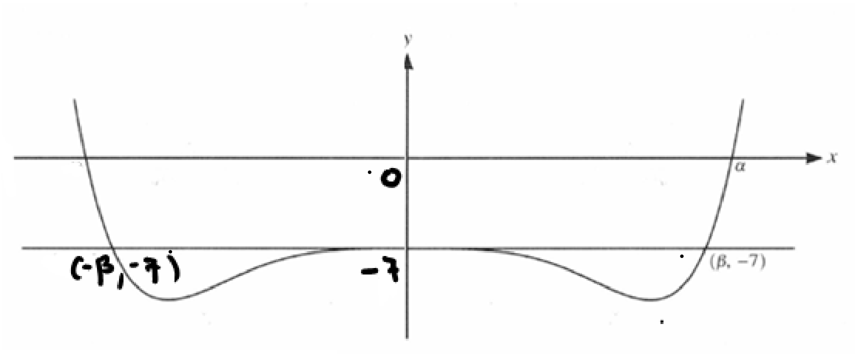
1.	i	$\mathrm{f}^2(x) = \mathrm{f}(\mathrm{f}(x))$ $= \mathrm{f}\left(\frac{1}{1-x}\right)$ $= \frac{1}{1-\left(\frac{1}{1-x}\right)}$ $= \frac{1-x}{-x}$ $= \frac{x-1}{x}$ $= 1-\frac{1}{x}$ $\therefore \mathrm{f}^2 : x \mapsto 1-\frac{1}{x}, x \in \mathbb{R}, x \neq 1, x \neq 0.$ (Since $\mathrm{D}_{\mathrm{f}^2} = \mathrm{D}_{\mathrm{f}}$) $\mathrm{R}_{\mathrm{f}} = \mathrm{D}_{\mathrm{f}^{-1}} = (-\infty, 0) \cup (0,1) \cup (1,\infty)$ Let $\mathrm{f}(x) = y$. $y = \frac{1}{1-x}$ $y(1-x) = 1$ $y - xy = 1$ $xy = y - 1$ $x = \frac{y-1}{y}$ $= 1-\frac{1}{y}$ $\therefore \mathrm{f}^{-1} : x \mapsto 1-\frac{1}{x}, x \in \mathbb{R}, x \neq 1, x \neq 0.$ Hence $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$.
	ii	Since $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$, $\mathrm{f}^3(x) = \mathrm{f}(\mathrm{f}^{-1}(x))$ $= x$

2		$C: x^2y + xy^2 + 54 = 0$ Differentiate w.r.t x $2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ When $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$, $2xy - x^2 + y^2 - 2xy = 0$ $x^2 = y^2$ $x = \pm y$ Substitute $x = y$ into C $y^3 + y^3 + 54 = 0$ $2y^3 = -54$ $y^3 = -27$ $y = -3$ Substitute $x = -y$ into C $y^3 - y^3 + 54 = 0$ $54 = 0$ (N.A) $\therefore$ Coordinates of the point at which the gradient is -1 is $(-3, -3)$ Hence there is only one such point.
3	i	**a** and **b** are parallel, or **a** or **b** is a zero vector.
	ii	Since $\mathbf{n} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \mathbf{0}$ $\mathbf{n}$ is parallel to the direction vector $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$. $\therefore \quad \hat{\mathbf{n}} = \frac{1}{\sqrt{1^2 + 2^2 + (-2)^2}}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ $= \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$
	iii	Let θ be the required angle. $\cos\theta = \frac{\left\| \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\|}{\left\|\sqrt{9}\right\|\left\|\sqrt{1}\right\|}$ $= \frac{2}{3}$

4		Out of syllabus
5	i	$z=1+2\mathrm{i}$ $z^2 = (1+2i)(1+2i)$ $=1+4i-4$ $=-3+4i$ $\frac{1}{z^3} = \frac{1}{(1+2\mathrm{i})^3}$ $=\left(\frac{1}{-11-2i}\right)\left(\frac{-11+2i}{-11+2i}\right)$ $=\frac{-11+2i}{121+4}$ $=\frac{-11+2i}{125}$ $=\frac{-11}{125}+\frac{2}{125}i$
	ii	From i $pz^2+\frac{q}{z^3} = p(-3+4i)+q(-\frac{11}{125}+\frac{2}{125}i)$ $=-3p-\frac{11}{125}q+(4p+\frac{2}{125}q)\mathrm{i}$ Since $pz^2+\frac{q}{z^3}$ is real, $(4p+\frac{2}{125}q)=0$ $q=-250p$ $\therefore\ pz^2+\frac{q}{z^3}=-3p-\frac{11}{125}(-250p)$ $=-3p+22p$ $=19p$

6	ai	Out of syllabus
	a ii	$\sum_{r=1}^{n} p_r = \sum_{r=1}^{n} \frac{1}{3}\left(7-4^r\right)$ $= \sum_{r=1}^{n} \frac{7}{3} - \frac{1}{3}\sum_{r=1}^{n} 4^r$ $= \frac{7}{3}n - \frac{4}{3}\left(\frac{4^n-1}{3}\right)$ $= \frac{7}{3}n + \frac{4}{9}\left(1-4^n\right)$
	b i	As n $\to \infty$, $S_n = \sum u_r$ approaches to 1, hence the series $\sum u_r$ converges. $S_\infty = 1$
	b ii	$u_n = S_n - S_{n-1}$ $= 1 - \frac{1}{(n+1)!} - \left(1 - \frac{1}{(n)!}\right)$ $= \frac{1}{(n)!} - \frac{1}{(n+1)!}$ $= \frac{n+1-1}{(n+1)!}$ $= \frac{n}{(n+1)!}$
7	i	$f(a) = 0$ $a^6 - 3a^4 - 7 = 0$ Using GC, $a = 1.88524$ ≈ 1.885 (to 3.d.p) $f(b) = -7$ $b^6 - 3b^4 - 7 = -7$ $b^6 - 3b^4 = 0$ $b^4(b^2 - 3) = 0$ $b = 0$ or $b^2 - 3 = 0$ $b = \pm\sqrt{3}$ Since $b > 0$, $b = \sqrt{3}$.

	ii	$\int_{\sqrt{3}}^{1.88524} \mathrm{f}(x)$ $= \int_{\sqrt{3}}^{1.88524} (x^6 - 3x^4 - 7)\ \mathrm{d}x$ $= -0.597272$ (Using GC) ≈ -0.597 (to 3.d.p)
	iii	$\int_0^{\sqrt{3}} (-7 - \mathrm{f}(x))\mathrm{d}x$ $= \int_0^{\sqrt{3}} -7 - (x^6 - 3x^4 - 7)\ \mathrm{d}x$ $= \int_0^{\sqrt{3}} -x^6 + 3x^4 \mathrm{d}x$ $= \left[-\frac{x^7}{7} + \frac{3x^5}{5}\right]_0^{\sqrt{3}}$ $= \left[-\frac{3^{\frac{7}{2}}}{7} + \frac{3(3^{\frac{5}{2}})}{5}\right] - 0$ $= \left[-\frac{3^{\frac{7}{2}}}{7} + \frac{3^{\frac{7}{2}}}{5}\right]$ $= 3^{\frac{7}{2}}\left(\frac{2}{35}\right)$ $= \left(\sqrt{3}\right)\left(3^3\right)\left(\frac{2}{35}\right)$ $= \frac{54}{35}\sqrt{3}$ units^2
	iv	$\mathrm{f}(-x) = (-x)^6 - 3(-x)^4 - 7$ $= x^6 - 3x^4 - 7$ $= \mathrm{f}(x)$ (shown)

		 Since f(x) = f(-x), the graph is symmetrical about the y-axis. From the graph above, it can be seen that the graph cuts the x-axis at 2 points; hence it has 2 real roots. Since the coefficients of the f(x) are all real, the equation f(x) = 0 has 2 real roots and 2 pairs of conjugate roots. In addition, since f(x) = f(-x), the roots form pairs within each which one root is the negative of the other. The four non-real roots are in the form $a+b\text{i}$, $-a-b\text{i}$, $a-b\text{i}$ and $-a+b\text{i}$.
8	i	$\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$ $=\sin^{-1}\left(\frac{x}{3}\right)+C,$ where C is an arbitrary constant
	ii	$(9-x^2)^{-\frac{1}{2}}$ $=9^{-\frac{1}{2}}\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}$ $=\frac{1}{3}\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x^2}{9}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x^2}{9}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x^2}{9}\right)^3+\dots\right)$ $=\frac{1}{3}\left(1+\frac{x^2}{18}+\frac{x^4}{216}+\frac{15x^6}{34992}+\dots\right)$ $\approx\frac{1}{3}+\frac{x^2}{54}+\frac{x^4}{648}+\frac{5x^6}{34992}$
	iii	From i,

		$\sin^{-1}\left(\dfrac{x}{3}\right) = \int \dfrac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$ $\approx \int \dfrac{1}{3}+\dfrac{x^2}{54}+\dfrac{x^4}{648}+\dfrac{5x^6}{34992}\,\mathrm{d}x$ $=\dfrac{1}{3}x+\dfrac{x^3}{162}+\dfrac{x^5}{3240}+\dfrac{5x^7}{244944}+c$ Sub x=0: LHS=0; RHS=c Thus, c=0 $\sin^{-1}\left(\dfrac{x}{3}\right)=\dfrac{x}{3}+\dfrac{x^3}{162}+\dfrac{x^5}{3240}+\dfrac{5x^7}{244944}$
9	i	$\prod_{\mathrm{p}}:\ \mathbf{r}\cdot\begin{pmatrix}1\\2\\-3\end{pmatrix}=12$ Since $\prod_{\mathrm{p}}$ and $\prod_q$ are perpendicular, the normal of $\prod_{\mathrm{p}}$ is parallel to $\prod_q$. Finding the vector equation of l: Let $\lambda=\dfrac{x-1}{2}=\dfrac{y+1}{-1}=\dfrac{z-3}{4}$, where $\lambda\in\mathbb{R}$ $\therefore x=1+2\lambda$ $y=-1-\lambda$ $z=3+4\lambda$ Hence $l:\mathbf{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\4\end{pmatrix},\lambda\in\mathbb{R}$ Since $\prod_q$ contains line *l* and the normal of $\prod_p$ is parallel to $\prod_q$,

		$n_q = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ $= \begin{pmatrix} -5 \\ 10 \\ 5 \end{pmatrix}$ $= 5\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\prod_q : \mathbf{r} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0$ $\therefore \prod_q : -x + 2y + z = 0$
	ii	Using GC to obtain the line m $m : \mathbf{r} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \omega \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \omega \in \mathbb{R}$
	iii	Since *B* is a general point on *m,*

$$\overrightarrow{OB} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \omega \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \text{ for some } \omega \in \mathbb{R}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 6+4\omega \\ 3+\omega \\ 2\omega \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5+4\omega \\ 4+\omega \\ 2\omega-3 \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(5+4\omega)^2 + (4+\omega)^2 + (2\omega-3)^2}$$

$$= \sqrt{25+40\omega+16\omega^2 + 16 + 8\omega + \omega^2 + 4\omega^2 - 12\omega + 9}$$

$$= \sqrt{50+36\omega+21\omega^2}$$

$$\left|\overrightarrow{AB}\right|^2 = 50+36\omega+21\omega^2$$

Hence method: Let $y = \left|\overrightarrow{AB}\right|$

Differentiate y with respect to ω

$$2y\frac{dy}{d\omega} = 36+42\omega$$

when $\frac{dy}{d\omega} = 0$,

$$36+42\omega = 0$$

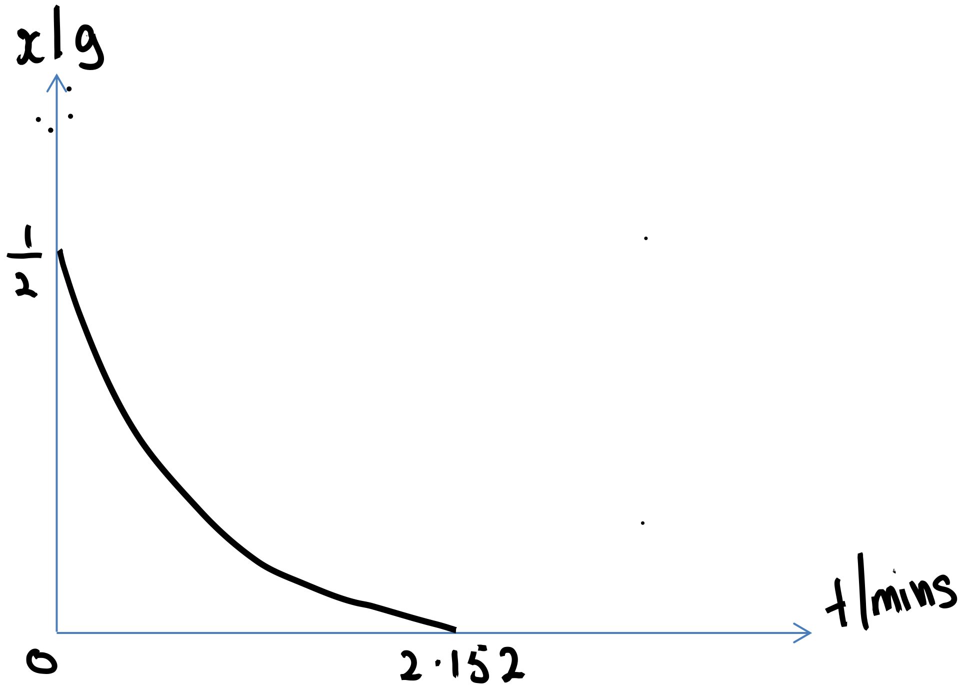
$$\omega = -\frac{6}{7}$$

Hence $\overrightarrow{OB} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \left(-\frac{6}{7}\right)\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 18 \\ 15 \\ -12 \end{pmatrix}$.

	iii	(Otherwise Method) $\overrightarrow{AB}$ must be perpendicular to the line m as it is the shortest distance between A and B. $\overrightarrow{AB}=\begin{pmatrix}5+4\omega\\4+\omega\\2\omega-3\end{pmatrix}$ (see above) $\therefore \overrightarrow{AB}\cdot m=0$ $\begin{pmatrix}5+4\omega\\4+\omega\\2\omega-3\end{pmatrix}\cdot\begin{pmatrix}4\\1\\2\end{pmatrix}=0$ $20+16\omega+4+w+4\omega-6=0$ $18+21\omega=0$ $\omega=-\frac{6}{7}$ Hence $\overrightarrow{OB}=\begin{pmatrix}6\\3\\0\end{pmatrix}+\left(-\frac{6}{7}\right)\begin{pmatrix}4\\1\\2\end{pmatrix}=\frac{1}{7}\begin{pmatrix}18\\15\\-12\end{pmatrix}$.
10	i	$\frac{\mathrm{d}x}{\mathrm{d}t}=k(1+x-x^2)$ When $x=\frac{1}{2}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{4}$, $-\frac{1}{4}=k\left(1+\frac{1}{2}-\frac{1}{4}\right)$ $k=-\frac{1}{5}$ (shown)

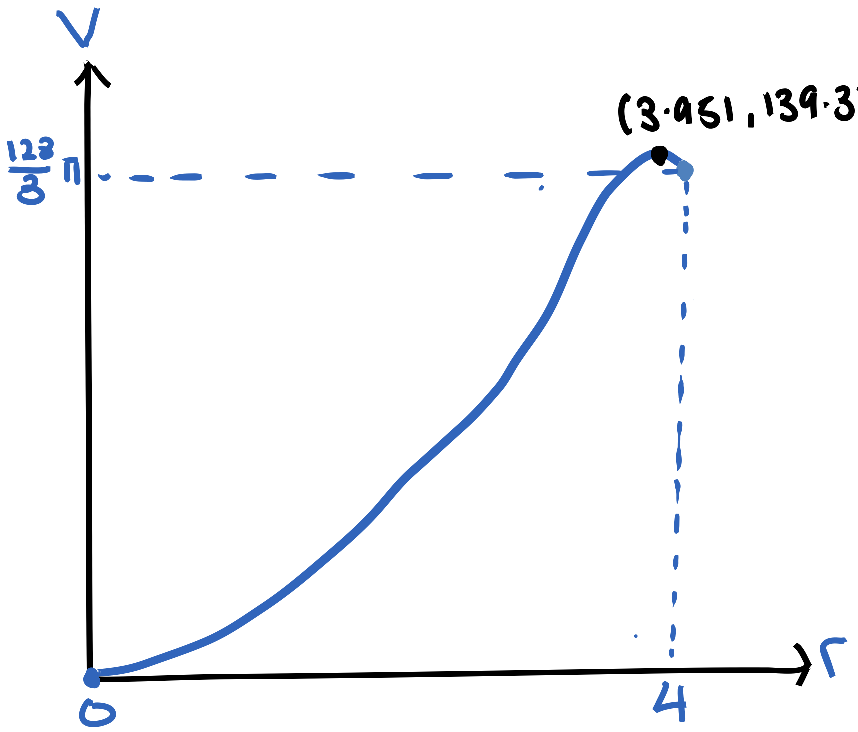
	ii	$\frac{\mathrm{d}x}{\mathrm{dt}} = -\frac{1}{5}(1+x-x^2)$ $= -\frac{1}{5}\left(\frac{5}{4}-\left(x-\frac{1}{2}\right)^2\right)$ $\int -\frac{1}{5}\mathrm{dt} = \int \frac{\mathrm{d}x}{\left(\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2\right)}$ $-\frac{1}{5}t = \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\ln\left\|\frac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}\right\| + \mathrm{C}$, where C is an arbitrary constant $\frac{1}{5}t = -\frac{1}{\sqrt{5}}\ln\left(\frac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}\right)+\mathrm{C}$ (since $0 \le x \le \frac{1}{2}$, $\frac{\sqrt{5}}{2}+x-\frac{1}{2} > 0$ and $\frac{\sqrt{5}}{2}-x+\frac{1}{2} > 0$, so $\frac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}} > 0$) when $t=0$, $x=\frac{1}{2}$, $0 = -\frac{1}{\sqrt{5}}\ln(1)+C$ $C = 0$ $\therefore \frac{1}{5}t = -\frac{1}{\sqrt{5}}\ln\left(\frac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}\right)$ $t = -\sqrt{5}\ln\left(\frac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}\right)$ $= \sqrt{5}\ln\left(\frac{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}\right)$

	iii a	When $x = \frac{1}{4}$, $t = -\sqrt{5}\ln\left(\frac{\frac{\sqrt{5}}{2}+\frac{1}{4}-\frac{1}{2}}{\frac{\sqrt{5}}{2}-\frac{1}{4}+\frac{1}{2}}\right)$ $= -\sqrt{5}\ln\left(\frac{\frac{\sqrt{5}}{2}-\frac{1}{4}}{\frac{\sqrt{5}}{2}+\frac{1}{4}}\right)$ $= \sqrt{5}\ln\left(\frac{2\sqrt{5}+1}{2\sqrt{5}-1}\right)$
	iii b	When $x = 0$, $t = -\sqrt{5}\ln\left(\frac{\frac{\sqrt{5}}{2}+0-\frac{1}{2}}{\frac{\sqrt{5}}{2}-0+\frac{1}{2}}\right)$ $= -\sqrt{5}\ln\left(\frac{\frac{\sqrt{5}}{2}-\frac{1}{2}}{\frac{\sqrt{5}}{2}+\frac{1}{2}}\right)$ $= -\sqrt{5}\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)$ ≈ 2.152 (to 3.d.p)

	iv	$t=-\sqrt{5}\ln\left(\dfrac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}\right)$ $\dfrac{1}{\sqrt{5}}t=\ln\left(\dfrac{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}\right)$ $e^{\frac{1}{\sqrt{5}}t}=\dfrac{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}$ $=\dfrac{\sqrt{5}-2x+1}{\sqrt{5}+2x-1}$ $\sqrt{5}e^{\frac{1}{\sqrt{5}}t}+2e^{\frac{1}{\sqrt{5}}t}x-e^{\frac{1}{\sqrt{5}}t}=\sqrt{5}-2x+1$ $x(2e^{\frac{1}{\sqrt{5}}t}+2)=\sqrt{5}+1+e^{\frac{1}{\sqrt{5}}t}-\sqrt{5}e^{\frac{1}{\sqrt{5}}t}$ $x=\dfrac{\sqrt{5}+1+e^{\frac{1}{\sqrt{5}}t}-\sqrt{5}e^{\frac{1}{\sqrt{5}}t}}{2e^{\frac{1}{\sqrt{5}}t}+2}$ 

11	i	$h^2 = 4^2 - r^2$ $h = \sqrt{4^2 - r^2}$ (reject $h = -\sqrt{4^2 - r^2}$ as $h > 0$) $V = \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$ Sub $h = \sqrt{4^2 - r^2}$ into V i.e. $V = \frac{1}{3}\pi r^2 \sqrt{4^2 - r^2} + \frac{2}{3}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{1}{3}\pi\left(2r\sqrt{4^2 - r^2} + r^2\left(\frac{1}{2}\left(4^2 - r^2\right)^{-\frac{1}{2}}(-2r)\right)\right) + 2\pi r^2$ when $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$ $0 = \frac{1}{3}\pi\left(2r\sqrt{4^2 - r^2} + r^2\left(\frac{1}{2}\left(4^2 - r^2\right)^{-\frac{1}{2}}(-2r)\right)\right) + 2\pi r^2$ $0 = \frac{1}{3}\pi r\left(2r\sqrt{4^2 - r^2} + r^2\left(\frac{1}{2}\left(4^2 - r^2\right)^{-\frac{1}{2}}(-2r)\right) + 6r^2\right)$ $0 = 2\sqrt{4^2 - r^2} - r^2\left(\left(4^2 - r^2\right)^{-\frac{1}{2}}\right) + 6r$ $r^2\left(\left(4^2 - r^2\right)^{-\frac{1}{2}}\right) = 2\sqrt{4^2 - r^2} + 6r$ $r^2 = 2(4^2 - r^2) + 6r\sqrt{4^2 - r^2}$ $3r^2 - 32 = 6r\sqrt{4^2 - r^2}$ (1) $(3r^2 - 32)^2 = 36r^2(4^2 - r^2)$ $9r^4 - 192r^2 + 1024 = 574r^2 - 36r^4$ $45r^4 - 768r^2 + 1024 = 0$ (shown)
	ii	$45r^4 - 768r^2 + 1024 = 0$ Using GC, r =1.20742168023789 or 3.95079733126849 $r \approx 1.207$ or 3.951 (to 3.d.p)

<table>
<tr><td></td><td>iii</td><td>Using GC,
$\left.\frac{\mathrm{d}V}{\mathrm{d}r}\right|_{r=3.95079733126849} \approx 0$

$\left.\frac{\mathrm{d}V}{\mathrm{d}r}\right|_{r=1.20742168023789} \approx 18.32 \neq 0$
(Notice that from (1), $r = 1.20742$ will give the
LHS a negative value and the RHS a positive value.
So $r = 1.20742$ cannot be the correct solution)
Hence, $r_1 = 3.951$
$\therefore h^2 = 16 - 3.950797^2$
$h^2 = 0.391203$
$h \approx 0.625$ (to 3s.f) (reject the negative answer since $h > 0$)</td></tr>
<tr><td></td><td>iv</td><td>Sketching the graph of $V = \frac{1}{3}\pi r^2\sqrt{4^2 - r^2} + \frac{2}{3}\pi r^3$.

</td></tr>
</table>