrm{P}(x,y)$ represent any point on the hyperbola, $x^2-y^2=\frac{1}{2}$.

If P_1 is the positive x-intercept of $x^2-y^2=\frac{1}{2}$,

(i) find the length of $\mathrm{F}_1\mathrm{P}_1$; [1]

(ii) find $\mathrm{F}_1\mathrm{P}_1-\mathrm{F}_2\mathrm{P}_1$. [1]

The point P represents the complex number z on the Argand diagram.

(iii) Deduce the geometrical meaning of $|z-1|$. [1]

(iv) Hence write down the values of $|z-1|-|z+1|$. [1]

The hyperbola with equation $x^2-y^2=\frac{1}{2}$ can be rotated about the origin to coincide with the curve defined by $y=\frac{p}{x}$, $p\in\mathbb{R}$.

(f) Find the possible values of p. [5]

2. [Maximum mark: 29]

In this question you will investigate how binomial expansion and limits are used to determine Euler's number and its upper bound.

Let $f_n(x) = \left(1+\frac{x}{n}\right)^n$, where $n \in \mathbb{Z}^+$ and $x \in [0, \infty]$.

(a) (i) Explain that for a given x and when n is large, $\frac{\mathrm{d}}{\mathrm{d}x}\left(f_n(x)\right) \approx f_n(x)$. [3]

(ii) Given that $y = 1$ when $x = 0$, solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y$. [5]

(iii) Determine the value of $f_n(0)$ and hence deduce, for large values of n, a relationship between y in (a)(ii) and $f_n(x)$. [2]

By considering the binomial expansion of $\left(1+\frac{x}{n}\right)^n$, it can be shown that $\left(1+\frac{x}{n}\right)^n$

can be written in the form

$$1 + x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \dots \frac{a_r x^r}{r!} \dots + \frac{a_n x^n}{n!}, \; 2 \le r \le n.$$

(b) (i) Show that $a_2 = 1 - \frac{1}{n}$ and $a_3 = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$. [4]

(ii) Find an expression for a_r where $2 \le r \le n$. [1]

(c) Show that for $2 \le r \le n$,

(i) $0 < a_r < 1$, [2]

(ii) $\frac{1}{r!} \le \frac{1}{2^{r-1}}$. [2]

(d) Hence show that $\left(1+\frac{x}{n}\right)^n < 1 + x + \frac{x^2}{2} + \frac{x^3}{2^2} + \dots + \frac{x^n}{2^{n-1}}$. [2]

(e) For $0 < x < 2$, show that $\left(1+\frac{x}{n}\right)^n < \frac{2+x}{2-x}$. [5]

(f) Using results from (a) and (e), show that $2 < \mathrm{e} < 3$ where e is Euler's number. [3]

End of Paper