2022 RVHS_ACJC_EJC_NJC Prelim Paper 2_Modified [82 marks]

Section A: Pure Mathematics [44 marks]

1. Mathematical Induction Question removed

2. Let $\mathbf{M}_{2\times 2}(\mathbb{R})$ denote the set of all 2×2 matrices with real number entries. Explain whether the following sets are subspaces of $\mathbf{M}_{2\times 2}(\mathbb{R})$ under the usual operations of addition and scalar multiplication defined on $\mathbf{M}_{2\times 2}(\mathbb{R})$.

(a) The set of all 2×2 matrices $\mathbf{A}$ such that $\det(\mathbf{A})\neq 0$.

(b) The set of all 2×2 matrices $\mathbf{B}$ such that $\mathbf{B}^T=-\mathbf{B}$. [7]

3. Use the substitution $x=\sec\theta$, where $0<\theta<\frac{\pi}{2}$, to show that the differential equation

$$\left(x^3-x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(2x^2-1\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{ky}{x}=\frac{2}{x^3},$$

where k is a positive integer, can be reduced to

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}+ky=2\cos^2\theta.$$

Hence, obtain the general solution for the differential equation in x and y for $k\neq 4$ in the form

$$y=A\cos\sqrt{k}\left(\sec^{-1}x\right)+B\sin\sqrt{k}\left(\sec^{-1}x\right)+\mathrm{f}(x),$$

where A and B are arbitrary constants and $\mathrm{f}(x)$ is a function of x to be determined.

[11]

4. A surface has the equation $\mathrm{f}(x,y)=\sqrt{x^2+y^2}, x\neq 0, y\neq 0$.

(a) Find the directional derivative in the direction of greatest ascent at the point where $x=1, y=2$. [3]

(b) Find the tangent plane at the point where $x=x_0, y=y_0$. Hence, show that all tangent planes to the surface pass through the origin. [3]

(c) Show algebraically that the normal at the point where $x=x_0, y=y_0$ does not intersect the surface again. [5]

5. **(i)** Find the roots of the equation $z^7 + 128 = 0$ exactly in the form $re^{i\theta}$, where $r > 0$ and $0 \leq \theta < 2\pi$. [3]

(ii) Represent these roots in a single Argand diagram, clearly illustrating any geometrical relationships between their moduli and arguments. [3]

(iii) Use the result in part **(i)** to find the exact value of

$$\sin^2\left(\frac{\pi}{14}\right) + \sin^2\left(\frac{3\pi}{14}\right) + \sin^2\left(\frac{5\pi}{14}\right),$$

showing your working clearly. [3]

(iv) In the same diagram in part **(ii)**, sketch the loci $\arg(z+2) = \frac{\pi}{14}$ and $\arg(z-2) = \frac{4\pi}{7}$. Find the area of the region bounded by these two loci and the real axis. [4]

(v) Let z_1, z_2, z_3 and z_4 be any 4 distinct roots of the equation in part **(i)** such that $0 \leq \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) < 2\pi$. Explain why

$$\arg\left(\frac{(z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}\right) = \pi.$$ [2]

Section B: Statistics [38 marks]

6. An international eSports website conducted a survey among its global pool of amateur gamers and a random sample of 600 responses on their preference for gaming keyboard colours were collected. The responses are summarised in the table below.

	White	**Green**	**Black**	**Blue**	**Red**	**Others**
Male	90	75	77	53	71	34
Female	48	38	21	27	55	11

(i) Carry out a chi-squared test, using a 5% significance level, to investigate whether the data give any reason to suppose that preferences for colours of gaming keyboards differ between male and female gamers. [5]

(ii) Give the contributions to the test statistic, and hence identify the significant differences in the colour preferences of gamers. [3]

7. An educational technologist is working for a tuition centre to develop a digital learning module that improves essay writing skills. The technologist administers a writing test before each student embarks on the module. He administers a similar writing test after the students have completed the module. He wishes to investigate whether or not the module is effective. The test results for a random sample of 12 of these students are shown in the table.

Student	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***	***I***	***J***	***K***	***L***
Before module	71	70	65	42	74	81	56	69	57	77	81	55
After module	51	72	46	93	95	31	81	84	88	67	70	82

(i) Explain whether it would be appropriate, in this case, to carry out a t-test on the data. [2]

(ii) Carry out a Wilcoxon matched-pairs signed rank test for these data, using a 1% significance level. State any assumption(s) required for the Wilcoxon test to be valid. [6]

8. The random variable X has probability density function $\mathrm{f}(x)$ defined as follows, where k is a real constant.

$$\mathrm{f}(x)=\begin{cases}\dfrac{k}{x^2-x+1} & \text{for } 0\le x\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

(i) Find the exact value of k. [3]

(ii) State the mean, median and mode of X. [1]

(iii) Obtain the cumulative distribution function for X. [2]

The random variable Y is such that $Y=50\tan^{-1}X$.

(iv) Find $\mathrm{E}(Y)$. [2]

(v) Evaluate $\mathrm{P}(Y>25)$. [2]

9. Removed 2 Sample t-test Question

10. MSW is a restaurant that provides both dine-in and food delivery services. Based on past data collected, the average number of customers arriving at the restaurant in a 10-minute interval is 2.

(i) State the assumptions needed for the number of customers arriving at the restaurant in a 10-minute interval to be well modelled by a Poisson distribution. [2]

For the rest of the question, it is given that these assumptions hold. It is also known that the number of food delivery orders that MSW receives in a 10-minute interval can be modelled by a Poisson distribution with mean 5.

(ii) Stating a necessary assumption, calculate the probability that in a given 10-minute interval, the total number of customers arriving at the restaurnt and food delivery orders received is at most 10. [3]

A 10-minute interval is considered busy if the total number of customers arriving and food delivery orders received is more than 10. The manager of the restaurant monitors the situation for successive 10-minute intervals from the start of their work day. Let B be the number of such intervals observed up to and including the first busy interval.

(iii) State the conditions needed for B to be well modelled by a geometric distribution. Explain why these conditions are met in the context provided. [3]

(iv) Find the value of $\mathrm{E}(B)$ and $\mathrm{Var}(B)$. [2]

(v) Let m be an integer such that there is a probability of less than 0.25 that more than m intervals are observed until the first busy interval.
Determine the least value of m. [2]

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