STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

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EXAMINATION CODE

8	8	2	4	-	7	1	1	1

ST JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2024

MATHEMATICS: ANALYSIS AND APPROACHES **12 August 2024**

HIGHER LEVEL

2 hours

PAPER 1

0900 – 1100 hrs

Monday

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and your teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided.
- Unless otherwise stated in the question, all numerical answers should to be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.
- This question paper consists of **11** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	TOTAL
											/ 110

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (55 marks)

Answer **all** questions in the spaces provided.

1 **[Maximum mark: 5]**

The functions *f* and *g* are defined by $f\left(x\right)=\dfrac{x+1}{3}$, $x\geq -1$ and $g\left(x\right)=18x^2-5$, $x\geq 0$.

(a) Show that $\left(g\circ f\right)\left(x\right)=2x^2+4x-3$. [2]

(b) Given that $\left(g\circ f\right)^{-1}\left(a\right)=2$, find the value of a. [3]

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2 **[Maximum mark: 5]**

(a) Expand $\dfrac{(1-x)^6}{\sqrt{1+4x}}$ in ascending powers of x, up to and including the term in x^3. [4]

(b) State the range of values of x for which the expansion in part (a) is valid. [1]

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3 **[Maximum mark: 6]**

Events A and B are such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B') = 0.4$ and $\mathrm{P}(A' \cap B') = 0.1$.

Find $\mathrm{P}(A \mid B)$.

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4 [Maximum mark: 7]

(a) Show that $\log_4(1+\sin 2x) = \log_2\sqrt{1+\sin 2x}$. [2]

(b) Hence, or otherwise, solve $\log_4(1+\sin 2x) = \log_2\left(\sqrt{2}\cos x\right)$ for $0 < x < \frac{\pi}{2}$. [5]

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5 [Maximum mark: 7]

Prove by mathematical induction that $(2n)! \geq 2^n (n!)^2$ for all natural numbers n.

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6 [Maximum mark: 8]

The graph of $y = f(x)$ cuts the axes at the points (0.5 , 0) and (0 , ln 2).

Given that $f'(x) = \dfrac{a(2x-1)}{2x^2 - 2x + 1}$ where a is a constant, find $f(x)$.

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7 **[Maximum mark: 8]**

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \dfrac{\pi}{6}\sin\left(\dfrac{\pi}{6}x\right) & , \quad 0 \le x \le 3 \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Show that its mean is less than its median.

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8 **[Maximum mark: 9]**

The function f is defined by $f(x)=\frac{1}{4}x^2-x+3$, $x\geq 2$.

(a) Find the x-coordinates of the points where the graphs of $y=f(x)$ and $y=f^{-1}(x)$ meet. [4]

(b) Find the area of the region enclosed by the graphs of $y=f(x)$ and $y=f^{-1}(x)$. [5]

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*Do **NOT** write solutions on this page.*

SECTION B (55 marks)

*Answer **all** questions on the writing paper provided. **Please start each question on a new page.***

9 [Maximum mark: 18]

(a) (i) Find the value of A and of B such that $\dfrac{x}{(2-x)^2}=\dfrac{A}{(2-x)^2}+\dfrac{B}{2-x}$.

(ii) Hence find $\displaystyle\int \frac{x}{(2-x)^2}\,\mathrm{d}x$ for $x<2$. [4]

The tangent at the point (x, y) on the curve with equation $y=f(x)$ passes through the point $(2, x)$.

(b) Show that $(2-x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=x-y$. [1]

(c) Find an integrating factor for the differential equation in part (b). [3]

It is given that the curve $y=f(x)$ passes through the origin and $x<2$.

(d) Show that $f(x)=(2-x)\ln\left(\dfrac{2-x}{2}\right)+x$. [5]

(e) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{1}{2-x}$ and hence find the Maclaurin's series for y, up to including the term in x^3. [5]

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*Do **NOT** write solutions on this page.*

10 [Maximum mark: 17]

(a) Consider an arithmetic sequence with 7 terms $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ where $x_k \in \mathbb{R}$, $1 \le k \le 7$. Let the common difference be *d*.

(i) Express *d* in terms of x_1 and x_7. Hence find an expression of x_3 in terms of x_1 and x_7.

(ii) Show that $x_2 + x_4 + x_6 = \frac{3}{2}(x_1 + x_7)$.

(iii) Given that the terms x_1, x_3, x_7 form a geometric sequence and $x_1 \ne x_7$, express x_7 in terms of x_1. [12]

(b) Consider a geometric sequence of 3 terms y_1, y_2, y_3 with common ratio *r* and $y_1 > 0$.

(i) Find $y_1 + y_2 + y_3$ in terms of y_1 and *r*.

(ii) Determine the minimum value of $y_1 + y_2 + y_3$ as *r* varies and state the corresponding value of *r*. [5]

11 [Maximum mark: 20]

Consider the three planes

$$\Pi_1 : 2x - 5y + 3z = 3$$
$$\Pi_2 : 3x + 2y - 5z = -5$$
$$\Pi_3 : x + \lambda y + z = \mu$$

where λ and μ are constants. The planes Π_1 and Π_2 intersect in a line *L*.

(a) (i) Verify that the point $\mathrm{P}(-1, -1, 0)$ lies on both Π_1 and Π_2.

(ii) Find a vector equation of line *L*.

(iii) When $\lambda = -1$, line *L* makes an angle θ with plane Π_3. Determine the value of $\sin\theta$. [8]

(b) Given that all three planes meet in the line *L*, find the value of λ and of μ. [4]

(c) (i) Suppose that the three planes have no point in common, what can be said about the values of λ and μ?

(ii) Let $\mu = 2$ and $\lambda = -2$. Find the distance between line *L* and plane Π_3. [8]

End of Paper