

2025 H3 Math Prelim Mark Scheme

	Solution
1	Let $P(n)$ be the statement $\frac{x_1 + x_2 + \dots + x_n}{n} \geq (x_1 x_2 \dots x_n)^{\frac{1}{n}}$ for $n \in \{2, 2^2, 2^3, \dots\}$. To prove $P(2)$ is true Since $(x_1 - x_2)^2 \geq 0$ $x_1^2 + 2x_1x_2 + x_2^2 \geq 4x_1x_2$ $(x_1 + x_2)^2 \geq 4x_1x_2$ $\frac{x_1 + x_2}{2} \geq \sqrt{x_1x_2}$ Therefore $P(2)$ is true Assume $P(2^m)$ is true for some $m \in \mathbb{N}$, i.e $\frac{x_1 + x_2 + \dots + x_{2^m}}{2^m} \geq (x_1 x_2 \dots x_{2^m})^{\frac{1}{2^m}} \text{ -----(1)}$ To prove $P(2^{m+1})$ is true, i.e. $\frac{x_1 + x_2 + \dots + x_{2^{m+1}}}{2^{m+1}} \geq (x_1 x_2 \dots x_{2^{m+1}})^{\frac{1}{2^{m+1}}} \text{ -----(2)}$ LHS = $\frac{x_1 + x_2 + \dots + x_{2^{m+1}}}{2^{m+1}}$ $= \frac{1}{2} \left[\frac{x_1 + x_2 + \dots + x_{2^m}}{2^m} + \frac{x_{2^m+1} + x_{2^m+2} + \dots + x_{2^{m+1}}}{2^m} \right]$ $\geq \sqrt{\left(\frac{x_1 + x_2 + \dots + x_{2^m}}{2^m} \right) \times \left(\frac{x_{2^m+1} + x_{2^m+2} + \dots + x_{2^{m+1}}}{2^m} \right)} \text{ ---(3)}$ $\geq \sqrt{\left((x_1 x_2 \dots x_{2^m})^{\frac{1}{2^m}} \right) \times \left((x_{2^m+1} x_{2^m+2} \dots x_{2^{m+1}})^{\frac{1}{2^m}} \right)} \text{ -----(4)}$ $= (x_1 x_2 \dots x_{2^m} x_{2^m+1} x_{2^m+2} \dots x_{2^{m+1}})^{\frac{1}{2^{m+1}}}$ Hence $P(2^{m+1})$ is true, by principle of mathematical induction, $P(n)$ is true for all $n \in \{2, 2^2, 2^3, \dots\}$.

	Solution
2(a)	$\sum_{i=1}^n S_i $ $= \sum_{i=1}^n \text{no. of permutations where the } i\text{th object is in the } i\text{th position}$ $= \sum_{i=1}^n (n-1)!$ $= n(n-1)!$ $= n!$ $\sum_{i < j} S_i \cap S_j $ $= \sum_{i < j} \text{no. of permutations where the } i\text{th \& } j\text{th object are in their original positions}$ $= \sum_{i < j} (n-2)!$ $= \binom{n}{2} (n-2)!$ $= \frac{n!}{2!(n-2)!} (n-2)!$ $= \frac{n!}{2!}$
2(b)	$D_n = \overline{S_1} \cap \overline{S_2} \cap \dots \cap \overline{S_n} $ $= n! - S_1 \cup S_2 \cup \dots \cup S_n $ $= n! - \sum_{i=1}^n S_i + \sum_{i < j} S_i \cap S_j - \sum_{i < j < k} S_i \cap S_j \cap S_k +$ $\dots + (-1)^n S_1 \cap \dots \cap S_n $ $= n! - n! + \frac{n!}{2} - \binom{n}{3} (n-3)! + \dots + (-1)^n 1$ $= n! - n! + \frac{n!}{2} - \frac{n!}{3!(n-3)!} (n-3)! + \dots + (-1)^n$ $= n! - n! + \frac{n!}{2!} - \frac{n!}{3!} + \dots + (-1)^n \frac{n!}{n!}$ $= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right)$

	Solution
2(c)	$\lim_{n \rightarrow \infty} \frac{D_n}{n!} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!} \right)$ $= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$ $= e^{-1}$

	Solution
3(a)	$1 + \frac{1}{(1+ki)^2}$ $= \frac{(1+ki)^2 + 1}{(1+ki)^2}$ $= \frac{(1+ki)^2 - i^2}{(1+ki)^2}$ $= \frac{[1+(k-1)i][1+(k+1)i]}{(1+ki)^2}$ $f(n) = 2 \left[1 + \frac{1}{(1+i)^2} \right] \left[1 + \frac{1}{(1+2i)^2} \right] \dots \left[1 + \frac{1}{(1+(n-1)i)^2} \right]$ $= 2 \times \frac{(1)(1+2i)}{(1+i)} \times \frac{(1+i)(1+3i)}{(1+2i)^2} \dots$ $\frac{[1+(n-3)i][1+(n-1)i][1+(n-2)i][1+ni][1+(n-1)i][1+(n+1)i]}{(1+n-2i)^2 (1+(n-1)i)^2 (1+ni)}$ $= \frac{2}{1+i} \left(\frac{1+(n+1)i}{1+ni} \right)$ $= \frac{2+2(n+1)i}{(1-n)+(n+1)i}$ $= \frac{[2+2(n+1)i][(1-n)-(n+1)i]}{(1-n)^2 + (n+1)^2}$ $= \frac{(n^2+n+2)}{n^2+1} + \frac{(-n^2-n)}{n^2+1} i$

	Solution
3(b)	$\tan \left[\sum_{k=0}^n \arg \left(1 + \frac{1}{(1+ki)^2} \right) \right] = -\frac{36}{37}$ $\Rightarrow \tan [\arg (f(n))] = -\frac{36}{37}$ $\Rightarrow \frac{-n^2 - n}{n^2 + n + 2} = -\frac{36}{37}$ $\Rightarrow n = 8$

	Solution
4(a)	Let $a_1 = \frac{x}{\sqrt{x+y}}, a_2 = \frac{y}{\sqrt{y+z}}, a_3 = \frac{z}{\sqrt{z+w}}, a_4 = \frac{w}{\sqrt{w+x}}$ and $b_1 = \sqrt{x+y}, b_2 = \sqrt{y+z}, b_3 = \sqrt{z+w}, b_4 = \sqrt{w+x}$ Applying Cauchy Schwarz inequality, $(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) \geq (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)^2$ Hence $\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \right) (2w+2x+2y+2z) \geq (w+x+y+z)^2 \text{ --(1)}$ $\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \geq \frac{w+x+y+z}{2}$
4(b)	Using Arithmetic-Geometric inequality, $\frac{w+x+y+z}{4} \geq (wxyz)^{\frac{1}{4}} \Rightarrow \frac{w+x+y+z}{2} \geq 2(wxyz)^{\frac{1}{4}}$ Hence $\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \geq 2(wxyz)^{\frac{1}{4}} \text{ -- (2)}$
4(c)	$wxyz = 81$ From (2), $\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \geq 6$

	Solution
	However, since for Arithmetic-Geometric Progression, equality only holds if $w = x = y = z = 3$. However since $wx = 27, yz = 3$, the equality condition is not met. $\therefore \frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} > 6$

	Solution
5(a)	G1 Correct $y = 2$ or $y = -1$ with endpoints G1 Correct $y = 1$ with endpoints G1 Correct $y = 0$ with endpoints G1 Correct points $\left(\frac{3\pi}{4}, -2\right)$ and $\left(\frac{7\pi}{4}, -2\right)$ indicated.
5(b)	For $2 \sin(nx)$, period = $\frac{2\pi}{n}$. From graph, $\therefore \int_0^{2\pi} \lceil 2 \sin(nx) \rceil dx = n \int_0^{\frac{2\pi}{n}} \lceil 2 \sin(nx) \rceil dx$

	Solution
	$\int_0^{2\pi} \lceil 2 \sin(nx) \rceil dx$ $= n \int_0^{\frac{2\pi}{n}} \lceil 2 \sin(nx) \rceil dx$ $= n \left\{ \left[\left(\frac{\pi}{6n} \right) + 2 \left(\frac{7\pi}{6n} - \frac{5\pi}{6n} \right) + \left(\frac{\pi}{6n} \right) \right] - \left(\frac{11\pi}{6n} - \frac{7\pi}{6n} \right) \right\}$ $= n \left\{ \frac{\pi}{n} \right\}$ $= \pi \quad (k=1)$
5(c)	From graph in (a), $\int_0^{2\pi} \lceil n \sin(nx) \rceil dx = \left(\frac{\pi}{n} \right) n$ $= \pi$

	Solution
6(a)(i)	Given $2f(x) - f\left(\frac{1}{x}\right) = 3x$ ————— (1) Replace x with $\frac{1}{x}$ in equation (1) $2f\left(\frac{1}{x}\right) - f(x) = \frac{3}{x} \quad \text{————— (2)}$
6(a)(ii)	$2 \times (1) + (2)$: $3f(x) = 6x + \frac{3}{x}$ $f(x) = 2x + \frac{1}{x}, \quad x \neq 0$
6(b)	$g(x) + g(-x) - g\left(\frac{1}{x}\right) = x$ ————— (1) Replace x with $-x$: $g(-x) + g(x) - g\left(-\frac{1}{x}\right) = -x \quad \text{————— (2)}$ (1) – (2): $g\left(-\frac{1}{x}\right) - g\left(\frac{1}{x}\right) = 2x$

	Solution
	Replace x with $\frac{1}{x}$: $g(-x) - g(x) = \frac{2}{x} \text{ ————— (3)}$ Taking (3) – (1): $2g(x) - g\left(\frac{1}{x}\right) = x - \frac{2}{x} \text{ ————— (4)}$ Replace x with $\frac{1}{x}$: $2g\left(\frac{1}{x}\right) - g(x) = \frac{1}{x} - 2x \text{ ————— (5)}$ Taking $2 \times (4) + (5)$: $3g(x) = -\frac{3}{x}$ $g(x) = -\frac{1}{x}$

	Solution
7(a)	$x^5 + 1 = (x + 1)(x^4 - x^3 + x^2 - x + 1)$ Assume that m has an odd divisor $2k + 1 > 1$. Let $m = (2k + 1)r$ for some positive integer r. Then $2^m + 1 = 2^{(2k+1)r} + 1$ $= (2^r)^{2k+1} + 1$ $= (2^r + 1)[2^{2kr} - 2^{(2k-1)r} + \dots + 2^{2r} - 2^r + 1]$ This contradicts the primality of $2^m + 1$. Hence m cannot have any odd divisor. So m must be a power of 2, i.e. $m = 2^k$, where $k \in \mathbb{Z}$.
7(b)	If p divides $2^{2^n} + 1$, then $2^{2^n} + 1 \equiv 0 \pmod{p}$ $2^{2^n} \equiv -1 \pmod{p}$ Squaring both sides, $2^{2^{n+1}} \equiv 1 \pmod{p}$ Let d be the smallest positive integer such that $2^d \equiv 1 \pmod{p}$.

	Solution
	So d divides 2^{n+1}, but does not divide 2^n (since $2^{2^n} \equiv -1 \pmod{p}$). So $d = 2^{n+1}$. By Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod{p}$ Since d divides $p-1$, so $p-1 = k2^{n+1}$ for some integer k. Hence $p = k2^{n+1} + 1$, where $k \in \mathbb{Z}$.
7(c)	Let p and q be prime factors of $F_n = 2^{2^n} + 1$. By (ii), $p = r2^{n+1} + 1$ and $q = s2^{n+1} + 1$ for some $r, s \in \mathbb{Z}$. Since $(r2^{n+1} + 1)(s2^{n+1} + 1)$ $= (rs2^{n+1} + r + s)2^{n+1} + 1$ is also of the form $k2^{n+1} + 1$. Hence any factor of $F_n = 2^{2^n} + 1$, whether prime or composite, must be of the form $k2^{n+1} + 1$, where $k \in \mathbb{Z}$.
7(d)	$641 = 5^4 + 2^4$ $\Rightarrow 5^4 + 2^4 \equiv 0 \pmod{641}$ $\Rightarrow 5^4 2^{28} + 2^{32} = 2^{28}(5^4 + 2^4) \equiv 0 \pmod{641}$ $641 = 5(2^7) + 1$ $\Rightarrow 5(2^7) + 1 \equiv 0 \pmod{641}$ $\Rightarrow 5^4 2^{28} - 1$ $= [5^2 2^{14} + 1] [5^2 2^{14} - 1]$ $= [5^2 2^{14} + 1] [5(2^7) + 1] [5(2^7) - 1] \equiv 0 \pmod{641}$ Subtracting, we get $2^{32} + 1 \equiv 0 \pmod{641}$ That is, 641 divides $F_5 = 2^{32} + 1$.

	Solution
8(a)	$D_q(x^n) = \frac{(qx)^n - x^n}{(q-1)x}$ $= \frac{(q^n - 1)x^n}{(q-1)x}$ $= \frac{(q-1)(q^{n-1} + q^{n-2} + \dots + 1)}{q-1} x^{n-1}$ $= (q^{n-1} + q^{n-2} + \dots + 1)x^{n-1}$ $\lim_{q \rightarrow 1} D_q(x^n) = \lim_{q \rightarrow 1} (q^{n-1} + q^{n-2} + \dots + 1)x^{n-1}$ $= nx^{n-1}$ $= f'(x)$
8(b)	$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ $\ln e = \ln \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ since $f(x) = \ln x$ is continuous, $1 = \lim_{n \rightarrow \infty} \ln \left(1 + \frac{1}{n}\right)^n$ Let $h = \frac{1}{n}$, we have $1 = \lim_{h \rightarrow 0} \ln (1+h)^{\frac{1}{h}}$ $1 = \lim_{h \rightarrow 0} \frac{\ln(1+h)}{h}$
8(c)	$\lim_{q \rightarrow 1} D_q(\ln x) = \lim_{q \rightarrow 1} \frac{\ln(qx) - \ln x}{(q-1)x}$ $= \lim_{q \rightarrow 1} \frac{\ln \frac{qx}{x}}{q-1} \frac{1}{x}$ $= \frac{1}{x} \lim_{q \rightarrow 1} \frac{\ln q}{q-1}$ Let $h = q-1$, we have

	Solution
	$\lim_{q \rightarrow 1} D_q (\ln x) = \frac{1}{x} \lim_{q \rightarrow 1} \frac{\ln q}{q-1}$ $= \frac{1}{x} \lim_{h \rightarrow 0} \frac{\ln(1+h)}{h}$ $= \frac{1}{x}$ $= f'(x)$
8(d)(i)	$D_q [f(x)g(x)] = \frac{f(qx)g(qx) - f(x)g(x)}{(q-1)x}$ $= \frac{f(qx)g(qx) - f(qx)g(x) + f(qx)g(x) - f(x)g(x)}{(q-1)x}$ $= \frac{f(qx)g(qx) - f(qx)g(x)}{(q-1)x} + \frac{f(qx)g(x) - f(x)g(x)}{(q-1)x}$ $= f(qx) \frac{g(qx) - g(x)}{(q-1)x} + g(x) \frac{f(qx) - f(x)}{(q-1)x}$ $= f(qx)D_q g(x) + g(x)D_q f(x)$ $D_q [f(x)g(x)] = \frac{f(qx)g(qx) - f(x)g(x)}{(q-1)x}$ $= \frac{f(qx)g(qx) - f(x)g(qx) + f(x)g(qx) - f(x)g(x)}{(q-1)x}$ $= \frac{f(qx)g(qx) - f(x)g(qx)}{(q-1)x} + \frac{f(x)g(qx) - f(x)g(x)}{(q-1)x}$ $= g(qx) \frac{f(qx) - f(x)}{(q-1)x} + f(x) \frac{g(qx) - g(x)}{(q-1)x}$ $= f(x)D_q g(x) + g(qx)D_q f(x)$
8(d)(ii)	The product rule for the first principle shows symmetry while the product rule of q-derivative is asymmetric.

Solution**8(e)(i)**

$$\begin{aligned}D_q \left[\frac{f(x)}{g(x)} \right] &= D_q \left[f(x) \frac{1}{g(x)} \right] \\&= f(qx) D_q \left[\frac{1}{g(x)} \right] + \frac{1}{g(x)} D_q f(x) \\&= f(qx) \frac{\frac{1}{g(qx)} - \frac{1}{g(x)}}{(q-1)x} + \frac{1}{g(x)} D_q f(x) \\&= f(qx) \frac{g(x) - g(qx)}{(q-1)x} \frac{1}{g(qx)g(x)} + \frac{g(qx) D_q f(x)}{g(qx)g(x)} \\&= \frac{-f(qx) D_q g(x)}{g(qx)g(x)} + \frac{g(qx) D_q f(x)}{g(qx)g(x)} \\&= \frac{g(qx) D_q f(x) - f(qx) D_q g(x)}{g(qx)g(x)}\end{aligned}$$

Or

$$\begin{aligned}D_q \left[\frac{f(x)}{g(x)} \right] &= D_q \left[f(x) \frac{1}{g(x)} \right] \\&= f(x) D_q \left[\frac{1}{g(x)} \right] + \frac{1}{g(qx)} D_q f(x) \\&= f(x) \frac{\frac{1}{g(qx)} - \frac{1}{g(x)}}{(q-1)x} + \frac{1}{g(qx)} D_q f(x) \\&= f(x) \frac{g(x) - g(qx)}{(q-1)x} \frac{1}{g(qx)g(x)} + \frac{g(x) D_q f(x)}{g(qx)g(x)} \\&= \frac{-f(x) D_q g(x)}{g(qx)g(x)} + \frac{g(x) D_q f(x)}{g(qx)g(x)} \\&= \frac{g(x) D_q f(x) - f(x) D_q g(x)}{g(qx)g(x)}\end{aligned}$$

Solution

Alternatively,

$$\begin{aligned}
D_q \left[\frac{f(x)}{g(x)} \right] &= \frac{\frac{f(qx)}{g(qx)} - \frac{f(x)}{g(x)}}{(q-1)x} \\
&= \frac{\frac{f(qx)g(x) - f(x)g(qx)}{g(qx)g(x)}}{(q-1)x} \\
&= \frac{f(qx)g(x) - f(x)g(qx)}{(q-1)x g(qx)g(x)} \\
&= \frac{f(qx)g(x) - f(qx)g(qx) + f(qx)g(qx) - f(x)g(qx)}{(q-1)x g(qx)g(x)} \\
&= \frac{\frac{f(qx)g(x) - f(qx)g(qx)}{(q-1)x} - \frac{f(x)g(qx) - f(qx)g(qx)}{(q-1)x}}{g(qx)g(x)} \\
&= \frac{\frac{f(qx)[g(x) - g(qx)]}{(q-1)x} - \frac{[f(x) - f(qx)]g(qx)}{(q-1)x}}{g(qx)g(x)} \\
&= \frac{-\frac{f(qx)[g(qx) - g(x)]}{(q-1)x} + \frac{[f(qx) - f(x)]g(qx)}{(q-1)x}}{g(qx)g(x)} \\
&= \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(qx)g(x)}
\end{aligned}$$

Or

	Solution
	$ \begin{aligned} D_q \left[\frac{f(x)}{g(x)} \right] &= \frac{\frac{f(qx) - f(x)}{(q-1)x}}{\frac{g(qx) - g(x)}{(q-1)x}} \\ &= \frac{f(qx)g(x) - f(x)g(qx)}{g(qx)g(x)} \\ &= \frac{f(qx)g(x) - f(x)g(qx)}{g(qx)g(x)} \\ &= \frac{f(qx)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(qx)}{(q-1)x} \\ &= \frac{f(qx)g(x) - f(x)g(x)}{(q-1)x} - \frac{f(x)g(qx) - f(x)g(x)}{(q-1)x} \\ &= \frac{g(x)[f(qx) - f(x)]}{(q-1)x} - \frac{[g(x) - g(qx)]f(x)}{(q-1)x} \\ &= \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)} \end{aligned} $
8(e)(ii)	As $q \rightarrow 1$, $ \begin{aligned} &\lim_{q \rightarrow 1} \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(qx)g(x)} \\ &= \frac{\lim_{q \rightarrow 1} g(qx)D_q f(x) - \lim_{q \rightarrow 1} f(qx)D_q g(x)}{\lim_{q \rightarrow 1} g(qx)g(x)} \\ &= \frac{\lim_{q \rightarrow 1} g(qx) \lim_{q \rightarrow 1} D_q f(x) - \lim_{q \rightarrow 1} f(qx) \lim_{q \rightarrow 1} D_q g(x)}{[g(x)]^2} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \end{aligned} $ Or

Solution

$$\begin{aligned} & \lim_{q \rightarrow 1} \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)} \\ &= \frac{\lim_{q \rightarrow 1} g(x)D_q f(x) - \lim_{q \rightarrow 1} f(x)D_q g(x)}{\lim_{q \rightarrow 1} g(qx)g(x)} \\ &= \frac{g(x) \lim_{q \rightarrow 1} D_q f(x) - f(x) \lim_{q \rightarrow 1} D_q g(x)}{[g(x)]^2} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \end{aligned}$$

As $q \rightarrow 1$, the quotient rule for q -derivative aligns with the quotient rule from the first principle.