1

Graphs and Transformations

SYLLABUS

- Use of a graphing calculator or a graphing software to graph a given function
- Important characteristics of graphs such as symmetry, intersections with the axes, turning points and asymptotes of the following:

  $y^2 = ax$; $x^2 = by$; $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$;

  $y = \dfrac{ax+b}{cx+d}$; $y = \dfrac{ax^2+bx+c}{dx+e}$
- Equations of asymptotes, axes of symmetry, and restrictions on the possible values of x and/or y
- Effect of transformations on the graph of $y = \mathrm{f}(x)$ as represented by $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x+a)$ and $y = \mathrm{f}(ax)$, and combinations of these transformations
- Relating the graphs of $y = |\mathrm{f}(x)|$, $y = \mathrm{f}(|x|)$ and $y = \dfrac{1}{\mathrm{f}(x)}$ to the graph of $y = \mathrm{f}(x)$
- Simple parametric equations and their graphs

CONTENTS

LECTURE 1

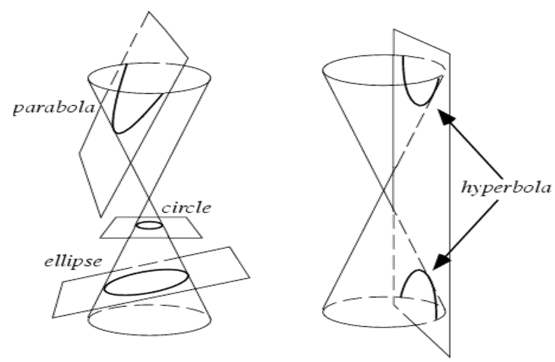
Lesson Outline

- Review of standard graphs & characteristics (Annex A)
- Graphs of conic sections: parabolas & ellipses

---

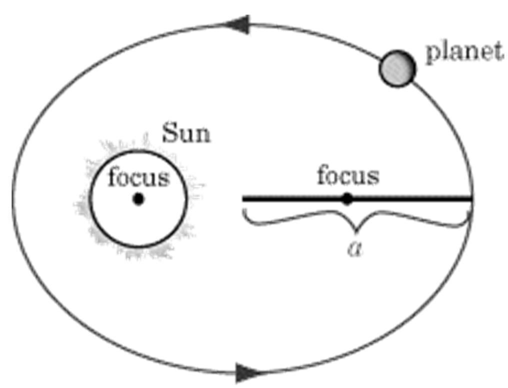
1 GRAPHS OF CONIC SECTIONS

The study of the set of curves known as conics is a good example of how mathematicians have the knack for studying something so seemingly useless, yet turned out to be of immense importance to the world centuries later.



Menaechmus (c. 375-325 BC), tutor to Alexander the Great, conceived the idea of conic sections in attempts to solve construction problems such as that of the famous geometrical problem of "doubling the cube". He was unsuccessful, and conic sections were cast aside until much later.

In the early 17th century, Johannes Kepler, a German mathematician, astronomer, and astrologer, best known for his laws of planetary motion, brought conics to life again. He discovered that the planets travel around the sun in an elliptical orbit (Kepler's First Law!).



a = semimajor axis

Galileo's usage of parabolas in his work on projectile motion allowed cannoneers in the 17th century to work out approximately where their cannon balls will land when fired at a specific angle.

Today, we utilise conic sections in a variety of applications, often employing their reflective properties in mirrors and satellite dishes. We shall now study the basic properties of conic sections.

The general equation of any conic section is of the form

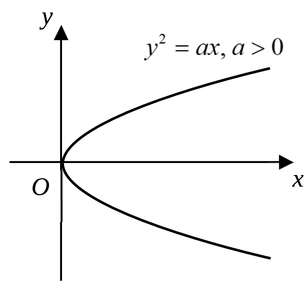
$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

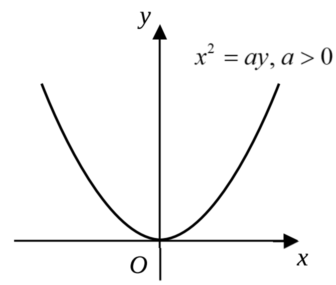
where a, b, c, d, e and f are constants and a, b and c are not all zero. We can identify different types of conic sections by observing the coefficients of the variables in the equation and/or rearranging the equation. Some of the conic sections you will learn in this chapter are **parabolas**, **ellipses**, **circles** and **hyperbolas**.

1.1 Graphs of Parabolas

Graphs of the form $y^2 = ax$ ($a \neq 0$) are **parabolas** with vertex at the origin and are symmetrical about the x-axis.

Graphs of the form $x^2 = ay$ ($a \neq 0$) are **parabolas** with vertex at the origin and are symmetrical about the y-axis.

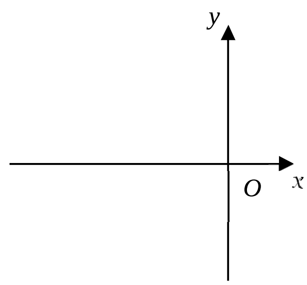


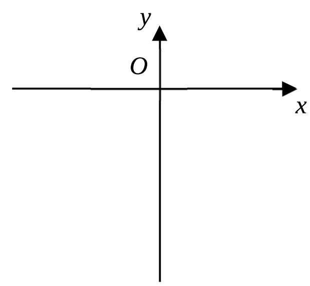


Example

Sketch the graphs with the equations $y^2 = ax$ and $x^2 = ay$, where $a < 0$.

Solution

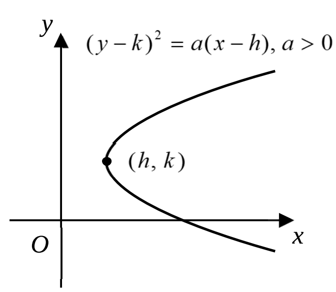


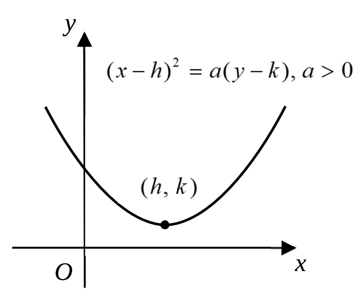


■

Graphs of the form $(y-k)^2 = a(x-h)$, where $a \neq 0$, are **parabolas** with vertex (h, k) and are symmetrical about the line $y = k$.

Graphs of the form $(x-h)^2 = a(y-k)$, where $a \neq 0$, are **parabolas** with vertex (h, k) and are symmetrical about the line $x = h$.

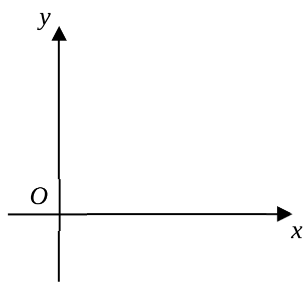


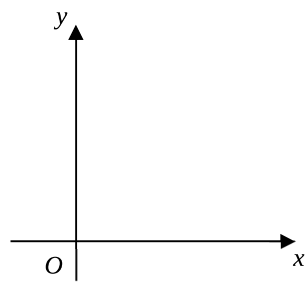


Self-Practice Exercise 1

Sketch the graphs given by the equations $(y-k)^2 = a(x-h)$ and $(x-h)^2 = a(y-k)$, where $a < 0$.

Solution

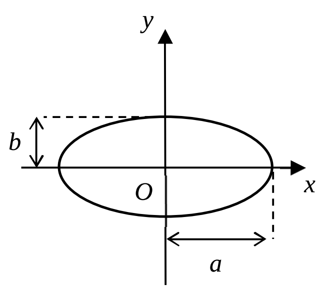




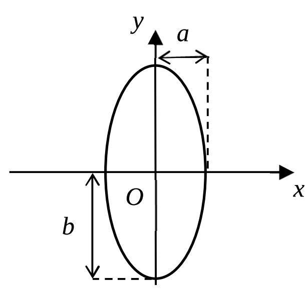
■

1.2 Graphs of Ellipses

Graphs of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > 0$, $b > 0$, are **ellipses** with the origin as the centre. These ellipses are symmetrical about the *x*- and *y*-axes.



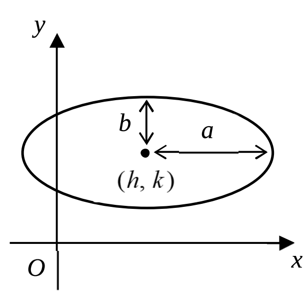
$a > b$



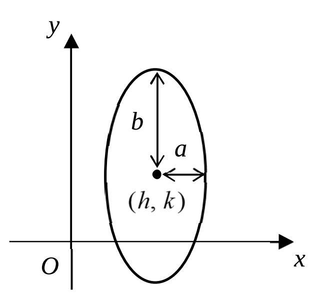
$a < b$

Graphs of the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ are **ellipses** with centre (*h*, *k*). These ellipses are symmetrical about the lines $x = h$ and $y = k$.

The above result can be obtained using simple transformations (see Section 5).



$a > b$



$a < b$

Example 1

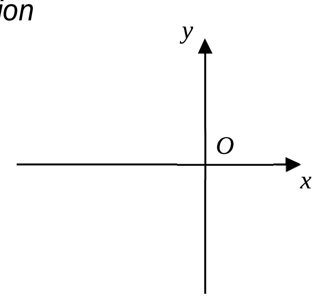
Sketch the graphs of

(a) $\dfrac{(x+1)^2}{9}+\dfrac{y^2}{4}=1$, **(b)** $4x^2+y^2-6y-16=0$,

(c) $(y-2)^2=4(x+1)$.

Solution

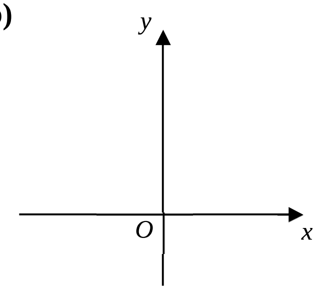
(a)



$$\frac{(x+1)^2}{9}+\frac{y^2}{4}=1$$

(b)

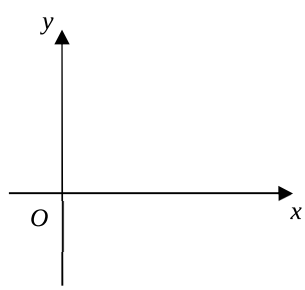
When the equation is in "expanded" form, we complete squares to obtain the standard form.



$$4x^2+y^2-6y-16=0$$

(c)

This equation only has a y^2 term and no x^2 term. Therefore it is a parabola.



$$(y-2)^2=4(x+1)$$

■

Special case: *a* = *b* (Graphs of circles)

Circles are special cases of ellipses, just as squares are special cases of rectangles.

When $a=b=r$, the general equation of the ellipse becomes

$$(x-h)^2+(y-k)^2=r^2,$$

which is the **equation of a circle with centre at (*h*, *k*) and radius *r*.**

Alternatively, it can be given in the general form

$$ax^2+ay^2+cx+dy+e=0 \text{ where } a\neq 0.$$

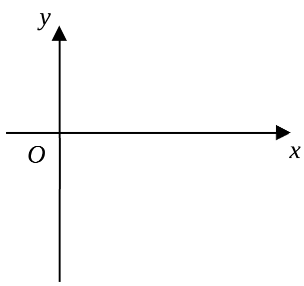
Compare this with the general equation of an ellipse

$$ax^2+by^2+cx+dy+e=0 \text{ where } a\neq 0 \text{ and } b\neq 0.$$

Example 2

Sketch the graph of $x^2 + y^2 - 6x + 2y + 1 = 0$.

Solution



$$x^2 + y^2 - 6x + 2y + 1 = 0$$

When the equation is in "expanded" form, we will complete the square to obtain the "standard" form to determine the features of the conic.

■

LECTURE 2

Lesson Outline

- Graphs of conic sections: hyperbolas
- Characteristics of graphs: axial intercepts and asymptotes

1.3 Graphs of Hyperbolas

Graphs of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ are **hyperbolas**. These hyperbolas are symmetrical about both the *x*- and *y*-axes.

Linear Asymptotes

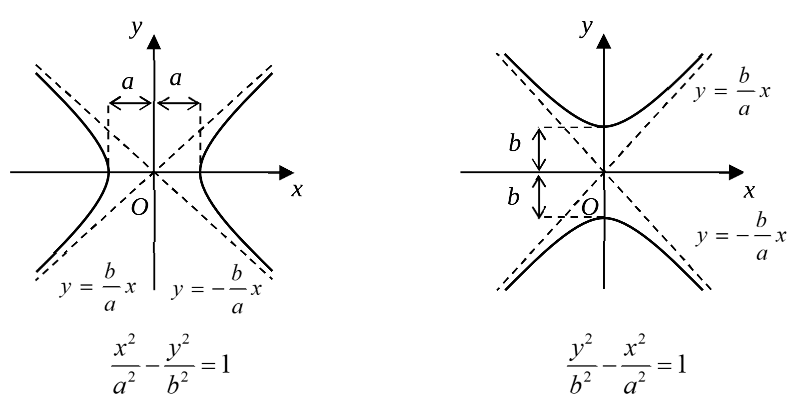
When a graph approaches a particular straight line as the value of *x* or *y* tends to infinity, then this straight line is called a **linear asymptote**.

To obtain the asymptotes of these hyperbolas, we first note that

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Leftrightarrow \frac{y^2}{b^2}=\frac{x^2}{a^2}-1 \qquad \text{or} \qquad \frac{y^2}{b^2}-\frac{x^2}{a^2}=1 \Leftrightarrow \frac{y^2}{b^2}=\frac{x^2}{a^2}+1$$

As $x \to \pm\infty$, $\frac{y^2}{b^2} \to \frac{x^2}{a^2}$.

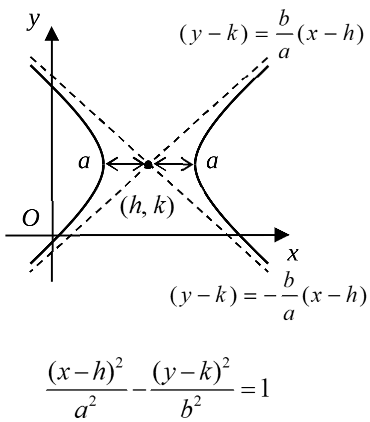
Thus we have $y \to \pm\frac{b}{a}x$ as $x \to \pm\infty$ and we get the lines $y=\pm\frac{b}{a}x$ as the asymptotes of the graphs.

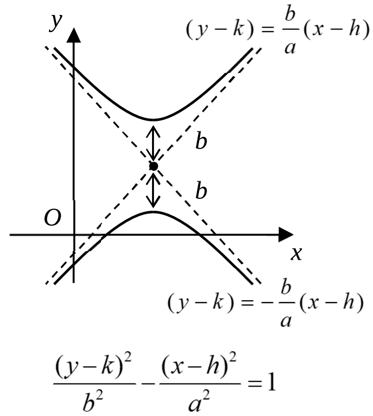


Graphs of the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ are the two previous **hyperbolas** translated *h* units in the positive *x*-direction and *k* units in the positive *y*-direction. These hyperbolas are symmetrical about the lines $x=h$ and $y=k$.

The asymptotes of these hyperbolas are $(y-k)=\pm\dfrac{b}{a}(x-h)$.

The above result can be obtained using simple transformations (see Section 5).





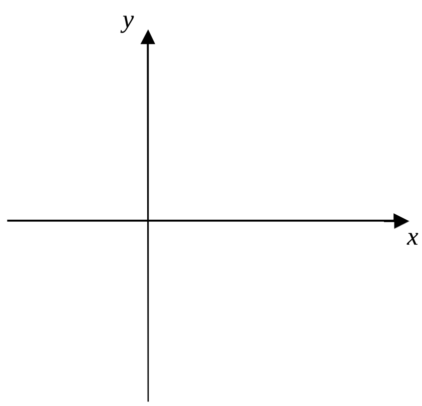
Example 3

Sketch the graph of $(x-2)^2-2y^2=4$ and state the equations of the asymptotes, if any.

Solution

$(x-2)^2-2y^2=4$

Asymptotes:



When sketching hyperbola, first indicate the **centre,** then sketch in the **asymptotes** which will guide the shape of the hyperbola.

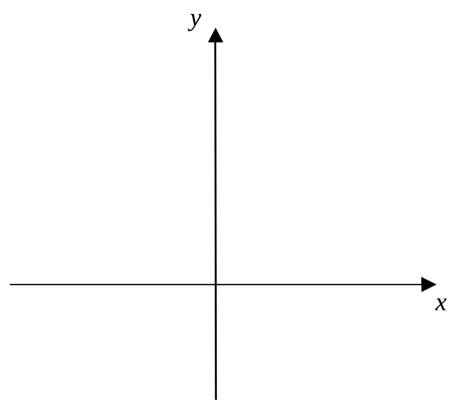
■

Self-Practice Exercise 2

Sketch the graph of $2y^2 - x^2 - 4y = 0$ and state the equations of the asymptotes, if any.

Solution

$2y^2 - x^2 - 4y = 0$



Asymptotes:

■

Note (See Annex B Section 2)

There are two ways of using the graphing calculator (GC) to sketch conic sections:

(a) by keying the equation of the curve directly into the **[Y=]** window

(b) by using the Conics App under **[APPS]**.

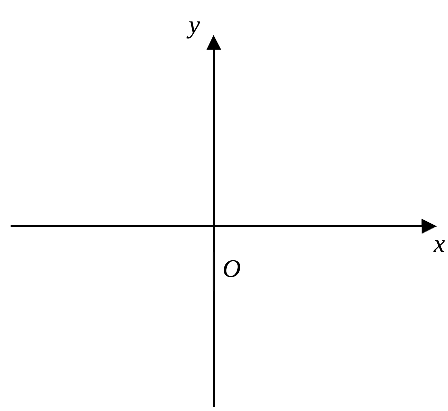
It is important to be familiar with method (a) for a number of uses, for example in finding intersection points between multiple graphs.

Example 4

Sketch the graphs of $x^2 - y^2 = 1$ and $y = x + 2$ and state the coordinates of the point of intersection of the two graphs.

Solution

Asymptotes:



■

Extra Practice

Annex C: Practice Question 1

2 CHARACTERISTICS OF GRAPHS

2.1 Intersections with Coordinate Axes

A graph $y = \mathrm{f}(x)$ will cut the y-axis at the point (0, k) if the pair of values satisfies the equation (that is, when $x = 0$, $y = k$). This point is called the ***y*-intercept**.

Similarly, it will cut the x-axis at the point (h, 0) if the pair of values satisfies the equation (that is, when $y = 0$, $x = h$). This point is called the ***x*-intercept**.

2.2 Asymptotes

When a graph approaches a particular straight line as the value of x or y tends to infinity, then this straight line is called a **linear asymptote**.
There are three types of linear asymptotes:

I. Vertical Asymptotes

If there exists a constant c such that $\mathrm{f}(x) \to \infty$ (or $-\infty$) as $x \to c$, then the line $x = c$ is a **vertical asymptote** of the graph $y = \mathrm{f}(x)$.

II. Horizontal Asymptotes

If there exists a constant k such that $\mathrm{f}(x) \to k$ as $x \to \infty$ (and/or $x \to -\infty$), then the line $y = k$ is a **horizontal asymptote** of the graph $y = \mathrm{f}(x)$.

Example 5

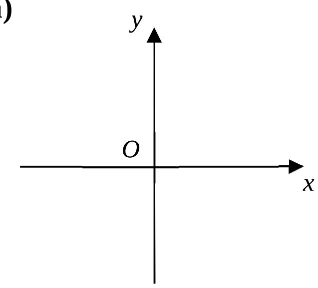
Find the asymptotes of the following graphs.

(a) $y=\dfrac{3}{x-1}$ **(b)** $y=\dfrac{2x-1}{x+3}=2-\dfrac{7}{x+3}$

Solution

These analyses of asymptotes are usually not necessary for standard functions. **Vertical asymptotes** are usually found by **setting the denominator to zero**.

(a)



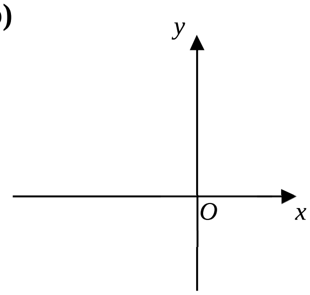
As $x\to 1^{-}$, $y\to$

As $x\to 1^{+}$, $y\to$

As $x\to\infty$, $y\to$

As $x\to-\infty$, $y\to$

(b)



As $x\to\infty$, $y\to$

As $x\to-\infty$, $y\to$

As $x\to-3^{-}$, $y\to$

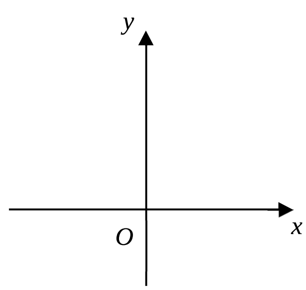
As $x\to-3^{+}$, $y\to$

■

Self-Practice Exercise 3

Sketch the graph of $y=\dfrac{3}{x^2+1}$ and state the equations of the asymptotes, if any.

Solution



■

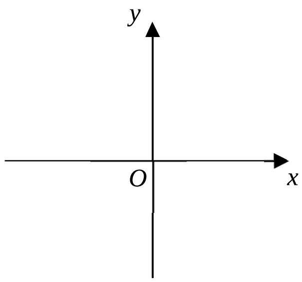
III. Oblique Asymptotes

If $y \to \mathrm{g}(x)$ as $x \to \infty$ (and/or $x \to -\infty$), where $\mathrm{g}(x)$ is a linear function, then the line $y = \mathrm{g}(x)$ is called an **oblique asymptote** of the graph.

Example 6

Find the asymptotes of $y = \dfrac{x(x+1)}{x-1} = x + 2 + \dfrac{2}{x-1}$.

Solution



As $x \to \infty$, $y \to$

As $x \to -\infty$, $y \to$

As $x \to 1^-$, $y \to$

As $x \to 1^+$, $y \to$

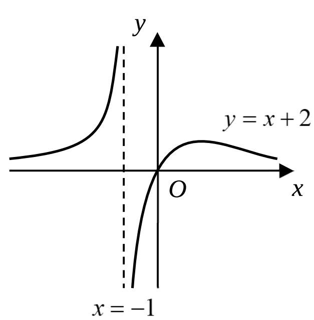
■

Note

Graphs may cross horizontal and oblique asymptotes, but never cross the vertical asymptotes.

Example

$y = \dfrac{5x}{x^3+1}$



■

LECTURE 3

Lesson Outline

- Characteristics of graphs: stationary points and symmetries
- Systematic curve sketching
- Graphs of rational functions: $y=\dfrac{ax+b}{cx+d},\ c\neq 0$

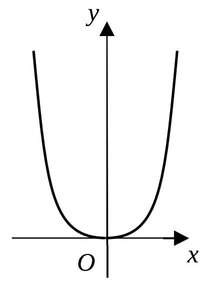
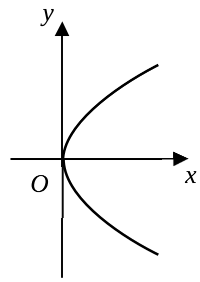
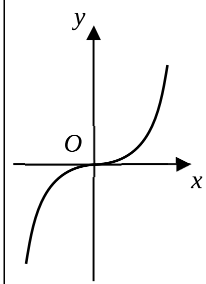
2.3 Stationary Points

Let $y=\mathrm{f}(x)$ be the equation of a curve. The **stationary points** of the curve are points on the curve where $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{f}'(x)=0$.

There are three types of stationary points: the maximum (turning) point, minimum (turning) point and the stationary point of inflexion. The nature of the stationary point can be determined using either the first derivative or the second derivative test. (see Section 3)

2.4 Symmetries

Axis/Point of Symmetry	*y*-axis	*x*-axis	origin
Graphically	If (x,y) is a point on the graph of $y=\mathrm{f}(x)$ then $(-x,y)$ is also a point on the same graph.	If (x,y) is a point on the graph of $y=\mathrm{f}(x)$ then $(x,-y)$ is also a point on the same graph.	If (x,y) is a point on the graph of $y=\mathrm{f}(x)$ then $(-x,-y)$ is also a point on the same graph.
Algebraically	Equation of the curve remains unchanged when x is replaced by $-x$.	Equation of the curve remains unchanged when y is replaced by $-y$.	Equation of the curve remains unchanged when x is replaced by $-x$ and y by $-y$.
Observations (general guidelines)	if the equation contains only even powers of x.	if the equation contains only even powers of y.	if every term of x **and** of y is of even power, **OR** if every term of x **and** of y is of odd power.

Graph	e.g. $y = x^4$ 	e.g. $y^2 = x$ 	e.g. $y = x^3$ 

Example 7

State the symmetries of these curves:

(a) $y = \dfrac{1}{x^2+5}$ **(b)** $x^2 - 2y^2 = 4$ **(c)** $y^2 = \dfrac{1}{x}$

(d) $y = \cos x$ **(e)** $y = \sin x$

Solution

(a) The equation contains only even powers of x. Therefore ________________________________.

(b) Every term of x and of y is of even power. Therefore ________________________________.

(c) The equation contains only even powers of y. Therefore ________________________________.

(d) $\cos(-x) = \cos x$. Therefore ____________________.

(e) $\sin(-x) = -\sin x$. Therefore ____________________. ■

3 SYSTEMATIC CURVE SKETCHING

The sketching of any curve involves a thorough analysis of the function to be sketched. The following steps need to be considered:

Step 1: Intercepts on Coordinate Axes

Let $x = 0$, find y.

Let $y = 0$, find x.

Step 2: Linear Asymptotes

For vertical asymptotes, check if y tends to ∞ or $-\infty$ as x approaches a particular value.

For horizontal asymptotes, check if y approaches a particular value, as x tends to ∞ and/or $-\infty$.

For oblique asymptotes, check if y approaches a straight line, as x tends to ∞ and/or $-\infty$.

Step 3: Stationary Points

The nature of the stationary point(s) (if any) can be determined using either the first derivative or the second derivative test.

First Derivative Test

Given $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ at point where $x=a$. We test the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at points around $x=a$.

x	a^-	a	a^+	a^-	a	a^+	a^-	a	a^+	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$+$	0	$-$	$-$	0	$+$	$+$	0	$+$	$-$	0	$-$
Shape												
Conclusion	$(a,\ \mathrm{f}(a))$ is a **maximum** point			$(a,\ \mathrm{f}(a))$ is a **minimum** point			$(a,\ \mathrm{f}(a))$ is a **stationary point of inflexion**			$(a,\ \mathrm{f}(a))$ is a **stationary point of inflexion**		

Second Derivative Test

If $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ at the point where $x=a$ and

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	<0	>0	$=0$
Conclusion	$(a,\ \mathrm{f}(a))$ is a **maximum** point	$(a,\ \mathrm{f}(a))$ is a **minimum** point	**Inconclusive. This does NOT imply a point of inflexion.** First derivative test must be done!

Notes

- Numerical values of a^- and a^+, and the corresponding values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ must be stated when using the first derivative test.
- Numerical value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ must be stated when using the second derivative test.
- If $\frac{\mathrm{d}y}{\mathrm{d}x}>0$, then f is increasing, i.e., y increases as x increases.
- If $\frac{\mathrm{d}y}{\mathrm{d}x}<0$, then f is decreasing, i.e., y decreases as x increases.

Example 8

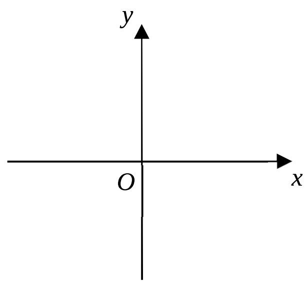
Sketch, on separate diagrams, the following graphs, and state the coordinates of axial intercepts and stationary points and the equations of the asymptotes (if any).

(a) $y = \ln(x^2 - 3)$ **(b)** $y = e^{x^2}$

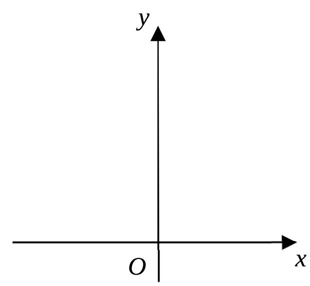
(c) $y = e^{-x^2}$ **(d)** $y = x \ln x$

Solution

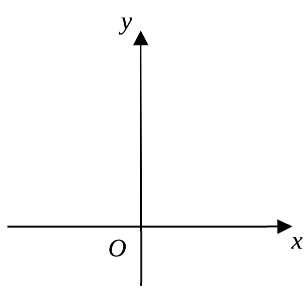
(a)



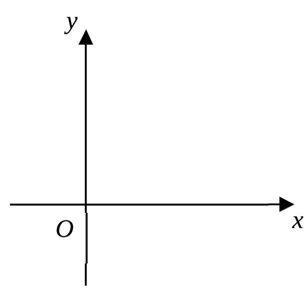
(b)



(c)



(d)

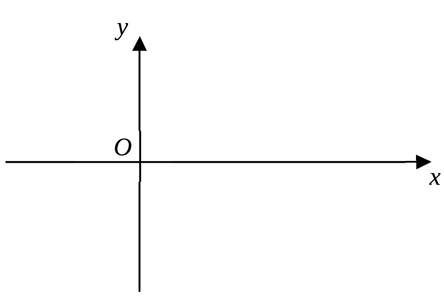


■

Example 9

Sketch the graph with equation $y = \dfrac{(x-3)^2}{(4-x)(x-2)}$ stating the equations of the asymptotes.

Solution



■

4 GRAPHS OF RATIONAL FUNCTIONS

A **rational function** is a function of the form $\mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ where P(x) and Q(x) are polynomials.

Consider rational functions where the degrees of P(x) and Q(x) are less than or equal to 2.

If **degree of P(x) < degree of Q(x)**, then $\mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is a proper fraction.

- Examples: $\mathrm{f}(x)=\dfrac{2}{x-1}$, $\mathrm{f}(x)=\dfrac{3x+1}{x^2+2x+2}$
- The graph of this function has a **horizontal asymptote which is the *x*-axis** (i.e. $y=0$).

If **degree of P(x) ≥ degree of Q(x)**, then $\mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is an improper fraction.

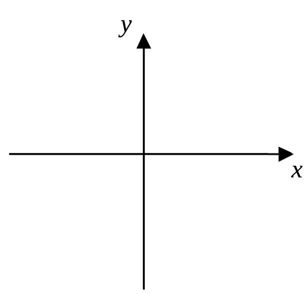
- Examples: $\mathrm{f}(x)=\dfrac{2x+3}{x-1}$, $\mathrm{f}(x)=\dfrac{x^2+3x+1}{x^2+2x+2}$
- Using long division, divide P(x) by Q(x) to obtain $\mathrm{f}(x)=ax+b+\dfrac{\mathrm{R}(x)}{\mathrm{Q}(x)}$, where $\dfrac{\mathrm{R}(x)}{\mathrm{Q}(x)}$ is a proper fraction
- If $a=0$, then the graph has a **horizontal asymptote** $y=b$.
- If $a\neq 0$, then the graph has an **oblique asymptote** $y=ax+b$.

4.1 Graphs of $y=\dfrac{ax+b}{cx+d}$, $c\neq 0$

Example 10

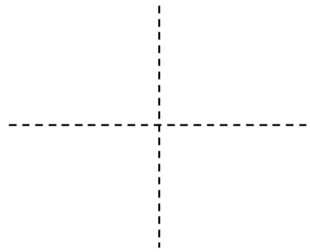
Sketch the graph of $y=\dfrac{x+2}{1-x}$ and state the equations of the asymptotes.

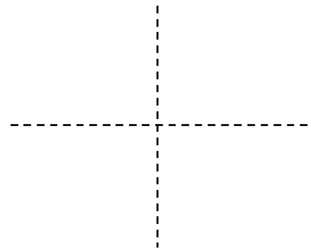
Solution



■

In general, the graph of $y=\dfrac{ax+b}{cx+d}$ (where $c \neq 0$) is a rectangular hyperbola and it is shaped either





To obtain the asymptotes of the graphs:

Express $y=\dfrac{ax+b}{cx+d}$ in the form $y=p+\dfrac{q}{cx+d}$ where p and q are constants.

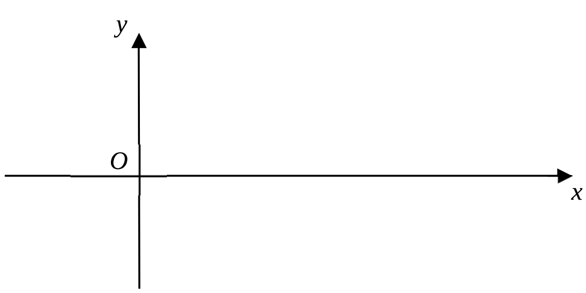
Vertical asymptote: $x=-\dfrac{d}{c}$

Horizontal asymptote: $y=p=\dfrac{a}{c}$

Self-Practice Exercise 4

Sketch the graph of $y=\dfrac{x+1}{x-100}$.

Solution



■

LECTURE 4

Lesson Outline

- Graphs of rational functions: $y=\dfrac{ax^2+bx+c}{dx+e}$, $a\neq 0$, $d\neq 0$
- Simple transformations

4.2 Graphs of $y=\dfrac{ax^2+bx+c}{dx+e}$, $a\neq 0$, $d\neq 0$

Example 11

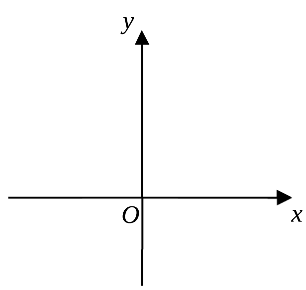
The equation of the curve C is given as $y=\dfrac{x^2+x+k}{x-1}$.

(a) Sketch, on separate diagrams, the curve C for which $k=-3$ and $k=1$.

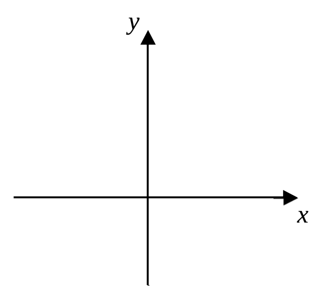
(b) Find the range of values of k for which the curve C has no stationary point.

Solution

(a) When $k=-3$



When $k=1$



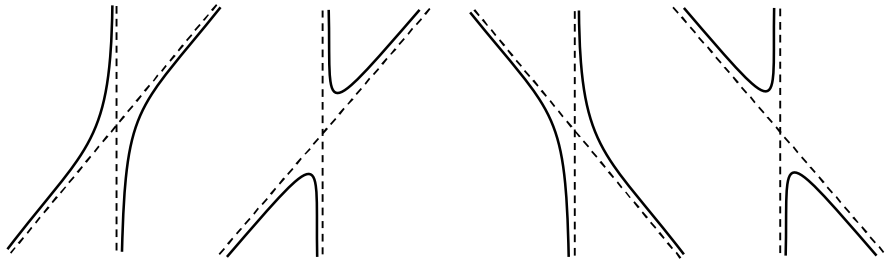
(b) $y=\dfrac{x^2+x+k}{x-1}=$

$\dfrac{dy}{dx}=$

To find stationary points,

■

In general, the graph of $y=\dfrac{ax^2+bx+c}{dx+e}$ where $a\neq 0$, $d\neq 0$, has two asymptotes, one vertical and the other oblique. The four possible shapes are



To obtain the asymptotes of the graphs:

Express $y=\dfrac{ax^2+bx+c}{dx+e}$ in the form $y=\left(px+q\right)+\dfrac{r}{dx+e}$ where p, q and r are constants, $p\neq 0$.

Vertical asymptote: $x=-\dfrac{e}{d}$

Oblique asymptote: $y=px+q$

Extra Practice

Annex C: Practice Question 2

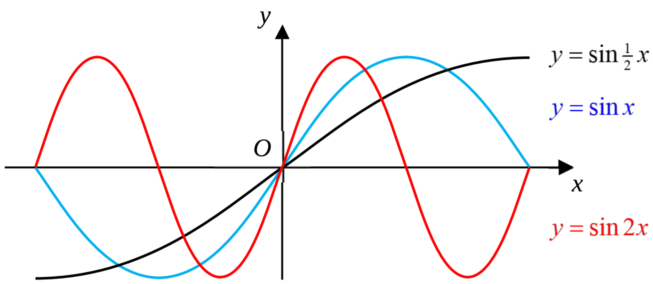
5 SIMPLE TRANSFORMATIONS

(Refer to Annex B Section 3.1 for the use of the GC)

Exploration

Using your graphing calculator, sketch, on the same diagram, the graphs of $y=\sin x$, $y=\sin 2x$ and $y=\sin\frac{1}{2}x$. How are the graphs related to each other?

Solution

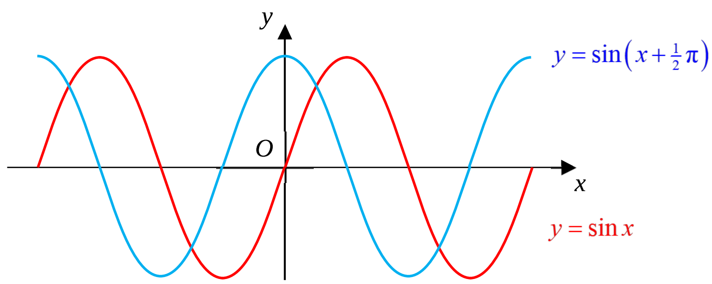


■

Self-Practice Exercise 5

Using your graphing calculator, sketch, on the same diagram, the graphs of $y=\sin x$ and $y=\sin\left(x+\frac{1}{2}\pi\right)$. What can you say about the graph of $y=\sin\left(x+\frac{1}{2}\pi\right)$?

Solution



■

Recall also (from O Level Maths) that any quadratic equation can be expressed in the form $y=(x-p)^2+q$. You can explore the relationship between the quadratic curves $y=x^2$ and $y=(x-p)^2+q$ for various values of p and q, using a graphing tool such as your GC or Desmos (available online at http://www.desmos.com/, or as a mobile app).

We can see from the above examples that it is possible to sketch several graphs through simple transformations on known graphs, with corresponding algebraic transformations on the equations of the graph. There are three main types of transformations: **reflection**, **translation** and **scaling** (or **stretching**).

Learning experience

In our syllabus, the transformations of graphs include translation, scaling and reflection. Explore the different types of transformations using Desmos graphing calculator (refer to Worksheet).

The following table summarises simple transformations for the case where $a > 0$,

Transformation	**Description of transformation**	**Geometrically**
Translation		
$y = f(x) \to y = f(x-a)$ Replace x by $x-a$.	The graph of $y = f(x-a)$ is the **translation** of the graph of $y = f(x)$ by a units in the **positive** x-direction.	$(x, y) \to (x+a, y)$
$y = f(x) \to y = f(x+a)$ Replace x by $x+a$.	The graph of $y = f(x+a)$ is the **translation** of the graph of $y = f(x)$ by a units in the **negative** x-direction.	$(x, y) \to (x-a, y)$
$y = f(x) \to y-a = f(x)$ i.e. $y = f(x)+a$ Replace y by $y-a$.	The graph of $y = f(x)+a$ is the **translation** of the graph of $y = f(x)$ by a units in the **positive** y-direction.	$(x, y) \to (x, y+a)$
$y = f(x) \to y+a = f(x)$ i.e. $y = f(x)-a$ Replace y by $y+a$.	The graph of $y = f(x)-a$ is the **translation** of the graph of $y = f(x)$ by a units in the **negative** y-direction.	$(x, y) \to (x, y-a)$
Scaling		
$y = f(x) \to y = f\left(\frac{x}{a}\right)$ Replace x by $\frac{x}{a}$.	The graph of $y = f\left(\frac{x}{a}\right)$ is the **scaling** of the graph of $y = f(x)$ **by factor *a* parallel to the *x*-axis**.	$(x, y) \to (ax, y)$
$y = f(x) \to \frac{y}{a} = f(x)$ i.e. $y = af(x)$ Replace y by $\frac{y}{a}$.	The graph of $y = af(x)$ is the **scaling** of the graph of $y = f(x)$ **by factor *a* parallel to the *y*-axis.**	$(x, y) \to (x, ay)$
Reflection		
$y = f(x) \to y = f(-x)$ Replace x by $-x$.	The graph of $y = f(-x)$ is the **reflection** of the graph of $y = f(x)$ in the ***y*-axis.**	$(x, y) \to (-x, y)$
$y = f(x) \to -y = f(x)$ i.e. $y = -f(x)$ Replace y by $-y$.	The graph of $y = -f(x)$ is the **reflection** of the graph of $y = f(x)$ in the ***x*-axis.**	$(x, y) \to (x, -y)$

Self-Practice Exercise 6

Given that $\mathrm{f}(x)=\dfrac{1}{x}$, state the corresponding equations.

(i) $y=\mathrm{f}(2x)$ **(ii)** $y=2\mathrm{f}(x)$

(iii) $y=\mathrm{f}(x+2)$ **(iv)** $y=\mathrm{f}(x)+2$

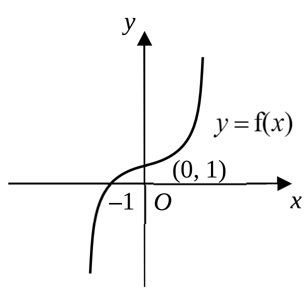
Solution

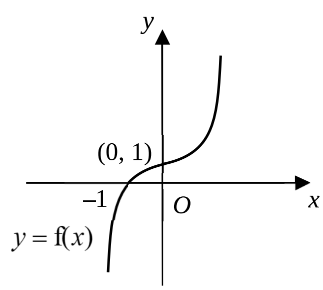
■

Example 12

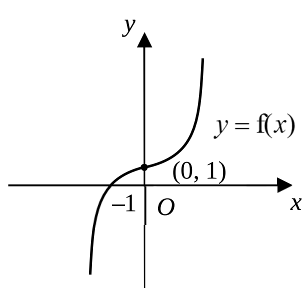
The graph of $y=\mathrm{f}(x)$ is shown in the diagrams below. Sketch the graphs of

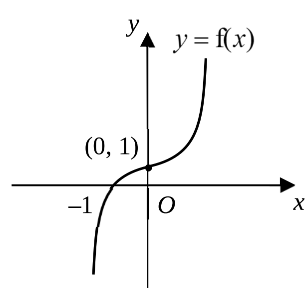
(a) $y=\mathrm{f}(x)+1$ **(b)** $y=\mathrm{f}(x)-1$



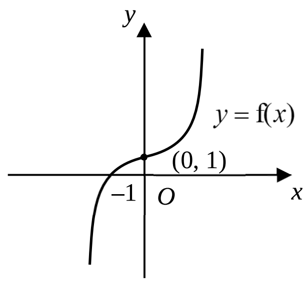


(c) $y=\mathrm{f}(x+1)$ **(d)** $y=\mathrm{f}(x-1)$

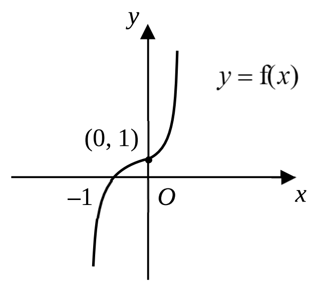




(e) $y = 2\mathrm{f}(x)$



(f) $y = \mathrm{f}(2x)$



■

LECTURE 5

Lesson Outline

- Simple transformations: combinations of successive transformations
- Graphs of modulus functions

---

6 THE SEQUENCE OF TRANSFORMATIONS

Very often, we would require a combination of two or more successive transformations to obtain the graph we need. A sequence of transformations is a combination of transformations in a specific order. A resulting equation or graph can be obtained from the original equation or graph after it undergoes a sequence of transformations.

(a) Composite Transformations of $y = \mathrm{f}(ax + b)$

The graph of $y = \mathrm{f}(x)$ can be transformed to the graph of $y = \mathrm{f}(ax + b)$, where $a > 0$ and $b > 0$, by the following methods.

$$y = \mathrm{f}(x) \xrightarrow[\text{translation of } b \text{ units in the negative } x\text{-direction}]{\text{replace } x \text{ by } (x+b)} y = \mathrm{f}(x+b) \xrightarrow[\text{scale (or stretch) parallel to the } x\text{-axis by factor } \frac{1}{a}]{\text{replace } x \text{ by } (ax)} y = \mathrm{f}(ax+b)$$

translation of b units in the negative ***x*-direction**

scale (or **stretch)** parallel to the ***x*-axis** by factor $\dfrac{1}{a}$

Alternatively, we can perform scaling first, followed by translation.

$$y = \mathrm{f}(x) \xrightarrow{\text{replace } x \text{ by } (ax)} y = \mathrm{f}(ax) \xrightarrow{\text{replace } x \text{ by } \left(x + \frac{b}{a}\right)} y = \mathrm{f}\left(a\left(x + \frac{b}{a}\right)\right) = \mathrm{f}(ax+b)$$

scale (or **stretch)** parallel to the ***x*-axis** by factor $\dfrac{1}{a}$

translation of $\dfrac{b}{a}$ units in the negative ***x*-direction**

(b) Composite Transformations of $y = c\mathrm{f}(x) + d$

The graph $y = \mathrm{f}(x)$ undergoes two successive transformations to become the graph $y = c\mathrm{f}(x) + d$, where $c > 0$ and $d > 0$, by the following methods:

$$y = \mathrm{f}(x) \xrightarrow{\text{replace } y \text{ by } \frac{y}{c}} y = c\mathrm{f}(x) \xrightarrow{\text{replace } y \text{ by } y-d} y = c\mathrm{f}(x) + d$$

scale (or **stretch)** parallel to the ***y*-axis** by factor *c*

translation of *d* units in the positive ***y*-direction**

Alternatively, we can perform translation first, followed by scaling.

$$y = \mathrm{f}(x) \xrightarrow{\text{replace } y \text{ by } y-\frac{d}{c}} y = \mathrm{f}(x) + \frac{d}{c} \xrightarrow{\text{replace } y \text{ by } \frac{y}{c}} y = c\left(\mathrm{f}\left(x\right) + \frac{d}{c}\right) = c\mathrm{f}(x) + d$$

translation of $\frac{d}{c}$ units in the positive ***y*-direction**

scale (or **stretch)** parallel to the ***y*-axis** by factor *c*

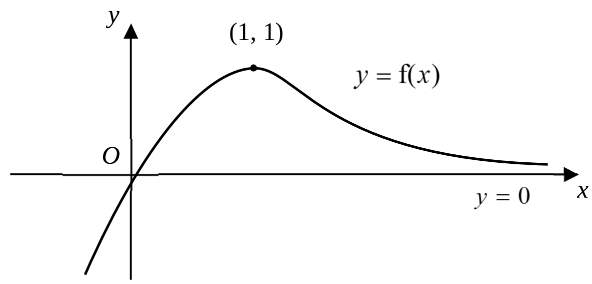
(c) Composite Transformations of both $y = \mathrm{f}(ax + b)$ and $y = c\mathrm{f}(x) + d$

When we perform the sequence of composite transformations for $y = c\,\mathrm{f}\left(ax + b\right) + d$, $a, b, c, d \in \mathbb{R}$ and $a, c \neq 0$, we may use the sequence as shown below:

$$y = \underset{\substack{\downarrow\\ 3^{\text{rd}}}}{c}\,\mathrm{f}(\underset{\substack{\downarrow\\ 2^{\text{nd}}}}{a}x + \underset{\substack{\downarrow\\ 1^{\text{st}}}}{b}) + \underset{\substack{\downarrow\\ 4^{\text{th}}}}{d}$$

Example 13

The diagram shows the graph of $y = \mathrm{f}(x)$. The curve passes through the origin, has a maximum point at $(1, 1)$ and the x-axis is the horizontal asymptote. Sketch, on separate diagrams, the graphs of the following functions, giving the coordinates of the turning point and the equation of the asymptote in each case.



(a) $y = \mathrm{f}(-2x)$ **(b)** $y = 1 - 2\mathrm{f}(x)$ **(c)** $y = \frac{1}{3}\mathrm{f}(x-1) + 2$

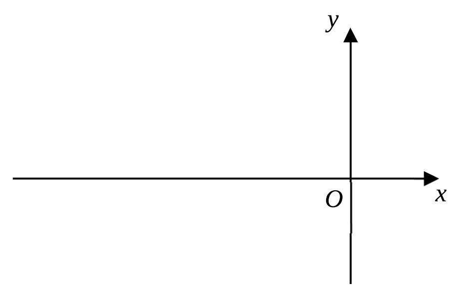
Solution

(a) $\mathrm{f}(x) \rightarrow$

$(1, 1) \rightarrow$

$(0, 0) \rightarrow$

$y = 0 \rightarrow$

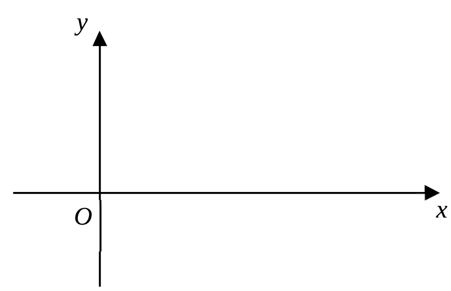


(b) $\mathrm{f}(x) \rightarrow$

$(1, 1) \rightarrow$

$(0, 0) \rightarrow$

$y = 0 \rightarrow$

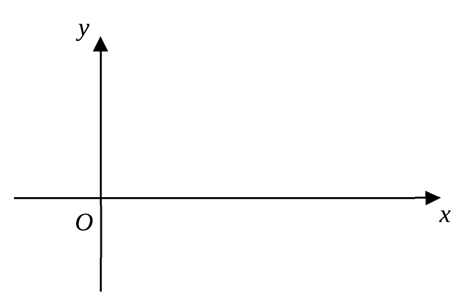


(c) $\mathrm{f}(x) \rightarrow$

$(1, 1) \rightarrow$

$(0, 0) \rightarrow$

$y = 0 \rightarrow$



■

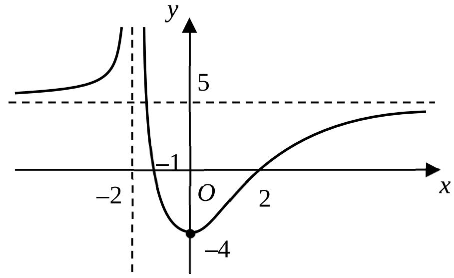
Example 14

The diagram shows the graph of $y = \mathrm{f}(x)$. Sketch, on separate diagrams, the graphs of

(a) $y = \mathrm{f}\left(\frac{1}{2}x + 3\right)$

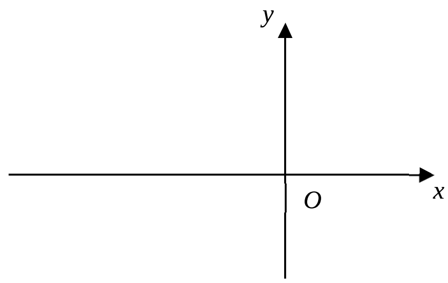
(b) $y = \mathrm{f}(1 - x)$

giving the coordinates of the turning point and the equations of the asymptotes in each case.



Solution

(a)



$y = 5 \rightarrow y = 5 \rightarrow y = 5$

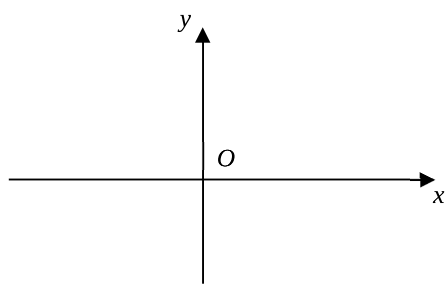
$x = -2 \rightarrow$

$(-1, 0) \rightarrow$

$(2, 0) \rightarrow$

$(0, -4) \rightarrow$

(b)



$y = 5 \rightarrow$

$x = -2 \rightarrow$

$(-1, 0) \rightarrow$

$(2, 0) \rightarrow$

$(0, -4) \rightarrow$

■

Example 15

Given that $\mathrm{g}(x)=1-x^2$, describe a transformation that will transform $\mathrm{g}(x)$ into each of the following curves:

(i) $\mathrm{g}_1(x)=1-4x^2$ **(ii)** $\mathrm{g}_2(x)=x^2-1$

(iii) $\mathrm{g}_3(x)=1-(x+1)^2$ **(iv)** $\mathrm{g}_4(x)=4-x^2$

Solution

(i) $\mathrm{g}_1(x)=1-4x^2=\mathrm{g}(2x)$

Therefore, ____

____

(ii) $\mathrm{g}_2(x)=x^2-1=-\mathrm{g}(x)$

Therefore, ____

____

(iii) $\mathrm{g}_3(x)=1-(x+1)^2=\mathrm{g}(x+1)$

Therefore, ____

____

(iv) $\mathrm{g}_4(x)=4-x^2=\mathrm{g}(x)+3$

Therefore, ____

____

■

Self-Practice Exercise 7

The curve whose equation is $y=x^2$ undergoes, in succession, the following transformations:

A: A translation of magnitude 3 units in the positive *x*-direction.

B: Stretch parallel to the *y*-axis by factor 2.

C: A translation of magnitude 5 units in the negative *y*-direction.

Give the equation of the resulting curve.

[A-level Maths Syllabus C Specimen Paper Modified]

Solution

$y=x^2 \xrightarrow{A}$

Resulting curve:

■

7 GRAPHS OF MODULUS FUNCTIONS

(Refer to Annex B Section 3.2 for the use of the GC)

Given a function $\mathrm{f}(x)$, we have the following results from the definition of modulus functions:

$$|\mathrm{f}(x)| = \begin{cases} \mathrm{f}(x), & \mathrm{f}(x) \geq 0 \\ -\mathrm{f}(x), & \mathrm{f}(x) < 0 \end{cases} \qquad \mathrm{f}(|x|) = \begin{cases} \mathrm{f}(x), & x \geq 0 \\ \mathrm{f}(-x), & x < 0 \end{cases}$$

Procedure to obtain the graph of $y = |\mathrm{f}(x)|$ from the graph of $y = \mathrm{f}(x)$

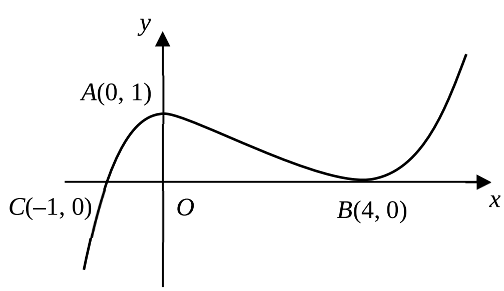
1. Keep the "positive *y* portions" of $y = \mathrm{f}(x)$.
2. Reflect the "negative *y* portions" of $y = \mathrm{f}(x)$ in the *x*-axis.
3. Delete the "negative *y* portions" of $y = \mathrm{f}(x)$.

Procedure to obtain the graph of $y = \mathrm{f}(|x|)$ from the graph of $y = \mathrm{f}(x)$

1. Keep the "positive *x* portions" of $y = \mathrm{f}(x)$. (i.e. keep those on the RHS of the *y*-axis)
2. Delete the "negative *x* portions" of $y = \mathrm{f}(x)$. (i.e. erase those on the LHS of the *y*-axis)
3. Reflect the "positive *x* portions" of $y = \mathrm{f}(x)$ in the *y*-axis.

Sketching the graphs of these functions would involve reflecting portions of the graph of $y = \mathrm{f}(x)$, as shown in Example 16 below.

Example 16



The diagram shows the graph of $y = \mathrm{f}(x)$. There is a maximum point at *A*(0, 1), a minimum point at *B*(4, 0) and the curve cuts the *x*-axis at the point *C*(–1, 0). Sketch on separate diagrams, the graphs of

(a) $y = |\mathrm{f}(x)|$ (b) $y = -|\mathrm{f}(x)|$

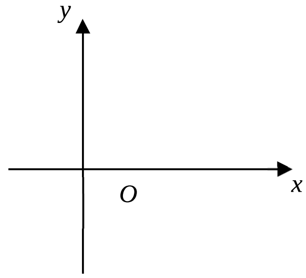
(c) $y = \mathrm{f}(|x|)$ (d) $y = \mathrm{f}(-|x|)$

(e) $y = \mathrm{f}(2|x|+1)$ (f) $y = \mathrm{f}(|2x+1|)$

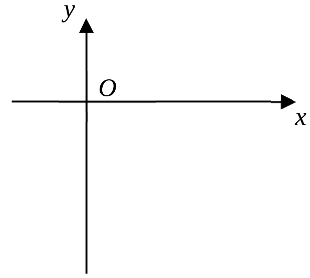
labelling each graph clearly and showing the coordinates of the points corresponding to A, B and C if applicable.

Solution

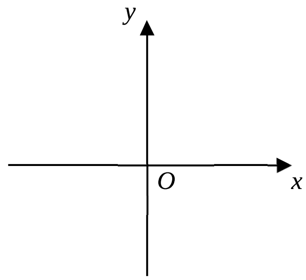
(a)



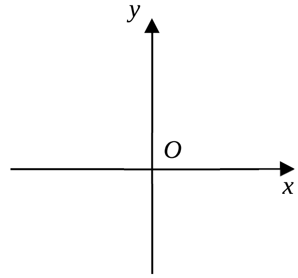
(b)



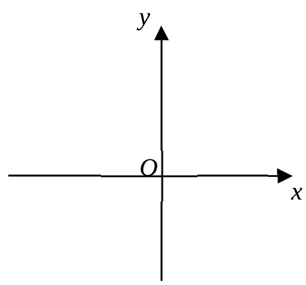
(c)



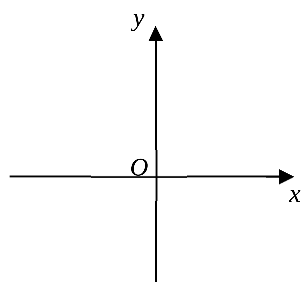
(d)



(e)



(f)



Extra Practice

Annex C: Practice Question 3

LECTURE 6

Lesson Outline

- Graph of $y = \dfrac{1}{\mathrm{f}(x)}$
- Miscellaneous examples

---

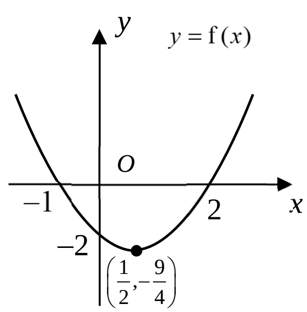
8 GRAPH OF $y = \dfrac{1}{\mathrm{f}(x)}$

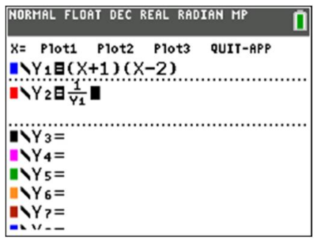
(Refer to Annex B Section 3.3 for the use of the GC)

Example

Using a graphing calculator, sketch the following graphs:

(1) $y = \mathrm{f}(x) = (x+1)(x-2)$





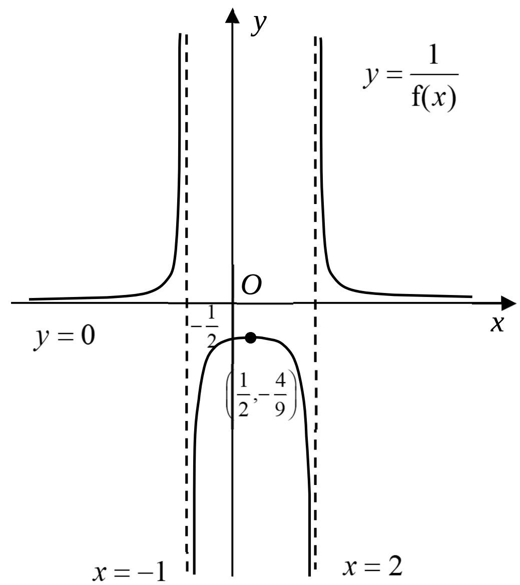
Question:

(a) State the coordinates of the minimum point for $y = \mathrm{f}(x)$: _______

(b) State the coordinates of the x-intercepts for $y = \mathrm{f}(x)$: __________

(c) State the coordinates of the y-intercept for $y = \mathrm{f}(x)$: ___________

(2) $y = \dfrac{1}{\mathrm{f}(x)} = \dfrac{1}{(x+1)(x-2)}$



Question:

(a) State the coordinates of the maximum point for $y = \dfrac{1}{\mathrm{f}(x)}$: ________

(b) State the coordinates of the x-intercepts for $y = \dfrac{1}{\mathrm{f}(x)}$: ___________

(c) State the coordinates of the y-intercept for $y = \dfrac{1}{\mathrm{f}(x)}$: ____________

(d) What do you notice about $y = \dfrac{1}{\mathrm{f}(x)}$ when $y = \mathrm{f}(x)$ is increasing?

____

(e) What do you notice about $y = \dfrac{1}{\mathrm{f}(x)}$ when $y = \mathrm{f}(x)$ is decreasing?

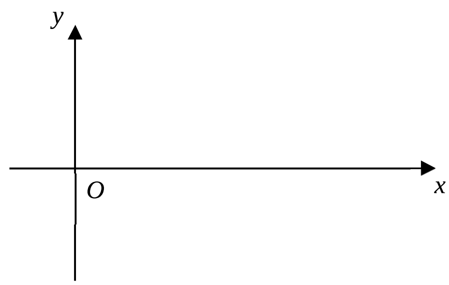
____

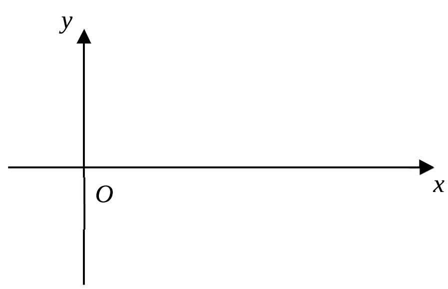
Graph of $y = \mathrm{f}(x)$	**Graph of** $y = \dfrac{1}{\mathrm{f}(x)}$
Let b $\neq$ 0:	
Has an x-intercept at $x = a$	Has a vertical asymptote $x = a$
Has a vertical asymptote $x = a$	Has an x-intercept $x = a$
Has a y-intercept at $y = b$	Has a y-intercept at $y = \dfrac{1}{b}$
Has a horizontal asymptote $y = b$	Has a horizontal asymptote $y = \dfrac{1}{b}$
(a) As $x \to \infty$ and/or $-\infty$, $\mathrm{f}(x) \to b^{+}$.	(a) As $x \to \infty$ and/or $-\infty$, $\dfrac{1}{\mathrm{f}(x)} \to \left(\dfrac{1}{b}\right)^{-}$.
(b) As $x \to \infty$ and/or $-\infty$, $\mathrm{f}(x) \to b^{-}$.	(b) As $x \to \infty$ and/or $-\infty$, $\dfrac{1}{\mathrm{f}(x)} \to \left(\dfrac{1}{b}\right)^{+}$.
Has an oblique asymptote	Has a horizontal asymptote $y = 0$
$\mathrm{f}(x) \to +\infty$	$\dfrac{1}{\mathrm{f}(x)} \to 0^{+}$
$\mathrm{f}(x) \to -\infty$	$\dfrac{1}{\mathrm{f}(x)} \to 0^{-}$
(a, b) is a maximum point	$\left(a, \dfrac{1}{b}\right)$ is a minimum point
(a, b) is a minimum point	$\left(a, \dfrac{1}{b}\right)$ is a maximum point
$\mathrm{f}(x)$ increases	$\dfrac{1}{\mathrm{f}(x)}$ decreases
$\mathrm{f}(x)$ decreases	$\dfrac{1}{\mathrm{f}(x)}$ increases
$\mathrm{f}(x) > 0$	$\dfrac{1}{\mathrm{f}(x)} > 0$
$\mathrm{f}(x) < 0$	$\dfrac{1}{\mathrm{f}(x)} < 0$

Example 17

Sketch the graph of $y = x\left(a-x\right)^2$. Hence sketch the graph of $y = \dfrac{1}{x\left(a-x\right)^2}$, whereby $0 < a < 1$.

Solution





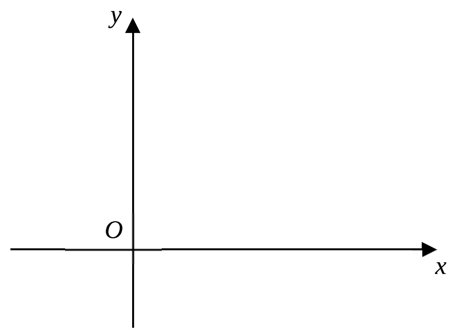
Example 18

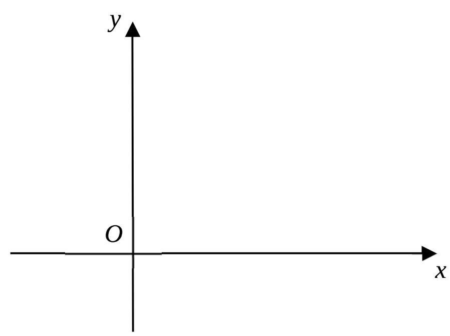
The graph of $y = \dfrac{2x(x-1)}{3x-4}$ has stationary points $\left(\dfrac{2}{3}, \dfrac{2}{9}\right)$ and $(2, 2)$. Sketch the graph of $y = \dfrac{2x(x-1)}{3x-4}$ and hence sketch the graph of $y = \dfrac{3x-4}{2x(x-1)}$, stating the equations of any asymptotes.

Solution

$y = \dfrac{2x(x-1)}{3x-4} =$

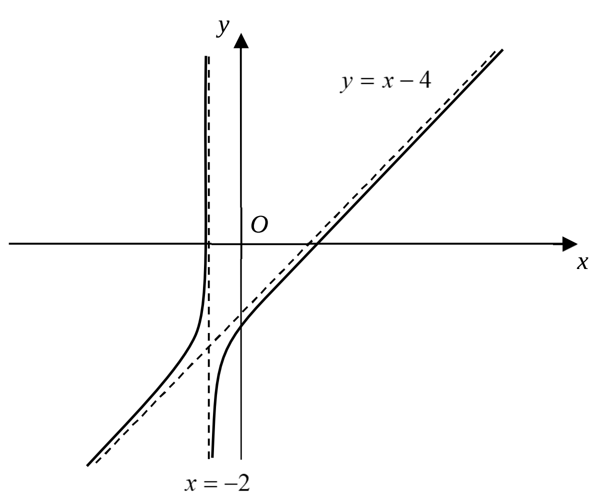
Intercepts: When $x = 0$, When $y = 0$,





Miscellaneous Example 19

The diagram below shows a sketch of the graph of $y = \dfrac{ax^2 + bx + c}{x + d}$ where a, b, c and d are constants. The asymptotes, also shown in the diagram, are $x = -2$ and $y = x - 4$.



(i) Write down the value of d and find the values of a and b. [4]

(ii) Given that the curve passes through the point $(0,-4.5)$, find the value of c. [1]

(iii) By sketching the graph of $(x-4)^2+y^2=k^2$, determine the range of values of k such that the equation $(x-4)^2+\left(\dfrac{ax^2+bx+c}{x+d}\right)^2=k^2$, where a, b, c and d are the values found above, has at least one negative root. [3]

[RVHS/2010/1/7]

Solution

(i) $x+d=0 \Rightarrow x=-d$ is the vertical asymptote.

Given $x=-2$ is a vertical asymptote, therefore $d=2$.

Method 1: Using long division

$$y=\frac{ax^2+bx+c}{x+d}=ax+(b-2a)+\frac{c-2(b-2a)}{x+2}$$

Therefore oblique asymptote is $y=ax+(b-2a)$.

Given $y=x-4$ is an asymptote, by comparing coefficients, we have

$a=1$ and $b-2a=-4 \Rightarrow b=-2$.

Method 2

Since $y=x-4$ is an asymptote, therefore

$$\frac{ax^2+bx+c}{x+d}=x-4+\frac{h}{x+2}=\frac{(x-4)(x+2)+h}{x+2}=\frac{x^2-2x-8+h}{x+2}$$

Comparing coefficients of the numerator, we have

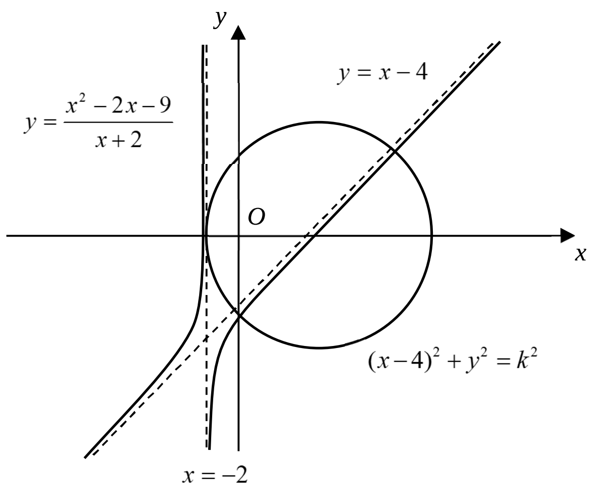
$a=1, b=-2$.

(ii) $y=\dfrac{x^2-2x+c}{x+2}$

Since the curve passes through $(0,-4.5)$,

$-4.5=\dfrac{c}{2} \Rightarrow c=-9$.

(iii) $(x-4)^2+y^2=k^2$ is a circle centred at $(4,0)$ with radius k. Solving this $(x-4)^2+\left(\dfrac{ax^2+bx+c}{x+d}\right)^2=k^2$ gives the intersection between the circle and the given curve.



Therefore the first negative root occur when the radius is larger than the distance from the centre of the circle to the y-intercept of the curve $y = \dfrac{x^2 - 2x - 9}{x + 2}$. Therefore,

$|k| > \sqrt{4^2 + 4.5^2} = 6.02 \Rightarrow k > 6.02$ or < -6.02.

Miscellaneous Example 20

The curves C_1 and C_2 have equations $9y^2 = (x+k)^2 - 9$ and $\dfrac{x^2}{k^2} + y^2 = 1$ respectively, where k is a real constant such that $k > 3$.

(a) Describe a sequence of transformations which transforms the graph of $x^2 - y^2 = 1$ to the graph of C_1. [2]

(b) On the same diagram, sketch the graphs of C_1 and C_2, stating clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes. [5]

[ACJC/2011/2/5 Modified]

Solution

(a) $9y^2 = (x+k)^2 - 9$

$\Rightarrow (x+k)^2 - 9y^2 = 9$

$\Rightarrow \dfrac{(x+k)^2}{9} - y^2 = 1 \rightarrow x^2 - y^2 = 1$

Method 1

(1) Scale parallel to the x-axis by factor 3,

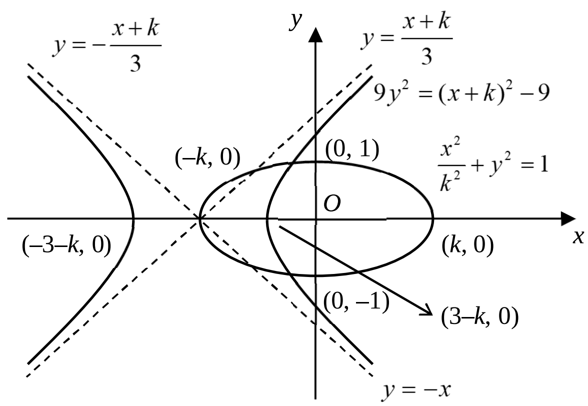
(2) translate in the negative x-direction by k units.

Method 2

(1) Translate in the negative x-direction by $\frac{k}{3}$ units,

(2) scale parallel to the x-axis by factor 3.

(b)



y-intercepts of C_1: $\left(0, \sqrt{\frac{k^2-9}{9}}\right), \left(0, -\sqrt{\frac{k^2-9}{9}}\right)$

Asymptotes of C_1: $\frac{(x+k)^2}{9}-y^2=0 \Rightarrow y=\pm\frac{x+k}{3}$

LECTURE 7

Lesson Outline

- Parametric equations and their graphs

---

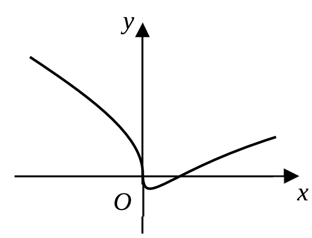
9 PARAMETRIC EQUATIONS AND THEIR GRAPHS

Some relationships between x and y are quite complicated so it is easier to express x and y each in terms of a third variable, called a **parameter**.

Consider the curve C defined by the equations $x=t^3$, $y=t^2-t$. The values of x and y depend on the value of the parameter t.

For example,

t	-2	-1	0	1	2
x					
y					



When $t=1$, the coordinates of the point on the curve C are $(1, 0)$.
We also say that the point on the curve C with parameter 1 is $(1, 0)$.

In general, the pair of equations for x and y given in terms of a parameter t are called **parametric equations** of the curve, i.e., we may write $x=\mathrm{g}(t)$ and $y=\mathrm{h}(t)$ instead of $y=\mathrm{f}(x)$.

The point P on the curve with parameter p has coordinates $(\mathrm{g}(p), \mathrm{h}(p))$.

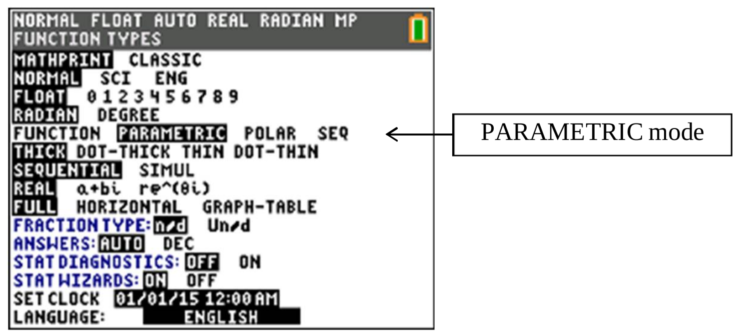
Examples of common curves and their parametric equations

Curve	**Parametric equations**	**Cartesian equation**
Circle	$x=a\cos\theta,\ y=a\sin\theta$	$x^2+y^2=a^2$
Ellipse	$x=a\cos\theta,\ y=b\sin\theta$	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$
Parabola	$x=t+p,\ y=at^2+q$	$y=a(x-p)^2+q$
Cycloid	$x=\theta-\sin\theta,\ y=1-\cos\theta$	-

10 Graphing a pair of parametric equations

To graph parametric equations, the calculator has to be in parametric graphing mode.

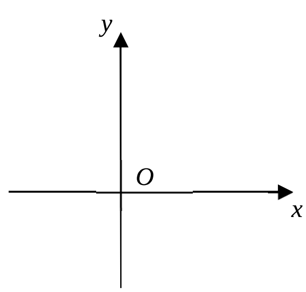
1) Press **[MODE]** to go to the Mode Menu.
2) Go down to "Function Types" by pressing ▼ repeatedly.
3) Select **PARAMETRIC**. (Setting should be highlighted.)
4) Press **[ENTER]**.
5) Press **[CLEAR]** or **[2ND] [MODE]** to go back to the Home Screen.



Example 21

The curve C is given by the parametric equations $x = 1 - t$, $y = t^2 - 4$. Sketch the curve C, showing clearly the axial intercepts.

Solution



When $x = 0$

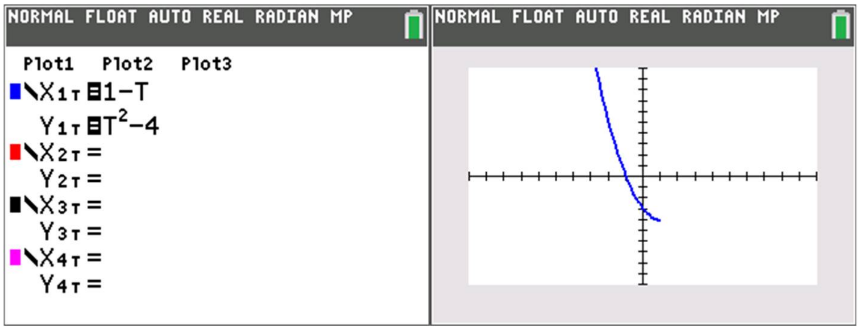
When $y = 0$

1) Press **[Y=]**.

2) For $\mathbf{X_{1T}}$, type **[1] [–] [X,T,θ,n]**. Press **[ENTER]**.

3) For $\mathbf{Y_{1T}}$, type **[X,T,θ,n] [x2] [–] [4]**. Press **[ENTER]**.

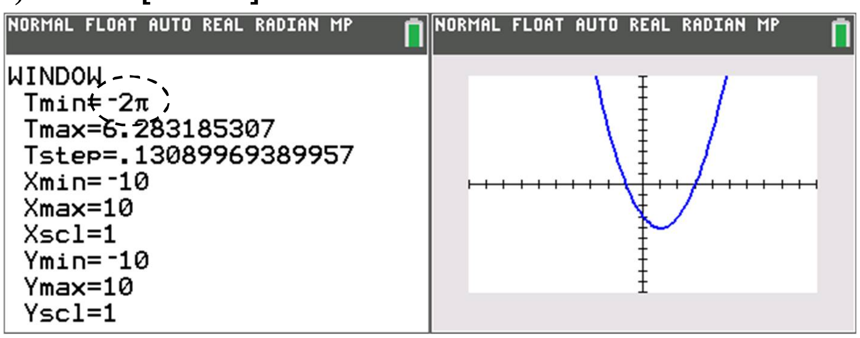
4) Press **[GRAPH]**. Insufficient points are plotted.



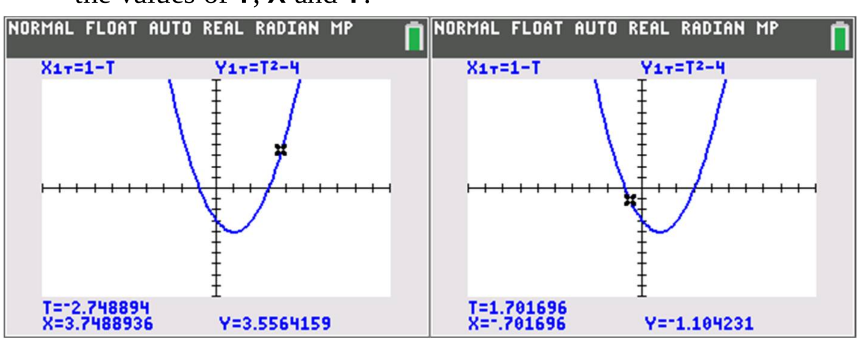
5) Press **[WINDOW]** and set **Tmin = -2π** to view the entire graph.

Note: The default value of t is from 0 to 2π. In order to see the entire graph, ensure that the range of t values includes both positive and negative values.

6) Press **[GRAPH]**.



7) (Optional) Press **[TRACE]** to move along the curve to observe the values of **T**, **X** and **Y**.



Example 22

Convert these parametric equations into Cartesian equations. Sketch the graphs of (b) and (c).

(a) $x = 1 - t,\ y = t^2 - 4$

(b) $x = a(1 + \sin\theta),\ y = a(1 - \cos\theta),\ a > 0$

(c) $x = t + \frac{1}{t},\ y = t - \frac{1}{t},\ t \neq 0$

Solution

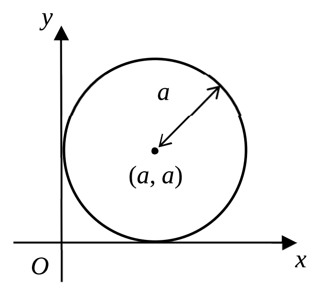
(a) $x = 1 - t \Rightarrow$

$y = t^2 - 4 =$

(b) $x = a(1 + \sin\theta) \Rightarrow$

$y = a(1 - \cos\theta) \Rightarrow$

Using $\sin^2\theta + \cos^2\theta = 1$, we get

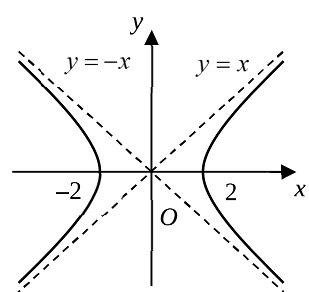


(c) $x = t + \frac{1}{t},\ y = t - \frac{1}{t}$

Adding x and y we get:

Subtracting x and y we get:

Multiply we get:

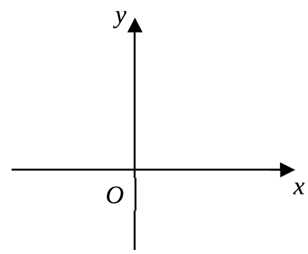


Example 23

The curve C is defined parametrically by $x=t^3$, $y=t^2-t$. Find the coordinates of the point of intersection between C and the line $y=x-1$.

Solution

Sub $x=t^3$ and $y=t^2-t$ into $y=x-1$:



Using GC,

Hence, the coordinates of the point of intersection is (__, __).

Example 24 - Application on Projectile Motion

A moving object is said to be in **projectile motion** if the only force that acts on it is gravity (assuming air resistance is negligible). Its position at any point in time t can be described by its horizontal displacement x and vertical displacement y from some reference point O. Since x and y each depend on t only, we can model the **trajectory** (i.e. path) of the projectile motion as a pair of parametric equations.

For an object that is launched into the air from O with an initial velocity $v \text{ m s}^{-1}$ and at an angle of $\theta°$ above the horizontal, its trajectory is given by the parametric equations

$$x=(v\cos\theta)t,\ y=(v\sin\theta)t-\frac{1}{2}gt^2,$$

where g is a constant known as the acceleration due to free fall.

A man on the ground level throws a rock from O into the air with an initial velocity of 10 m s^{-1} and at angle of 60° above the horizontal. Taking g = 10 m s^{-2},

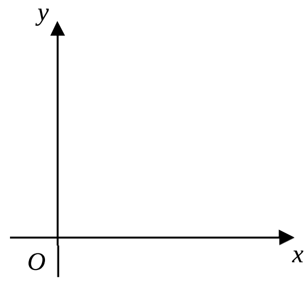
(i) sketch the trajectory of the rock,

(ii) find the time taken for the rock to reach the ground,

(iii) find the distance from O where the rock lands.

Solution

(i)



(ii) When $y = 0$,

(iii) Sub $t =$ ___________ ,

Therefore, the distance from O where the rock lands is ______ m.

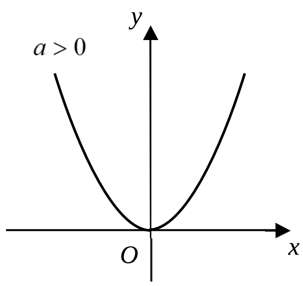
Extra Practice

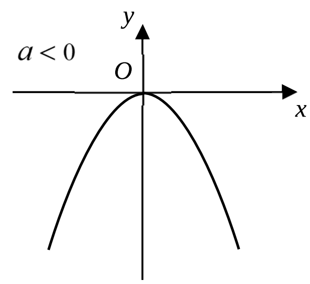
Annex C: Practice Question 4 and 5

ANNEX A: BASIC STANDARD GRAPHS

1 Graphs of $y = ax^n$ where a is a constant

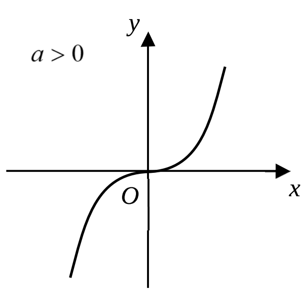
Case 1: n is a positive even integer (i.e., n = 2, 4, 6 ...)

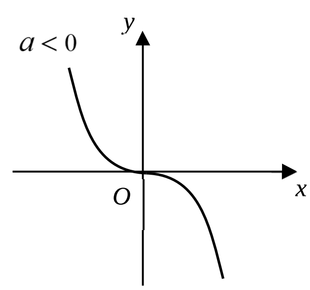




Axial intercept(s):	Origin (0, 0)
Asymptote(s):	None
Stationary point(s):	(0, 0) is a minimum point for $a > 0$ and maximum point for $a < 0$
Symmetry**:**	Symmetrical about the y-axis ($x = 0$)

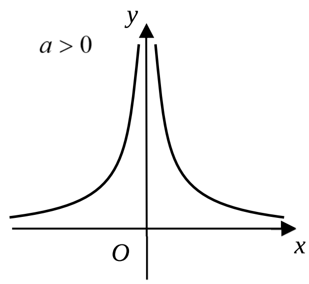
Case 2: n is a positive odd integer except 1 (i.e., n = 3, 5, 7 ...)

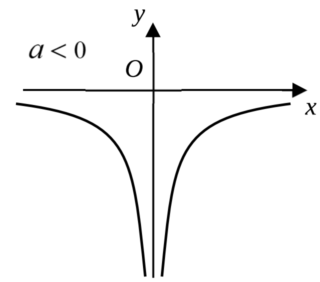




Axial intercept(s):	Origin (0, 0)
Asymptote(s):	None
Stationary point(s):	(0, 0) is a stationary point of inflexion in both graphs
Symmetry:	Symmetrical about the origin

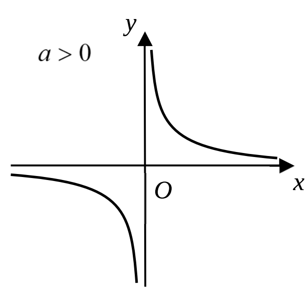
Case 3: *n* is a negative even integer (i.e., *n* = –2, –4, –6 ...)

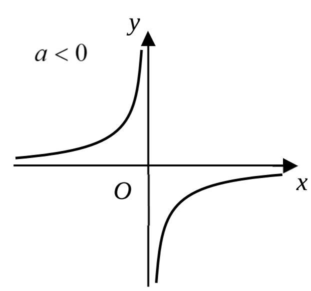




Axial intercept(s):	None
Asymptote(s):	*x*-axis ($y=0$) and *y*-axis ($x=0$)
Stationary point(s):	None
Symmetry:	Symmetrical about the *y*-axis ($x=0$)

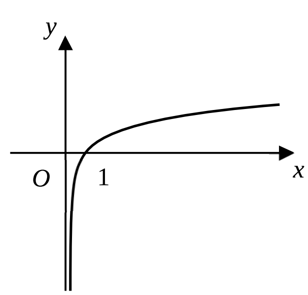
Case 4: *n* is a negative odd integer (i.e., *n* = –1, –3, –5 ...)





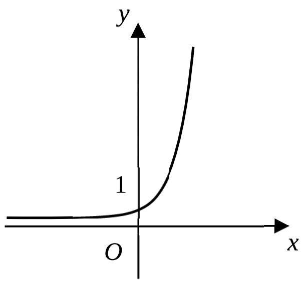
Axial intercept(s):	None
Asymptote(s):	*x*-axis ($y=0$) and *y*-axis ($x=0$)
Stationary point(s):	None
Symmetry:	Symmetrical about the origin

2 Graphs of Logarithm Functions $y = \log_a x$ ($a>1$)



Axial intercept(s):	(1, 0)
Asymptote(s):	*y*-axis ($x=0$)
Stationary point(s):	None
Symmetry:	None

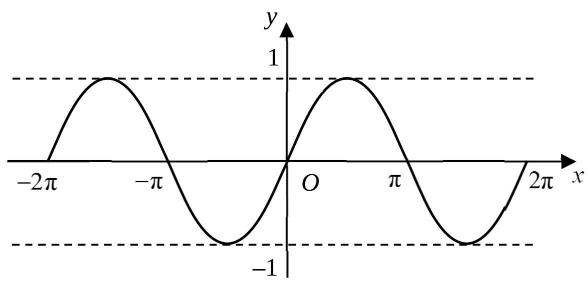
3 Graphs of Exponential Functions $y = a^x$ (a>1)



Axial intercept(s):	(0, 1)
Asymptote(s):	x-axis ($y = 0$)
Stationary point(s):	None
Symmetry:	None

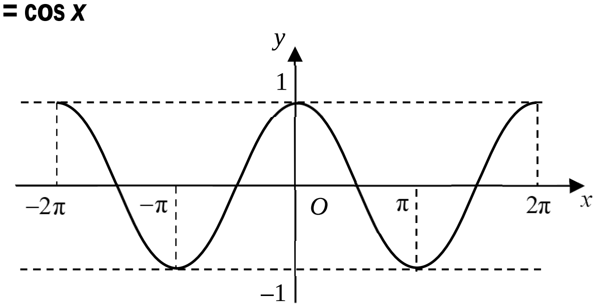
4 Graphs of Trigonometric Functions

Graph of $y = \sin x$



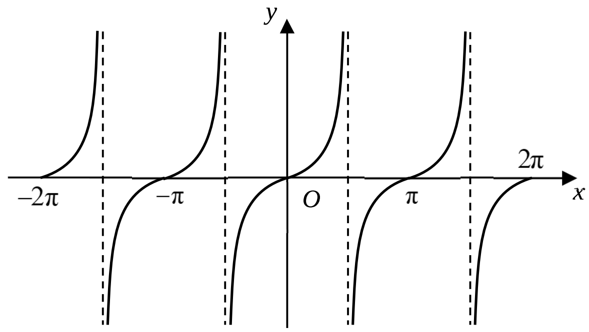
Axial intercept(s):	x-intercepts at $(k\pi, 0)$, for all integers k
Asymptote(s):	None
Stationary point(s):	Max. points at $\left(\frac{\pi}{2} + 2k\pi, 1\right)$,
	Min. points at $\left(-\frac{\pi}{2} + 2k\pi, -1\right)$
Symmetry:	Symmetrical about the origin

Graph of $y = \cos x$



Axial intercept(s):	x-intercepts at $\left((2k+1)\frac{\pi}{2}, 0\right)$, for all integers k
	y-intercept at (0, 1)
Asymptote(s):	None
Stationary point(s):	Max. points at $\left(2k\pi, 1\right)$
	Min. points at $\left((2k+1)\pi, -1\right)$
Symmetry:	Symmetrical about the y-axis ($x = 0$)

Graph of *y* = tan *x*



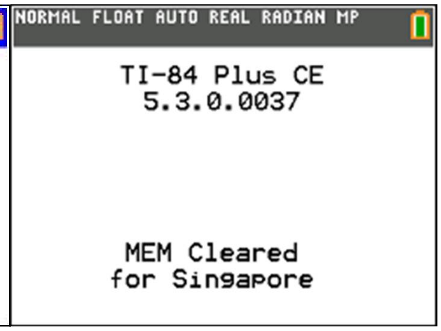
Axial intercept(s):	x-intercepts at $(k\pi, 0)$, for all integers k
Asymptote(s):	Vertical asymptotes $x = (2k+1)\frac{\pi}{2}$, for all integers k
Stationary point(s):	None
Symmetry:	Symmetrical about the origin

ANNEX B: GRAPHING CALCULATOR ACTIVITIES

*** Resetting the Graphic Calculator (Before Test Or Exam)

1) Start with the GC off.
2) Press **[2]**, **[8]** and **[ON]** simultaneously.
3) Press **[OK]**.

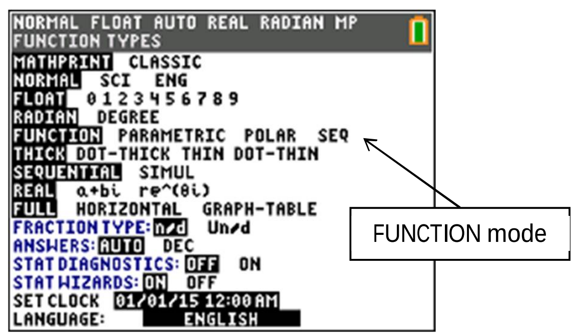
"**MEM Cleared for Singapore**" should be seen on the screen of the GC.



1 Graphing and the Use of the [CALC] Operations

1.1 Defining and Displaying a Graph

To graph a function, the calculator has to be in **FUNCTION** Graphing Mode:



1) Press **[MODE]** to go to the Mode Menu.
2) Scroll down to "Function Types" by pressing ▼ repeatedly.
3) Select **FUNCTION**. (The setting should be highlighted.)
4) Press **[ENTER]**.
5) Press **[CLEAR]** or **[2ND] [MODE]** to go back to the Home Screen.

1.2 Use of [GRAPH], [WINDOW], [TRACE], [ZOOM] And [CALC]

Example 1

Sketch the graphs of $y = x^2 - 4$ and $y = x + 3$ and investigate the graphs.

Solution

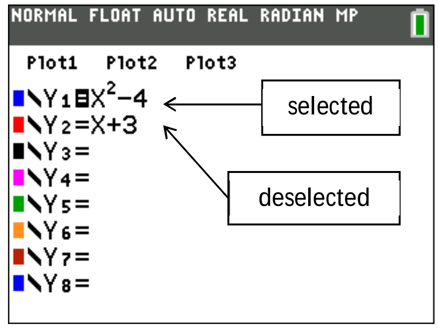
To enter equations:

1) Press **[Y=]**.

2) For **Y1**, type **[X,T,θ,n] [x2] [−] [4]**. Press **[ENTER]**.

Note: For negative numbers or variables, **[(-)]** is used. For subtraction, **[−]** is used.

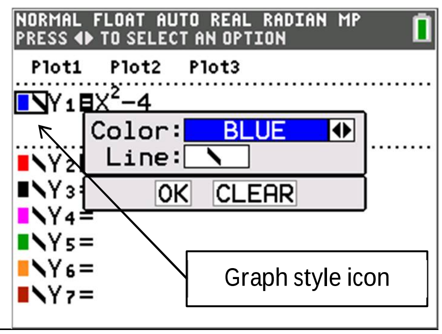
3) For **Y2, type [X,T, θ,n] [+] [3]**. Press **[ENTER]**.



A graph is selected when the '**=**' sign is highlighted. To **deselect (or select) a graph**, move the cursor over the '**=**' sign and press **[ENTER]** to change the selection status. Deselected graphs will not appear in the **GRAPH** window.

Press **[GRAPH]** to display the graphs of the functions.

To set the **graph style,** press ◀ to move the cursor left until the cursor is on the graph style icon. Press **[ENTER]** and scroll through until your preferred colour and line style are chosen.



Press **[WINDOW]** to change the window settings:

Xmin: the minimum value of x.

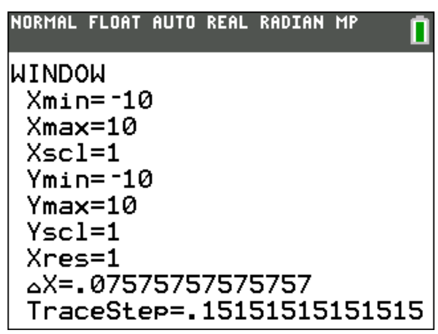
Xmax: the maximum value of x.

Xscl: the distance between the tick marks on the x-axis.

Ymin: the minimum value of y.

Ymax: the maximum value of y.

Yscl: the distance between the tick marks on the y-axis.



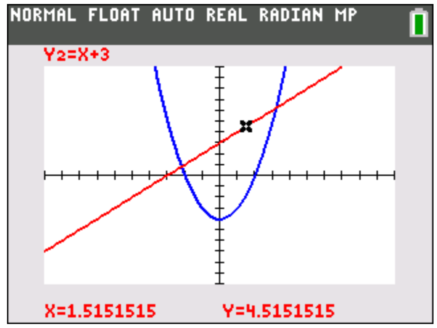
Xres: pixel resolution – not usually changed except by advanced users.

Press **[TRACE]** to move along the graph:

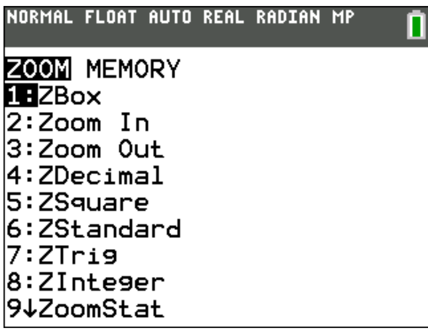
Use ▼ and ▲ to move from one curve to another.

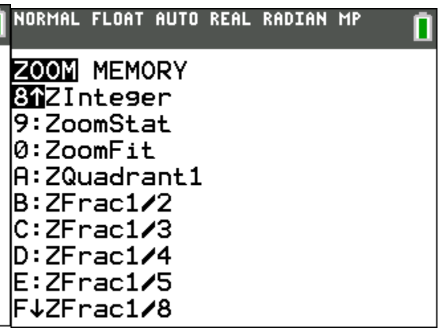
The graph on which the cursor lies is indicated on the top left hand corner of the screen.

Use ◀ and ▶ to move along each curve.



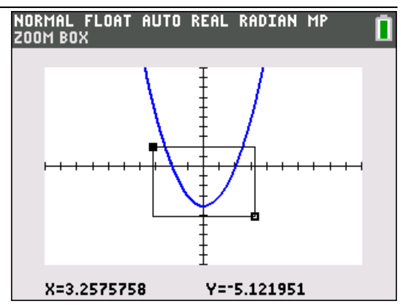
Press **[ZOOM]** to adjust the graph window to a predetermined level of magnification:





1:ZBox specifies a region and magnifies it for viewing.

A blinking cursor will appear on the screen. Shift the cursor to a position intended to have as a corner of the box and press **[ENTER]**. Next move the cursor to the corner diagonally opposite to form the box and press **[ENTER]**.



2:Zoom In magnifies the graph around the cursor.

3:Zoom Out displays a greater portion of the graph, centred on the cursor.

Note: The default position of the cursor is at the origin. You have to move the cursor near to the region which you want to zoom in.

5:ZSquare ensures that the same scale is used on both axes. It ensures that the line $y = x$ drawn makes an angle of 45^0 with the vertical. A circle will not look distorted.

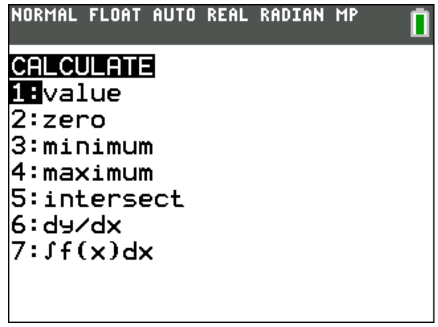
6:ZStandard sets the window to the standard size of

x values : -10 to 10

y values: -10 to 10

0:ZoomFit recalculates **YMin** and **YMax** to include the maximum and minimum y values of the selected functions between the current **XMin** and **XMax**.

To display the **[CALC]** menu, press **[2ND] [TRACE]**:



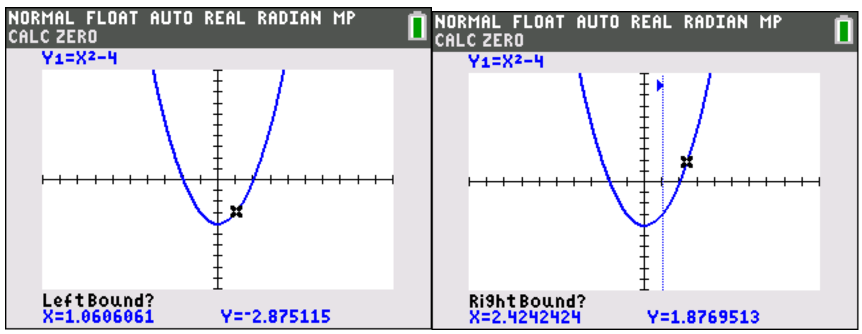
1:value evaluates y for a given value of x.

Enter a value for x. The corresponding y value and the function are displayed on the screen.

The menu can be used to find the y-intercept by entering $x = 0$.

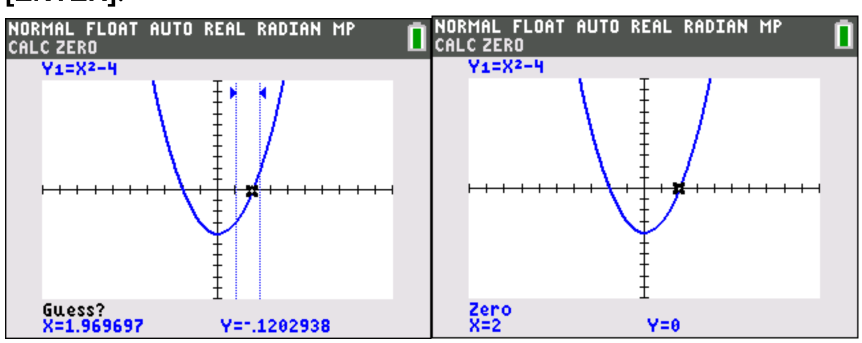
2:zero finds the x-intercept of a function.

Graph is displayed with "**Left Bound?**" in the bottom left hand corner. Move the cursor along the curve to a point to the left of the x-intercept, and then press **[ENTER]**.



Graph is displayed with "**Right Bound?**" in the bottom left hand corner. Move the cursor along the curve to the right of the x-intercept, and then press **[ENTER]**.

Graph is displayed with "**Guess?**" in the bottom left hand corner. Move the cursor to a point estimated to be the x-intercept, and then press **[ENTER]**.



Do the same for the other intercept.

3:minimum finds the minimum of a function;

4:maximum finds the maximum of a function.

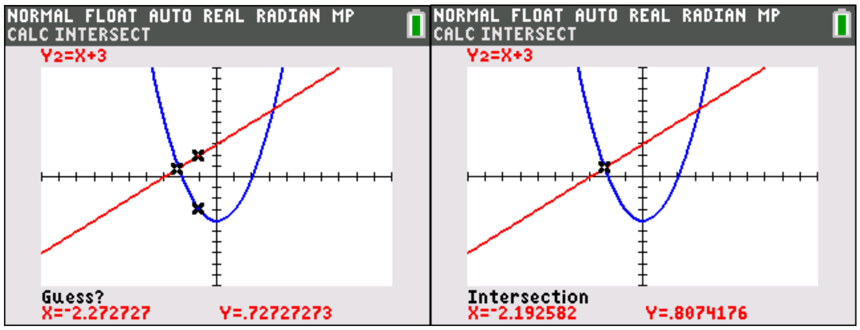
Steps required are similar to finding zero.

5:intersect finds an intersection of two functions.

Move the cursor to the first function and press **[ENTER]**.

Cursor will have moved to the second function. Press **[ENTER]**.

Graph is displayed with "**Guess?**" in the bottom left hand corner. Move the cursor to a point close to the point of intersection, and then press **[ENTER]**.



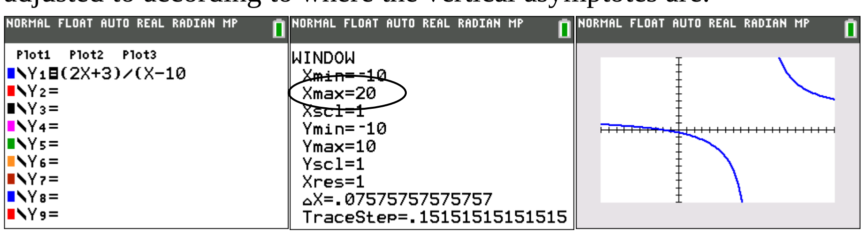
Example 2

Sketch the graph of $y = \dfrac{2x+3}{x-10}$.

Solution

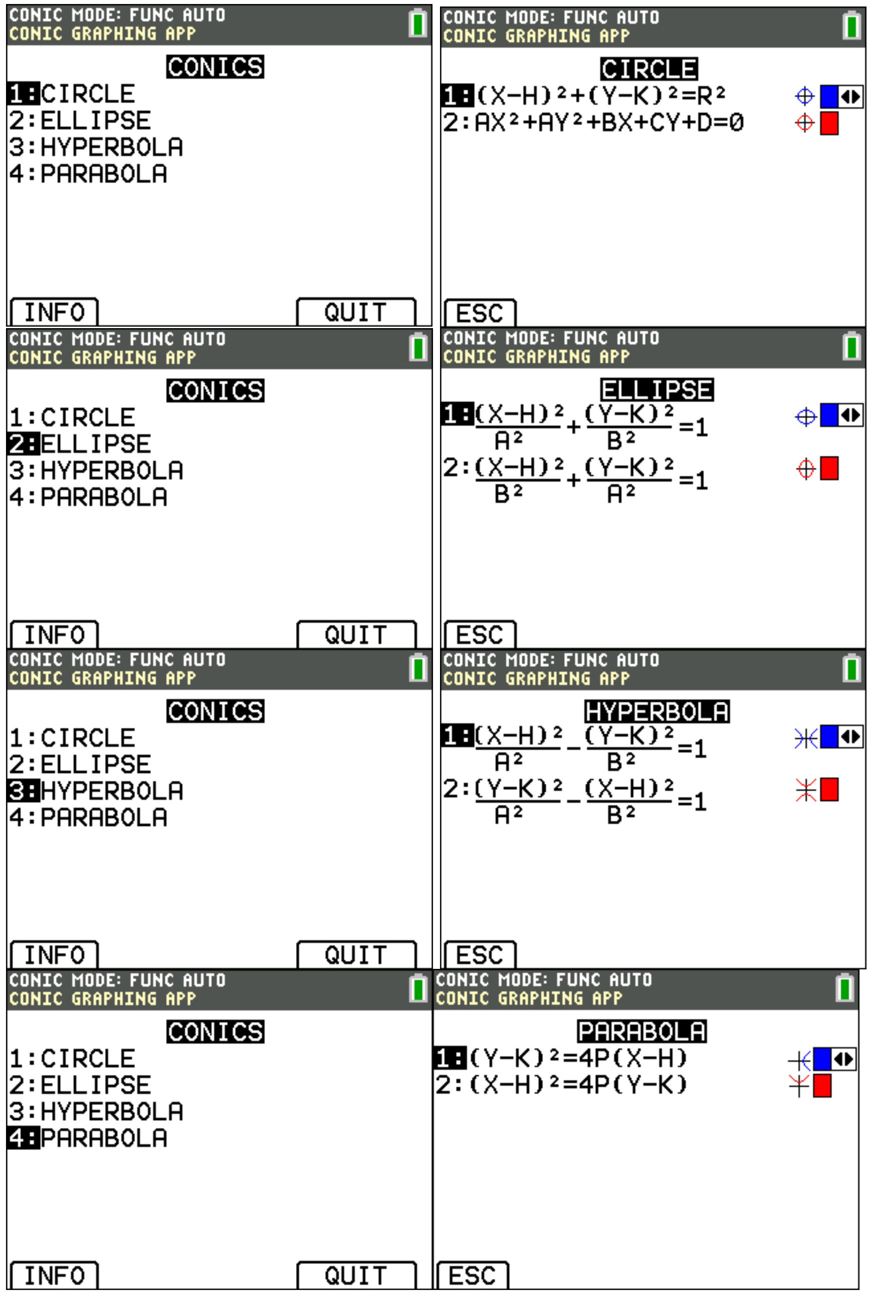
1) Press **[Y=]**.
2) For **Y1, type [(] [2] [X,T, θ,n] [+] [3] [)] [÷] [(] [X,T, θ,n] [–] [10] [)]**. Press **[ENTER]**.
3) Press **[WINDOW]** and set **Xmax = 20** to view the entire graph.
4) Press **[GRAPH]**.

* In the graphic calculator, asymptotes are not shown. Asymptotes, if any, should be found before using the GC and the windows should be adjusted to according to where the vertical asymptotes are.



2 GRAPHING CONICS

1) Press **[APPS]** and select **[2:Conics]**.



2) Select the relevant conics section to sketch.
3) Fill in the appropriate values for the constants.
4) Press **[ALPHA] [ENTER]** for the characteristics of the graph.
5) Press **[GRAPH]** to view the graphical representation of the function.
6) Press **[2ND] [MODE]** to quit **APPS** and exit to Home Screen.

Example 3

Sketch the graph of $\dfrac{x^2}{9}-\dfrac{y^2}{4}=1$.

Solution

Method (a): Rearranging the equation, we get

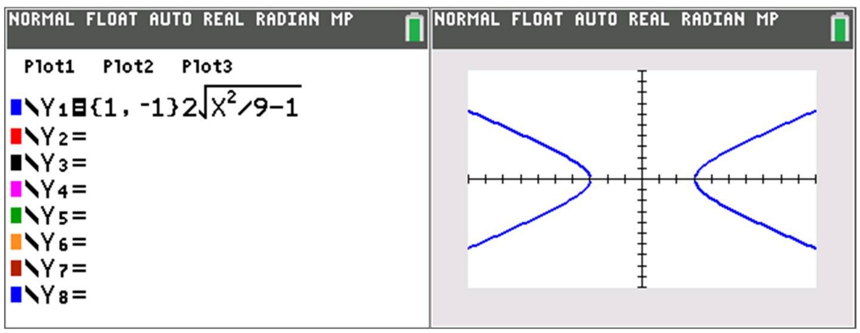
$$\frac{y^2}{4}=\frac{x^2}{9}-1 \Rightarrow y=\pm 2\sqrt{\frac{x^2}{9}-1}.$$

The “$\pm$” symbol can be represented in the GC by the keystrokes “**{1,-1}**”.

1) Press **[Y=]**.

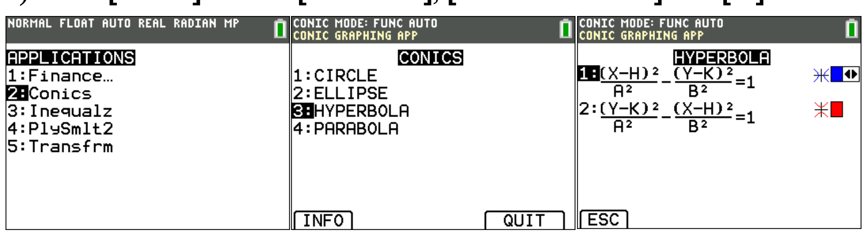
2) For **Y1**, type **[2ND] [(] [1] [,] [(-)] [1] [)] [2] [2ND] [x²] [X,T, θ ,n] [x²] [÷] [9] [−] [1]**. Press **[ENTER]**.

3) Press **[GRAPH]**.

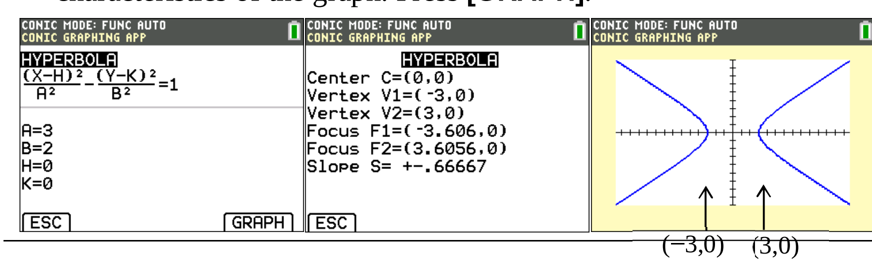


Method (b):

1) Press **[APPS]**. Select **[2:Conics]**, **[3:HYPERBOLA]** and **[1:]**.



2) Key in **A=3**, **B=2**, **H=0**, **K=0**. Press **[ALPHA] [ENTER]** for the characteristics of the graph. Press **[GRAPH]**.



3 Graphing Multiple Functions Using [ALPHA] Function

The **[ALPHA]** key allows you access to shortcut menus **F1-F4** via the top row of keys, from which you can find various useful functions.

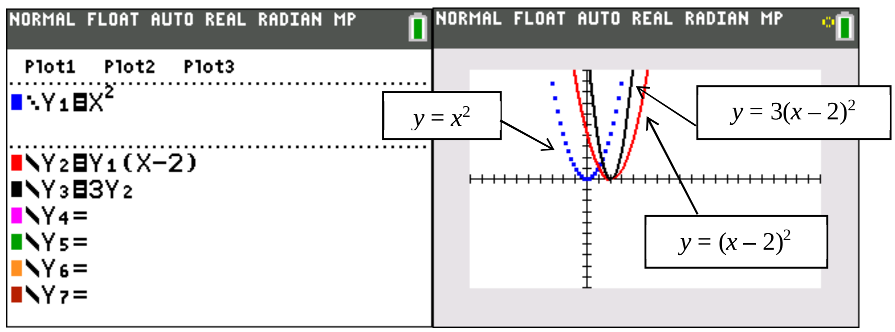
3.1 Transformation of Curves

Example 4

Sketch the graphs of $y = x^2$, $y = (x-2)^2$ and $y = 3(x-2)^2$.

Solution

1) Press **[Y=]**.
2) For **Y1**, type **[X,T, θ ,n] [x²]**. Press **[ENTER]**.
3) For **Y2**, press **[ALPHA] [TRACE]**, select **[Y1]**. Type **[(] [X,T, θ ,n] [−] [2] [)]**. Press **[ENTER]**.
4) For **Y3**, type **[3]**, press **[ALPHA] [TRACE]**, select **[Y2]**. Press **[ENTER]**.
5) Set graph style and change window settings.
6) Press **[GRAPH]**.

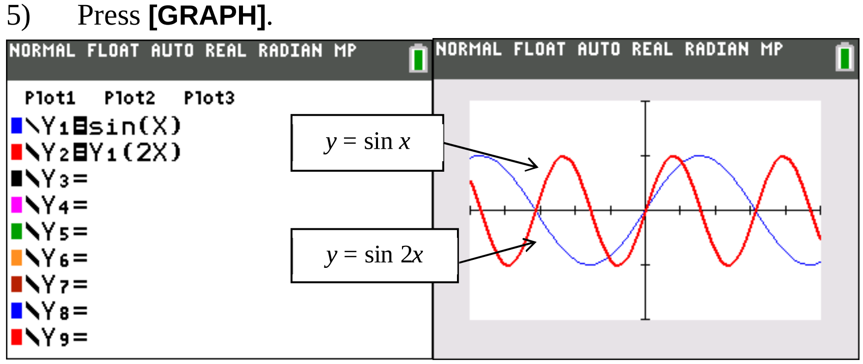


Example 5

Sketch the graphs of $y = \sin x$ and $y = \sin 2x$.

Solution

1) Press **[Y=]**.
2) For **Y1**, type **[SIN] [X,T, θ ,n] [)]**. Press **[ENTER]**.
3) For **Y2**, press **[ALPHA] [TRACE]**, select **[Y1]**. Type **[(] [2] [X,T, θ ,n] [)]**. Press **[ENTER]**.
4) Set graph style and change window settings.
5) Press **[GRAPH]**.



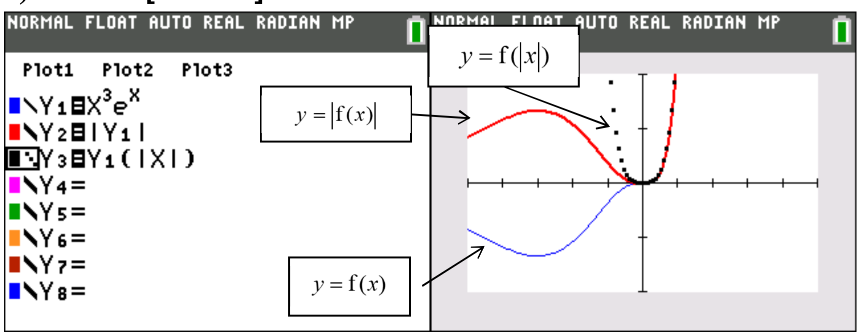
3.2 SKETCHING $y = |f(x)|$ AND $y = f(|x|)$

Example 6

Sketch the graphs of $y = f(x) = x^3 e^x$, $y = |f(x)|$ and $y = f(|x|)$.

Solution

1) Press **[Y=]**.
2) For **Y1**, type **[X,T, θ,n] [^] [3] [2ND] [LN] [X,T, θ,n]**. Press **[ENTER]**.
3) For **Y2**, press **[ALPHA] [WINDOW]**, select **[1:abs(]**. Press **[ALPHA] [TRACE]**, select **[Y1]**. Press **[ENTER]**.
4) For **Y3**, press **[ALPHA] [TRACE]**, select **[Y1]**. Type **[(]**, press **[ALPHA] [WINDOW]**, select **[1:abs(]**. Type **[X,T, θ,n] [)]**. Press **[ENTER]**.
5) Set graph style and change window settings.
6) Press **[GRAPH]**.

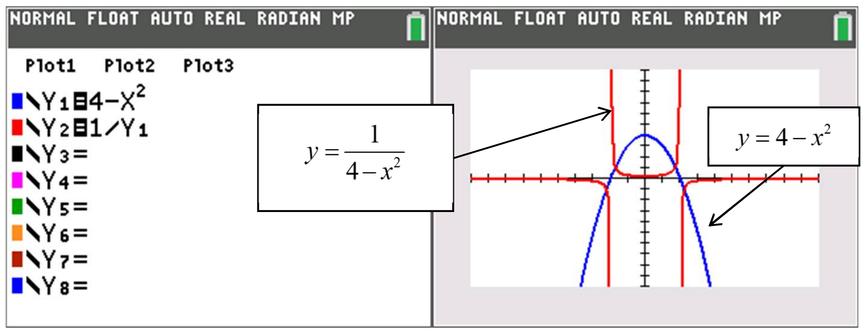


3.3 SKETCHING $y = \dfrac{1}{\mathrm{f}(x)}$

Example 7

Sketch the graphs of $y = 4 - x^2$ and $y = \dfrac{1}{4-x^2}$.

1) Press **[Y=]**.
2) For **Y1**, type **[4] [−] [X,T, θ ,n] [x²]**. Press **[ENTER]**.
3) For **Y2**, type **[1] [÷]**. Press **[ALPHA] [TRACE]**, select **[Y1]**.
4) Set graph style and change window settings.
5) Press **[GRAPH]**.



ANNEX C: PRACTICE QUESTIONS

Do not use the graphing calculator for the questions in this annex.

1 Sketch the graphs with the following equations.

(a) $y=3x^4$ (b) $y=-x^5$ (c) $y=(x-1)^2-2$

(d) $y=-x^2+4x$ (e) $y=\dfrac{1}{x^3}$ (f) $y=-\dfrac{4}{x^4}$

(g) $y=\ln x$ (h) $y=\mathrm{e}^x$ (i) $y^2=2x$

(j) $x^2+y^2=4$ (k) $4x^2+y^2=4$ (l) $\dfrac{(x-1)^2}{9}+(y+1)^2=1$

(m) $x^2-y^2=1$ (n) $y^2-x^2=1$ (o) $4x^2-y^2=4$

2 For each of the following graphs, write down the equations of any vertical, horizontal and oblique asymptotes.

(a) $y=\dfrac{1}{x-1}$ (b) $y=\dfrac{1}{3-2x}$ (c) $y=1+\dfrac{2}{x}$

(d) $y=-1-\dfrac{2}{2x-1}$ (e) $y=\dfrac{2x-3}{x+1}$ (f) $y=\dfrac{1-x}{2x+1}$

(g) $y=x+\dfrac{1}{x}$ (h) $y=\dfrac{x^2+1}{x+1}$ (i) $y=\dfrac{x^2+3x+1}{x+2}$

3 Sketch the graphs of $y=\mathrm{f}(x)$, $y=\left|\mathrm{f}(x)\right|$ and $y=\mathrm{f}(|x|)$ on separate diagrams for each of the following functions.

(a) $\mathrm{f}(x)=2-x$ (b) $\mathrm{f}(x)=1+\dfrac{2}{x}$

4 Given $\mathrm{f}(x)=\mathrm{e}^x$, sketch each of the following graphs separately.

(a) $y=\mathrm{f}(x-2)$ (b) $y=\mathrm{f}(x+2)$

(c) $y=\mathrm{f}(x)+2$ (d) $y=\mathrm{f}(x)-2$

(e) $y=-\mathrm{f}(x)$ (f) $y=\mathrm{f}(-x)$

5 (a) Find the coordinates of the point on the curve $x=3t^2$, $y=6t$

(i) when $t=3$, (ii) when $t=-1$.

(b) Find the coordinates of the point on the curve $x=1+\dfrac{1}{t}$, $y=1-\dfrac{1}{t}$

(i) when $t=3$, (ii) when $t=-1$.

Answers

2 (a) $x=1$, $y=0$ (b) $x=1.5$, $y=0$ (c) $x=0$, $y=1$

(d) $x=0.5$, $y=-1$ (e) $x=-1$, $y=2$ (f) $x=-0.5$, $y=-0.5$

(g) $x=0$, $y=x$ (h) $x=-1$, $y=x-1$ (i) $x=-2$, $y=x+1$

5 (a)(i) $(27,18)$ (ii) $(3,-6)$

(b)(i) $\left(\frac{4}{3},\frac{2}{3}\right)$ (ii) $(0,2)$