Differential Equations

1. **AJC/2009/II/3**

(a) By using the substitution $z = x + y$, show that the differential equation $(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + xy + x + 1$ can be reduced to the equation $z\frac{\mathrm{d}z}{\mathrm{d}x} = (x+1)(z+1)$.
Hence find the general solution of the differential equation

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + xy + x + 1 . \qquad [4]$$

(b) John jogs 8 km every morning. As a mathematical model, he assumes that his speed during the jog is proportional to the distance he has yet to complete.

(i) He starts off at an initial speed of 10 km per hour, and after t hours, he has jogged x km. Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{5}{4}(8-x)$. [1]

(ii) Find an expression for x in terms of t and hence find the time taken for John to complete 6 km. [3]

(iii) Comment on the suitability of the model. [1]

2. **EJC Prelim/2020/01/Q9**

(a) (i) Given x and y are related by the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy = k$ for $k \in \mathbb{R}$, show that $y = \frac{k(\ln x + \alpha)}{x}$ is a solution of the differential equation where α is an arbitrary constant. [2]

(ii) Hence, show that $\left(\mathrm{e}^{1-\alpha}, k\mathrm{e}^{\alpha-1}\right)$ is a stationary point of the curve $y = \frac{k(\ln x + \alpha)}{x}$. [2]

(b) It is given that x and y are related by the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x} + x = \sqrt{x^2 + y^2}$ and that $y = 0$ when $x = -2$.

(i) By substituting $v = x^2 + y^2$, show that the differential equation can be written as $\frac{\mathrm{d}v}{\mathrm{d}x} = 2\sqrt{v}$. [2]

(ii) Find v in terms of x and hence show that $y^2 = \mathrm{f}(x)$ where $\mathrm{f}(x)$ is to be determined. [4]

3. **NYJC/II/2**

In a controlled experiment, the rate at which liquid fuel is pumped into a tank is twice the remaining volume of the tank, and the rate at which the fuel in the tank is used up is proportional to the square of the volume of fuel in the tank. The capacity of the tank is $\frac{3}{2}$ m^3 and the volume of fuel in the tank at time t is x m^3.

(i) Given that the volume of fuel in the tank stabilizes at 1 m^3, show that the increase of the fuel is given by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = -\left(x^2 + 2x - 3\right)$.

(ii) Given that the tank is initially empty, find x in terms of t.

(iii) Sketch the graph of your solution curve in **(ii)**.

4. **SAJC/II/1**

The population size of a colony of ants increases at a rate which is inversely proportional to the population size and decreases at a rate which is proportional to the population size. The population size of the colony (in thousands) is x at time t days, where $x > 2$. Given that when the population size is 2000, it remains at this value, show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(\frac{x^2-4}{x}\right)$$

where k is a negative constant.

Hence find the general solution of the differential equation by expressing x in terms of t. [5]

5. **YJC/1/6**

In a chemical reaction, the amount of substance X and substance Y are being observed. There are x grams of substance X and y grams of substance Y at time t minutes. Using a mathematical model, the variable x and y satisfy the equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2 - x \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = e^x \text{, for } 0 \le t < 1.$$

(i) Initially, there are no traces of substance X and substance Y present in the reaction. Express y in terms of x. [4]

(ii) The amount of substance Y during the reaction is the maximum at time k minutes. Find the value of k. [4]

6. **NYJC/2010/I/10**

An innovation is introduced into a community of 100 farmers at time $t = 0$. Let x denote the number of farmers who have adopted the innovation at time t. Assume that x is a continuous function of time. The rate at which the number of farmers in that community who adopted the innovation at a particular instant is proportional to the product of the number of farmers who have already adopted and the number of farmers who have not adopted the innovation.

Initially, one farmer adopted the innovation and the rate at which the number of farmers who adopted the innovation is one farmer per unit time.

(i) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = k(100x - x^2)$, where k is to be determined. [1]

(ii) Find the particular solution of x, in terms of t. [5]

(iii) Sketch the graph of x versus t, for $t \geq 0$. [2]

(iv) Using the graph in (iii) or otherwise, find the time taken for 75% of the population of farmers to adopt the innovation, leaving your answer to 2 decimal places. [1]

(v) Give a reason why the model may not be suitable. [1]

7. **NYJC/2011/I/6**

(a) The variables y and x are related by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} - 2x^2$. By using the substitution $y = ux$, solve this equation and hence find y in terms of x, given that $y = 0$ when $x = 1$.

(b) In a pathological investigation of the spread of disease in a country, it is found that the rate at which the proportion of area infected at any time (hours) is proportional to the product of the proportion of area infected and the proportion of area not infected. Initially, half of the area of the country is infected. When proportion of area infected increases at a rate of $\frac{1}{54}$ per hour, the proportion of area infected is $\frac{1}{3}$. The proportion of area infected at any time t is given by x.

(i) Obtain a differential equation relating x and t and show that the particular solution of the differential equation is of the form $x = \frac{A}{1 + Be^{-\frac{t}{12}}}$, where A and B are positive constants to be determined.

(ii) Find the proportion of the area infected after 12 hours.

8. **DHS/2012/I/11**

Given that $z = \mathrm{e}^{2x}\frac{\mathrm{d}y}{\mathrm{d}x}$, express $\frac{\mathrm{d}z}{\mathrm{d}x}$ in terms of x, $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Hence show that the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{1-4x}$ can be reduced to $\frac{\mathrm{d}z}{\mathrm{d}x} = \mathrm{e}^{1-2x}$.

Solve this differential equation, expressing y in terms of x. [5]

9. **MJC/2012/I/10**

(a) By means of the substitution $y = xz$, show that the differential equation

$$\left(e^x + 1\right)\left(\frac{dy}{dx} - \frac{y}{x}\right) = \frac{x^2}{y}\left(e^x - 1\right)$$

can be reduced to the form

$$z\frac{dz}{dx} = \frac{e^x - 1}{e^x + 1}.$$

Hence find the general solution for y^2 in terms of x. [5]

(b) There was an island where initially there was no one living on it. The total capacity of the island is 9 000. The population increases at a rate which is inversely proportional to the remaining capacity of the island. At the same time, the rate at which the population decreases is $\frac{1}{20}$ of the population size. When the population reaches 4 000, it remains at this value. The population size (in thousands) is x at time t months, show that

$$\frac{dx}{dt} = \frac{(x-4)(x-5)}{20(9-x)}.$$

Find t in terms of x. Hence find the time when the population reaches 2 000. [7]

10. **VJC Prelim 9758/2023/01/Q11**

Statistics show that the population of rodents in a small area follows a *density-dependent population growth model*. In this model, as the population approaches its carrying capacity, the growth rate slows down due to increased competition for limited resources, such as food, space, and mates.

In a particular small farm, a virus spread has a significant impact on the population of rodents. The population t days after the start of the virus spread is denoted by N (in hundreds). The *density-dependent population growth model* states that the population of rodents increases at a rate inversely proportional to the square of the population. At the same time rodents die due to the virus spread at a rate proportional to the population. When $N = 3$, the number of rodents remains constant.

(a) Show that $\frac{dN}{dt} = \frac{k(27 - N^3)}{N^2}$, where k is a positive constant. [3]

(b) It is further given that when the virus spread started, there were 500 rodents. Show that $N^3 = 27 + Ce^{-3kt}$, where C is a positive constant to be determined. [6]

(c) Sketch the graph of N against t and explain what will happen to the population of rodents in the farm in the long run. [3]

11. **EJC Prelim 9758/2023/02/Q3**

The variables x and y are related by the differential equation $\pi\frac{\mathrm{d}y}{\mathrm{d}x}+y\left(3-\pi\tan x\right)=0$.

(a) Using the substitution $y=z\sec x$, show that $\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{-3z}{\pi}$. Hence, show that the particular solution for which $y=2$ when $x=\frac{\pi}{3}$ can be expressed in the form $y\cos x=\mathrm{e}^{a+bx}$ where a and b are constants to be determined. [6]

(b) For the graph of the solution found in part **(a)**, find the equations of the two vertical asymptotes closest to the y-axis. [2]

12. **DHS/2018/I/6**

(a) By using the substitution $y = zx^2$, find the general solution of the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}=2xy-y^2, \quad \text{where } x\neq 0.$$

[4]

(i) Sketch the solution curve that passes through $\left(2\,,-4\right)$, indicating any stationary points and asymptotes clearly. [4]

(ii) State the particular solution for which y has no turning point. [1]

(b) A differential equation is of the form $\frac{\mathrm{d}y}{\mathrm{d}x}+y=px+q$, where p and q are constants. Its general solution is $y=4x-1+D\mathrm{e}^{-x}$, where D is an arbitrary constant. Find the values of p and q. [2]

13. **SAJC/2019/II/Q2**

A water tank contains 1 cubic meter of water initially. The volume of water in the tank at time t seconds is V cubic metres. Water flows out of the tank at a rate proportional to the volume of water in the tank and at the same time, water is added to the tank at a constant rate of k cubic metres per second.

(i) Show that $\frac{\mathrm{d}V}{\mathrm{d}t}=k\left(1-\frac{a}{k}V\right)$, where a is a positive constant. [2]

Hence find V in terms of t. [5]

(ii) Sketch the solution curve for V against t, such that

(a) $a<k$.

(b) $a>k$.

For cases (a) & (b), describe and explain what would happen to the volume of water, V in the tank eventually. [4]

14. **ACJC/2021/I/Q10**

The **Gompertz differential equation** provides a good model for gauging lung cancer growth. The differential equation is given as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = aV \ln\left(\frac{K}{V}\right),$$

where V mm^3 is the volume of the tumour at time t days after an early discovery, and a and K are positive real constants.

(i) Describe the behaviour of $\frac{\mathrm{d}V}{\mathrm{d}t}$ as $V \to K$. [1]

(ii) By using the substitution $u = \ln\left(\frac{K}{V}\right)$, solve the given differential equation and show that $V = K\mathrm{e}^{-A\mathrm{e}^{-at}}$, where A is an arbitrary constant. [5]

(iii) What happens to V as $t \to \infty$? State the significance of K in the context of this question. [2]

For the rest of this question, you may assume that $a = 0.01$ and $K = 8000$.

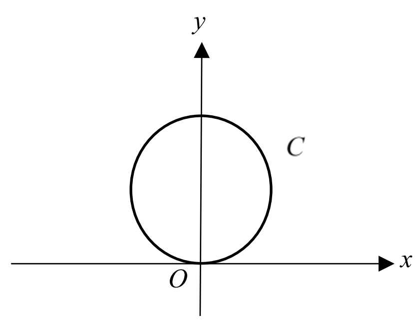
A patient was discovered early to have a lung tumour of volume $100\ \mathrm{mm}^3$.

(iv) Find the size of the tumour after 100 days. Give your answer to the nearest mm^3. [2]

(v) Sketch the graph of V against t. [2]

15. **EJC/2022/I/Q10**

A spherical container of radius 5 m is formed by rotating the following circle C about the y-axis.



The container has negligible thickness, and the circle C passes through the origin O.

(a) State a cartesian equation of C. [1]

Initially the spherical container is completely filled with water. Two engineers are calculating the time needed for the container to be completely drained from a small circular hole at the bottom. The volume of water in the container at time t seconds is denoted by V m^3.

[The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.]

(b) The first engineer proposes that the rate of change of V with respect to t is a constant k.

(i) Write down a differential equation relating V, t and k. [1]

(ii) Determine, with justification, the sign of k. [1]

(iii) Find V in terms of t and k, leaving your answer in exact form. [2]

(c) The second engineer argues that the rate at which water flows out from the hole will be at its greatest in the beginning, and decreases as the depth of water in the container decreases. He suggests using Torricelli's law, which says that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\alpha\sqrt{20h}\,,$$

where α m^2 is the area of the circular hole at the bottom of the container, and h m is the depth of water in the container at time t seconds. The radius of the hole at the bottom of the container is found to be constant at 1 cm.

(i) Show that $\left(10h^{\frac{1}{2}} - h^{\frac{3}{2}}\right)\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\sqrt{5}}{5000}$. [4]

(ii) Hence find the numerical value of t when the container is completely drained. [4]

16. **ACJC Prelim 9758/2024/01/Q10**

Second-hand smoking in public spaces have resulted in negative effects such as coronary heart disease, lung cancer, and other diseases. Designated smoking rooms are often being built to contain the smoke. A room containing 30 m^3 of air is originally free of carbon monoxide. Let V m^3 be the volume of carbon monoxide in the room at time t minutes after the smoke starts entering the room. Let C be the concentration of carbon monoxide in the room at time t.

Initially, there is no carbon monoxide in the room. However, smoke containing 5% of carbon monoxide is blown into the room at the rate of 0.002 m^3/min. The rate at which the carbon monoxide leaves the room is $\frac{C}{500}$ m^3/min.

(i) Express $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of C. [1]

(ii) Hence, given that $C = \frac{V}{30}$, show that $\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{1}{15000}\left(0.05 - C\right)$. [2]

(iii) By solving the differential equation in part **(ii)**, show that the concentration of carbon monoxide in the room at time t is $C = 0.05\left(1 - \mathrm{e}^{-\frac{1}{15000}t}\right)$. [4]

(iv) Explain in context what will happen to the concentration of carbon monoxide in the long run? [1]

(v) Sketch the curve of C against t. [2]

(vi) Medical research has shown that when the volume of carbon monoxide in the room reaches 0.0036 m^3, a person exposed to it can lead to loss of consciousness. Find the time for the concentration of carbon monoxide to reach this level, giving your answer to nearest integer. [2]

17. **ASRJC Prelim 9758/2024/01/Q11**

In a large town, the number of people infected by a particular virus t days after the virus was first discovered is x. It is assumed that the rate of infection is proportional to x. Initially there are 5 people who are infected by the virus, and there are 5120 people who are infected by the virus 30 days after the virus was first discovered.

(i) Show that $x = 5(2)^{\frac{t}{3}}$. [5]

A cure and vaccine for the virus were discovered and administered to the population 30 days after the virus was discovered. Individuals who were cured are not at risk of reinfection. The number of people infected by the virus p days after the cure and vaccine were administered is represented by y. It is believed that the new rate of infection from then on is proportional to $6400y - y^2$.

It is given that 30 days after the cure or vaccine was administered, 3200 people remain infected with the virus.

(ii) Show that $y = \dfrac{6400}{1+2^{\left(\frac{p}{15}+H\right)}}$, where H is a constant to be determined. [6]

(iii) By finding the number of people in the town that will be infected by the virus in the long term, comment on the effectiveness of the cure and vaccine administered. [2]

18. **MI Prelim 9758/2024/02/Q3**

(i) Use the substitution $w = x + y$, show that the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3-x-y}{1+x+y}$ can be reduced into $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{4}{1+w}$. Solve this differential equation and deduce that the general solution can be written as $(x+y)^2 = 2(3x-y) + A$, where A is an arbitrary constant. [5]

(ii) Find the particular solution for the above differential equation, given that the curve passes through y-intercept $(0,5)$. Hence or otherwise, find the coordinates of the other y-intercept. [2]

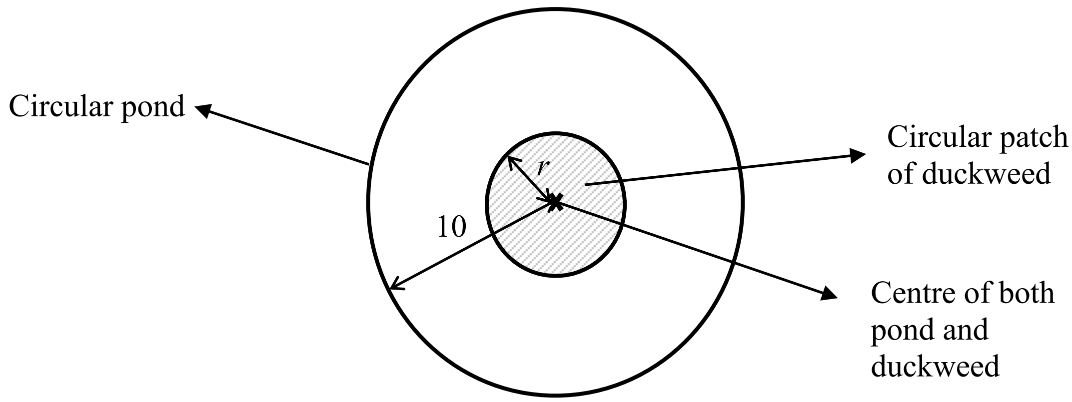
19. **YIJC Prelim 9758/2024/01/Q4**

(a) It is given that $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy(\ln x + \ln y) - y$. Using the substitution $w = xy$, show that the differential equation can be transformed to $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \mathrm{f}(w)$, where the function $\mathrm{f}(w)$ is to be found. [3]

(b) Hence, given that $y = \dfrac{1}{2}\mathrm{e}^3$ when $x = 2$, solve the differential equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy(\ln x + \ln y) - y$, to find y in terms of x. [5]

20. **CJC Prelim 9758/2025/P1/Q9**

A Koi pond farmer introduces a circular patch of duckweed at the center of his circular pond to enhance water purification and provide supplemental nutrition for his Koi, as shown in the figure below.



The duckweed grows over time t days, forming a continuous circular patch on the pond's surface. It is assumed that the duckweed forms a thin, floating layer with negligible thickness. The rate of increase of the area covered by duckweed is directly proportional to the area of the pond that remains uncovered by the duckweed.

(a) Given that the radius of the pond is 10 m, show that the radius, r m of the circular patch of duckweed satisfies the differential equation $\dfrac{\mathrm{d}r}{\mathrm{d}t}=\dfrac{k\left(100-r^2\right)}{2r}$, $k>0$. [2]

(b) The initial radius of the circular patch of duckweed is 1 m. Solve the differential equation, expressing r in the form $r=\sqrt{P-Q\mathrm{e}^{-kt}}$, where P and Q are constants to be determined. [6]

(c) Sketch the graph of r against t. [3]

21. **DHS Prelim 9758/2025/P2/Q5**

A heated metal block at 160°C is left to cool in a laboratory with an ambient temperature of 30°C. The temperature, T in $^\circ\text{C}$, of the metal block t minutes after it is left in the laboratory is model as a function $T(\text{t})$. The rate of cooling is proportional to the difference between the object's temperature and the ambient temperature, with a positive constant of proportionality k.

(a) Write down a differential equation for the situation and find the expression of $T(t)$ in terms k. [3]

An experiment to study the cooling of the heated metal block starts at 10:00 am. However, a protective thermal casing is used to delay the cooling of the metal block for 5 minutes, resulting in the temperature of the metal block remaining constant at 160°C during this time. At 10:05 am, the casing is removed and cooling begins immediately. At 10:15 am, the temperature of the metal block is measured to be 100°C.

(b) In order to use the solution obtained in part **(a)** to model the cooling process now, the time t needs to be replaced with $t+a$, where a is a constant. State the value of a. [1]

(c) Due to the protective thermal casing, the temperature of the metal block $T_p(t)\,^\circ\text{C}$, t minutes after 10:00 am can be modelled by the following piecewise function

$$T_p(t) = \begin{cases} 160 & \text{for } 0 \le t < 5, \\ T(t+a) & \text{for } t \ge 5. \end{cases}$$

Using your answers from parts **(a)** and **(b),** show that $T(t+a) = 30 + 177\mathrm{e}^{-0.0619t}$. [3]

(d) Find the time it takes for the metal block to cool to 60°C. [1]

(e) Sketch the graph of $T_p(t)$ against t for $t \ge 0$. [2]

22. **SAJC Prelim 9758/2025/P1/Q12**

Welding fumes contain a dangerous amount of particulates. To protect workers in a workshop, an extraction device remove particulates in the air continuously. Let V mg represent the mass of particulates in the air in the workshop at time t min after the extraction device is activated. Particulates are produced at a constant rate of 0.32 mg/min. The rate at which particulates are removed is proportional to the mass of particulates in the air.

When the mass of particulates in the air in the workshop is 18.75 mg, the mass of particulates in the air increases at a rate of 0.12 mg/min.

(i) Show that $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \dfrac{4}{375}(30 - V)$. [2]

(ii) Solve the differential equation, given that the workshop is initially free of particulates. [5]

(iii) Sketch the graph of V against t. [2]

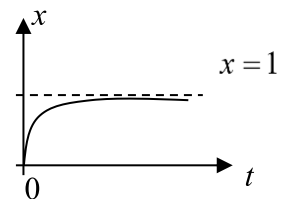
(iv) It is recommended that the mass of particulates in this workshop should be kept below 32mg. Comment on whether the extraction device is effective enough to meet the recommendation. [1]

The extraction device becomes less efficient and now removes particulates at **half** its original rate.

(v) Write down a new differential equation to model this. [1]

(vi) The mass of particulates in the air in the workshop is said to reach a steady-state when it remains constant. Find the value of V at steady-state. [1]

Answers

1	(a) $x+y-\ln\lvert x+y+1\rvert=\frac{1}{2}(x+1)^2+c$ (bii) $x=8-8e^{-\frac{5}{4}t}$. When $x=8$ (i.e. he completes his 8 km jog), $t\to\infty$; Model predicts infinite amount of time to complete jog, therefore model is NOT suitable.
2	(b)(ii) $\mathrm{f}(x)=8x+16$
3	(ii) $x=\frac{12}{3+\mathrm{e}^{-4t}}-3$ (ii) 
4	$x=\sqrt{\left(Ae^{2kt}+4\right)}$
5	(i) $y=xe^{-x}-e^{-x}+1$ (ii) $k=1-e^{-2}$
6	(ii) $x=\frac{100e^{\frac{100}{99}t}}{99+e^{\frac{100}{99}t}}$ (iv) 5.64 yr (v) The farmers may be influenced by adoption of innovation from other sources
7	(a) $y=-x^3+x$; (b) $x=\frac{e^{\frac{t}{12}}}{1+e^{\frac{t}{12}}}$, (ii) $x=\frac{e}{1+e}=0.731$
8	(a) $\frac{dz}{dx}=e^{2x}\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}\right)$; $y=\frac{1}{8}e^{1-4x}-\frac{1}{2}Ce^{-2x}+D$
9	(a) $y^2=4x^2\ln(e^x+1)-2x^2\ln(e^x)+Dx^2$, (b) $t=-100\ln\left\lvert\frac{x-4}{4}\right\rvert+80\ln\left\lvert\frac{x-5}{5}\right\rvert$; $t=28.4$ mths
10	$N^3=27+98e^{-3kt}$
11	(a) $y\cos x=\mathrm{e}^{1-\frac{3}{\pi}x}$ (b) $x=\pm\frac{\pi}{2}$
12	(a) $y=\frac{x^2}{x-C}$ (a)(ii) $y=x$ (b) $p=4,\ q=3$

13	(i) $V=\frac{1}{a}(k-(k-a)e^{-at})$. Last part: The volume of water in the water tank, V, decreases from one cubic meter and approach $\frac{k}{a}$ cubic meters eventually
14	(i) $\frac{\mathrm{d}V}{\mathrm{d}t}\to 0$ (iii) $V\to K$ (iv) 1596
15	(a) $x^2+(y-5)^2=25$ (b)(i) $\frac{\mathrm{d}V}{\mathrm{d}t}=k$; (ii) $k<0$; (iii) $V=kt+\frac{500}{3}\pi$ (c)(i) $t=189000$ seconds (3 s.f.)
16	(i) $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{10000}-\frac{C}{500}$ (iv) The concentration of carbon monoxide increases and approaches to 5%. (vi) 36 minutes
17	(ii) $H=-2$ (iii) effective
18	(ii) $(x+y)^2=2(3x-y)+35$, $(0,-7)$
19	(a) $\frac{\mathrm{d}w}{\mathrm{d}x}=w(\ln w)$ (b) $y=\frac{\mathrm{e}^{3\mathrm{e}^{x-2}}}{x}$
20	(b) $r=\sqrt{100-99e^{-kt}}$
21	(a) $T(t)=30+130\mathrm{e}^{-kt}$ (b) $a=-5$ (d) 28.7 min
22	(ii) $V=30-30\mathrm{e}^{-\frac{4}{375}t}$ (v) $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{2}{375}(60-V)$ (vi) $V=60$