5. PROJECTILE MOTION

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Learning Objectives

5.1 Free fall

(a) Describe and use the concept of weight as the force experienced by a mass in a gravitational field.

(b) Describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

5.2 Gravitational potential energy in a uniform field

(c) Derive, from the definition of work done by a force, the equation $\Delta E_p = mg\Delta h$ for gravitational potential energy changes in a uniform gravitational field (e.g. near the Earth's surface).

(d) Recall and use the equation $\Delta E_p = mg\Delta h$ to solve problems.

5.3 Effects of air resistance

(e) Describe qualitatively, with reference to forces and energy, the motion of bodies falling in a uniform gravitational field with air resistance, including the phenomenon of terminal velocity.

5.1 FREE FALL

5.1.1 What is weight?

(a) Describe and use the concept of weight as the force experienced by a mass in a gravitational field.

Weight is the force experienced by an object due to the gravitational attraction exerted on it by a massive body, such as the Earth. It can be described using the following formula:

$$\boldsymbol{W} = m\boldsymbol{g}$$

Where:

- $\boldsymbol{W}$ is the weight of the object (in newtons, N),
- m is the mass of the object (in kilograms, kg),
- $\boldsymbol{g}$ is the gravitational field strength (in N kg^{-1}), which is approximately 9.81 N kg^{-1}, near the Earth's surface.

Question : Show gravitational field strength is numerically equal to acceleration of free fall.

For an object to be in free fall, the only force acting on it is the gravitational force acting on the body or its weight.

$$\text{Resultant Force} = \text{Weight}$$

$$ma = mg$$

$$a = g$$

Where:

- a is the acceleration of free fall.

Therefore, gravitational field strength is numerically equal to acceleration of free fall.

The concept of weight arises because gravity pulls objects toward the center of a massive body, like Earth. The force of gravity, on an object, depends on two things:

1. **Mass of the object (*m*)**: The more massive an object, the greater the gravitational pull it will experience.
2. **Gravitational field strength (*g*)**: The gravitational field strength at a point in a gravitational field is defined as the **gravitational force exerted per unit mass** acting on a small mass placed at that point.

The weight of an object is a force, so it is measured in newtons (N), which are a unit of force in the International System of Units (SI).

An object is said to be in free fall when the only force acting on it is the gravitational force (weight)

5.1.2 Projectile Motion

Self-study resources		
A SLS lesson on Projectile Motion	https://for.edu.sg/projectile-motion	https://for.edu.sg/projectile-motion

(b) Describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

In this section we will study how an object moves in a plane under the action of a constant force. An example is the motion of a ball thrown at an angle to the vertical, as shown in Fig. 5.1. The constant force acting on it is its weight (or gravitational force). The path of the ball is a parabola. This type of motion is also known as **projectile motion**.

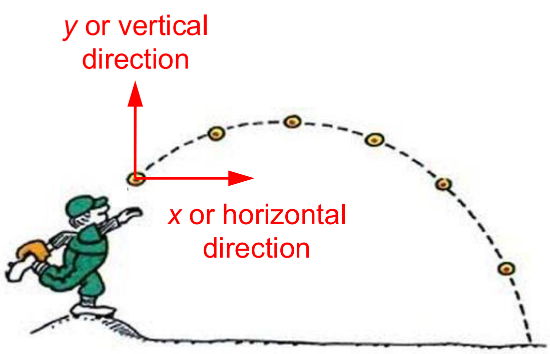


Fig. 5.1

Analysis of a projectile motion

Galileo first gave an accurate analysis of this motion by splitting the motion into its vertical and horizontal components, and consider these separately.

The photograph in Fig. 5.2 was taken while a lamp emitted regular flashes of light. The ball on the left was dropped from rest and the other, a projectile, was *thrown sideways* at the same time. The images of each ball were captured at equal time interval apart.

- The horizontal distance travelled by the projectile is equal at every time interval. So, the horizontal motion is due to uniform velocity.
- Their vertical accelerations (due to gravity) are equal, showing that a projectile falls like a body which is dropped from rest. Its horizontal velocity does not affect its vertical motion.

Therefore, the horizontal and vertical motions of a body are independent and can be treated separately.

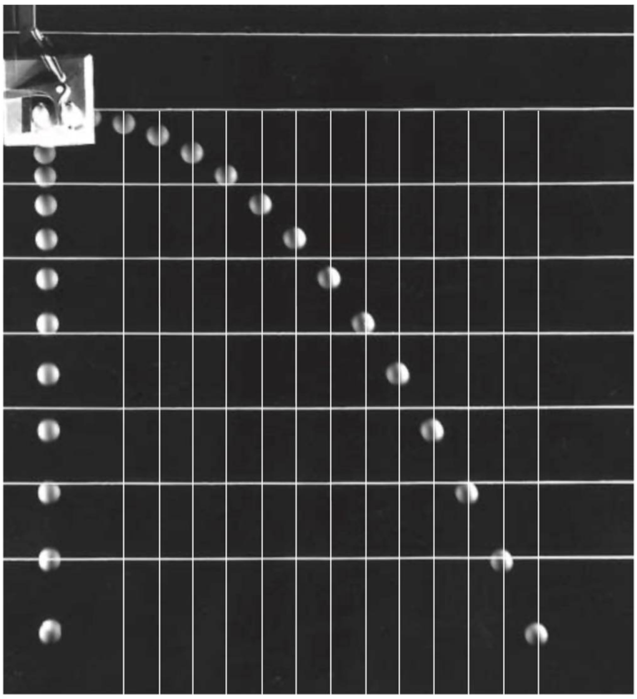


Fig. 5.2

Fig. 5.3 shows the trajectory (or path) of a particle (in black) projected with an initial velocity u at an angle θ to the horizontal from the origin. Air resistance is negligible.

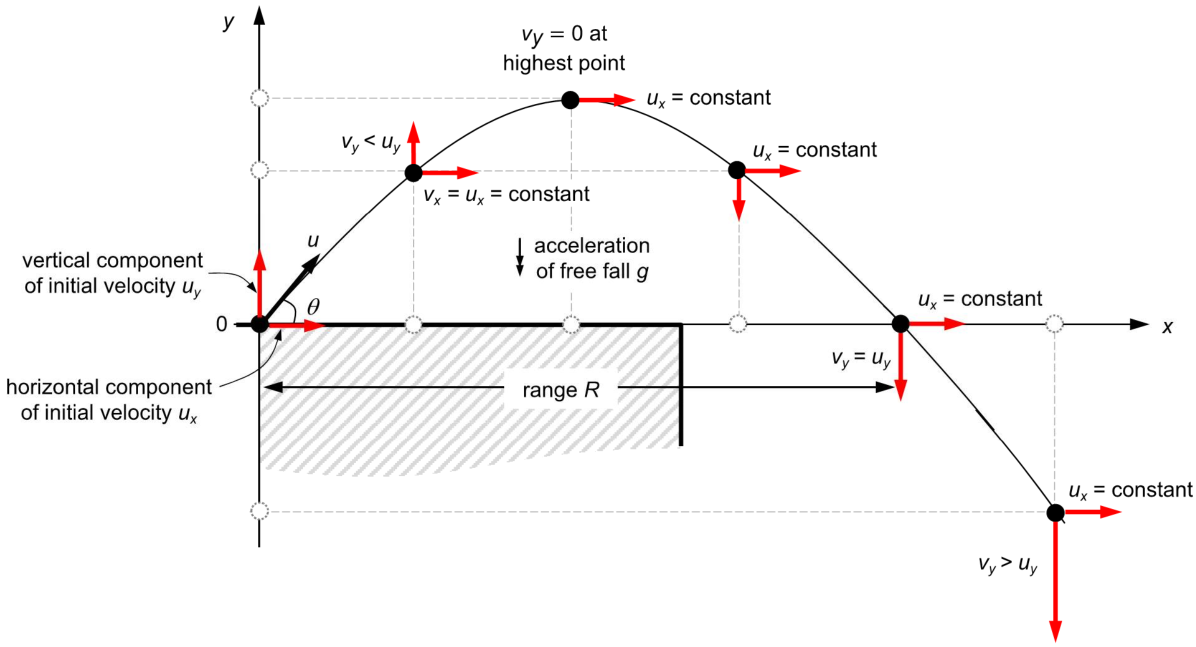


Fig. 5.3

- The initial velocity can be resolved into its horizontal component u_x and vertical component u_y.
- In the horizontal motion, the horizontal velocity $v_x = u_x$ is constant throughout the motion.
- In the vertical motion, it is equivalent to a free fall motion. Therefore, the vertical velocity v_y changes its the magnitude and/or direction with constant acceleration g.

At the highest point of the trajectory, $v_y = 0$ (but the horizontal speed ≠ 0).

Steps to Solving Projectile Motion Problems

Step 1 Resolve the initial velocity (of projection) into its horizontal and vertical components.

Step 2 Determine the initial and final positions of the object for your calculation in step 3.

Step 3 Select the equations of motion to determine the unknown value:

Horizontal motion: $s_x = u_x t$

Note: If two out of the three quantities are unknown, consider vertical motion.

Vertical motion:

$$v_y = u_y + a_y t$$

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

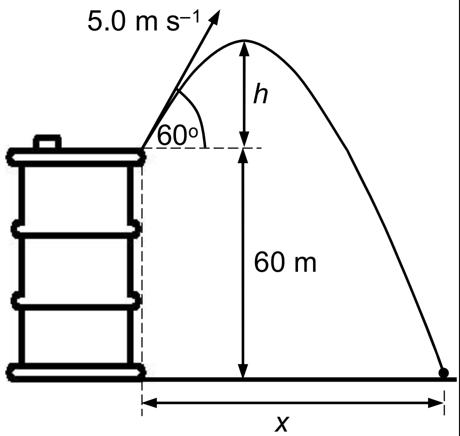
$$v_y^2 = u_y^2 + 2a_y s_y$$

Note:

1. *Remember to assign either ↑ or ↓ positive when using these equations.*
2. *When the motion is in a uniform gravitational field,* $a_y = g = 9.81$ m s^{-2}.

Example 5.1

A ball is kicked from the top of a 60 m building. The initial velocity of the ball is 5.0 m s^{-1} at an angle to the horizontal of 60°. Air resistance is negligible.



(a) The ball will reach a maximum height h before falling to the ground. Calculate

(i) the time taken for the ball to reach h, and

(ii) the height h.

(b) When the ball to hit the ground, determine

(i) the time from leaving the building to hitting the ground,

(ii) the horizontal displacement x, and

(iii) the velocity v of the ball just before it hits the ground.

Solution

Resolve initial velocity: horizontal velocity u_x = 5.0 cos 60° = 2.5 m s^{-1}

vertical velocity u_y = 5.0 sin 60° = 4.3 m s^{-1}

(a) Consider the vertical motion of the ball from the point it is kicked to the point it reaches h.

At the top of the trajectory, vertical velocity v_y becomes 0.

Take vectors directed upwards (↑) as positive,

u_y = +5.0 sin 60° (↑), s = +h (↑), a_y = −9.81 (↓), v_y = 0, t = ?

(i) Use $v_y = u_y + a_y t$ to determine t

0 = 5.0 sin 60° + (-9.81)t = 20.766

t = 0.4414 = 0.441 s

(ii) Use $v_y^2 = u_y^2 + 2a_y s_y$ to determine h

$0^2 = (5.0 \sin 60°)^2 + 2(-9.81)h$

h = 0.9557 = 0.956 m

Note: You can also use $s_y = u_y t + \frac{1}{2} a_y t^2$ and the value of t in **(i)** *to solve for h*

(b)

(i) Consider the vertical motion of the ball from the point it is kicked to the point it hits the ground.

Take vectors directed upwards (↑) as positive,

u_y = +5.0 sin 60° (↑), s = −60 m (↓), a_y = −9.81 (↓), v_y = ? (↓), t = ?

Use $s_y = u_y t + \frac{1}{2} a_y t^2$ to determine t

$-60 = (5.0 \sin 60°)t + \frac{1}{2}(-9.81)t^2$

$\underline{t = 3.967 = 3.97 \text{ s}}$ or t = -3.084 s (rejected)

Note: Since the horizontal displacement x is not known, we have to consider the vertical motion to determine the time of flight.

(ii) Consider the horizontal motion of the ball from the point it is kicked to the point it hits the ground.

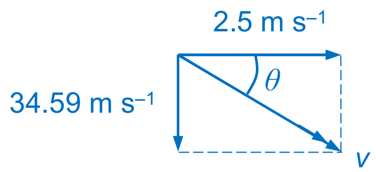
$$s_x = u_x t$$

$$x = 2.5 \times 3.967 = 9.92 \text{ m}$$

(iii) Horizontal velocity is constant, so $v_x = u_x = 2.5 \text{ m s}^{-1}$

Use $v_y = u_y + a_y t$ to determine v_y

$v_y = 5.0 \sin 60° + (-9.81)(3.967) = -34.59 \text{ m s}^{-1}$ (negative value, so downwards)



magnitude of $v = \sqrt{2.5^2 + 34.59^2} = 34.7 \text{ m s}^{-1}$

$\tan q = \frac{34.59}{2.5} \Rightarrow q = 85.9°$

Velocity is <u>34.7 m s⁻¹ at 85.9° below the horizontal</u>

Note: You may use conservation of energy to determine the magnitude of v

gain in kinetic energy = loss in gravitational potential energy

$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mgh \Rightarrow v^2 - u^2 = 2gh \Rightarrow v = \sqrt{5.0^2 + 2 \times 9.81 \times 60} = 34.7 \text{ m s}^{-1}$

5.2 GRAVITATIONAL POTENTIAL ENERGY IN A UNIFORM FIELD

(c) Derive, from the definition of work done by a force, the equation $\Delta E_p = mg\Delta h$ for gravitational potential energy changes in a uniform gravitational field (e.g. near the Earth's surface).

(d) Recall and use the equation $\Delta E_p = mg\Delta h$ to solve problems.

Gravitational potential energy, E_p of an object is the energy it possesses by virtue of its position in a gravitational field.

Consider a body of mass m being lifted from height h_1 to height h_2 on the surface of the Earth.

An external force F_{ext} that is equal and opposite to mg is applied to the mass (This can be achieved by moving the object very slowly so that it can be modelled as in equilibrium with a constant velocity)

Since there is no change in kinetic energy, F_{ext} is equal to weight or gravitational force.

$W_{field} + W_{ext} = 0$

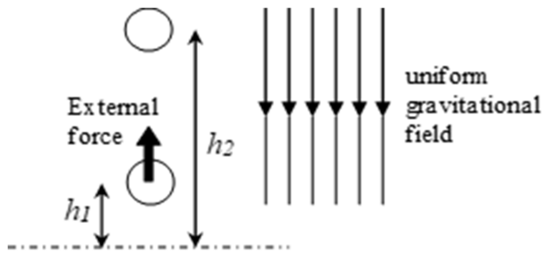
$-\Delta E_p + W_{ext} = 0$

So, $\Delta E_p = W_{ext}$

Since $W_{ext} = F_{ext}\,\Delta h$

$\Delta E_p = F_{ext}\,\Delta h$

$\Delta E_p = mg\Delta h$



https://for.edu.sg/d6eb4y

Example 5.2

A mass of weight 5.0 N is lifted by a vertical force of 7.0 N through a vertical distance of 2.0 m.

a) Calculate the work done by the lifting force.
b) Calculate the change in gravitational potential energy.
c) Assuming it was initially at rest, calculate the final speed of the mass.

Solution

a. Work done by lifting force = 7(2) = 14 J
b. change in GPE = - work done by gravitational force = -(-5)(2) = +10 J
c. By conservation of energy

Initial total energy + work done = final total energy

$$(0 + 0) + 14 = 10 + ½ (5/9.81) v^2$$

$$v = 4.0 \text{ m s}^{-2}$$

5.3 EFFECTS OF AIR RESISTANCE

(e) Describe qualitatively, with reference to forces and energy, the motion of bodies falling in a uniform gravitational field with air resistance, including the phenomenon of terminal velocity.

Air resistance / drag force / viscous force / viscous drag acts on all objects moving relative to the air.

Some facts about drag force

- Drag force always acts opposite to the object's velocity.
- The magnitude of the drag force *F* depends on many factors.
  - **Shape of the object.** Therefore, car manufacturers spend a lot of time and money researching on the best shape for a car to reduce air resistance.
  - **Speed *v* of the object relative to the air.** The faster it moves through the air, the greater the resistance. Drag force is modelled by $F \propto v$ or $F \propto v^2$ depending on the conditions.
  - **Viscosity of the air.** The viscosity of the air is affected by the temperature of the air. The higher the viscosity, the greater the air resistance.
- Besides air, drag force is also present in an object moving in any fluid which includes both gases and liquids.

Consider an object, of mass m, falling vertically from great height from rest near Earth's surface. A downward gravitational force or weight W acts on it. The drag force F depends on the speed of the object. Fig. 5.4 shows the forces (W and F) and velocities v of the object at different instances.

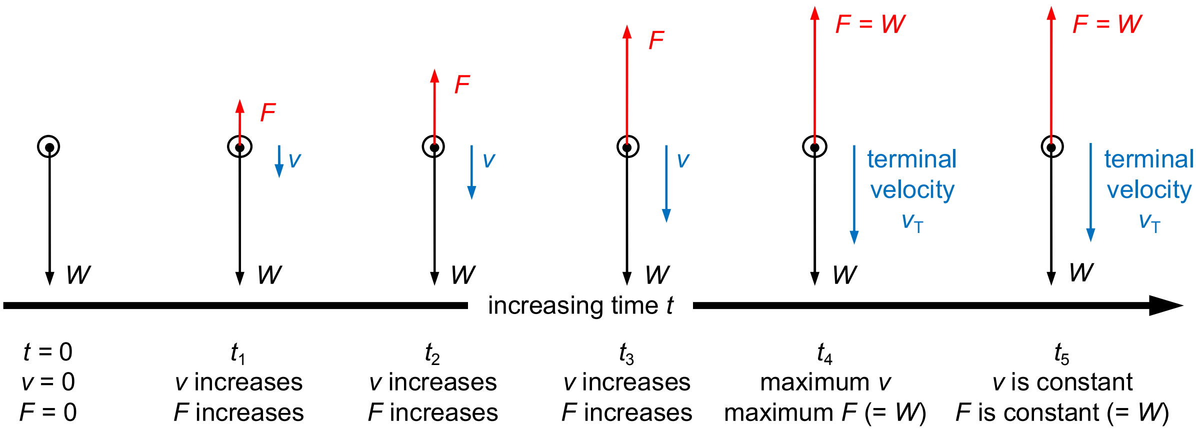


Fig. 5.4

- Between $t = 0$ and t_4, resultant downward force $F_{\text{resultant}} = W - F = mg - F > 0$

From Newton's 2nd law, acceleration $a = \frac{F_{\text{resultant}}}{m} = g - \frac{F}{m}$

Resultant downward force causes the object to accelerate and its speed to increase. The larger speed leads to a larger drag force F. Therefore resultant force and acceleration decrease with time.

- From t_4 onwards, drag force F is equal to the weight W, so resultant force is zero and the object stops accelerating. It will now travel at a constant, maximum velocity known as the **terminal velocity**.

displacement-time, velocity-time and acceleration-time graphs

Fig 5.5 shows the *s-t*, *v-t* and *a-t* graphs of the object falling from a great height in air. When terminal velocity is reached, the *s-t* graph is a straight line and acceleration is zero.

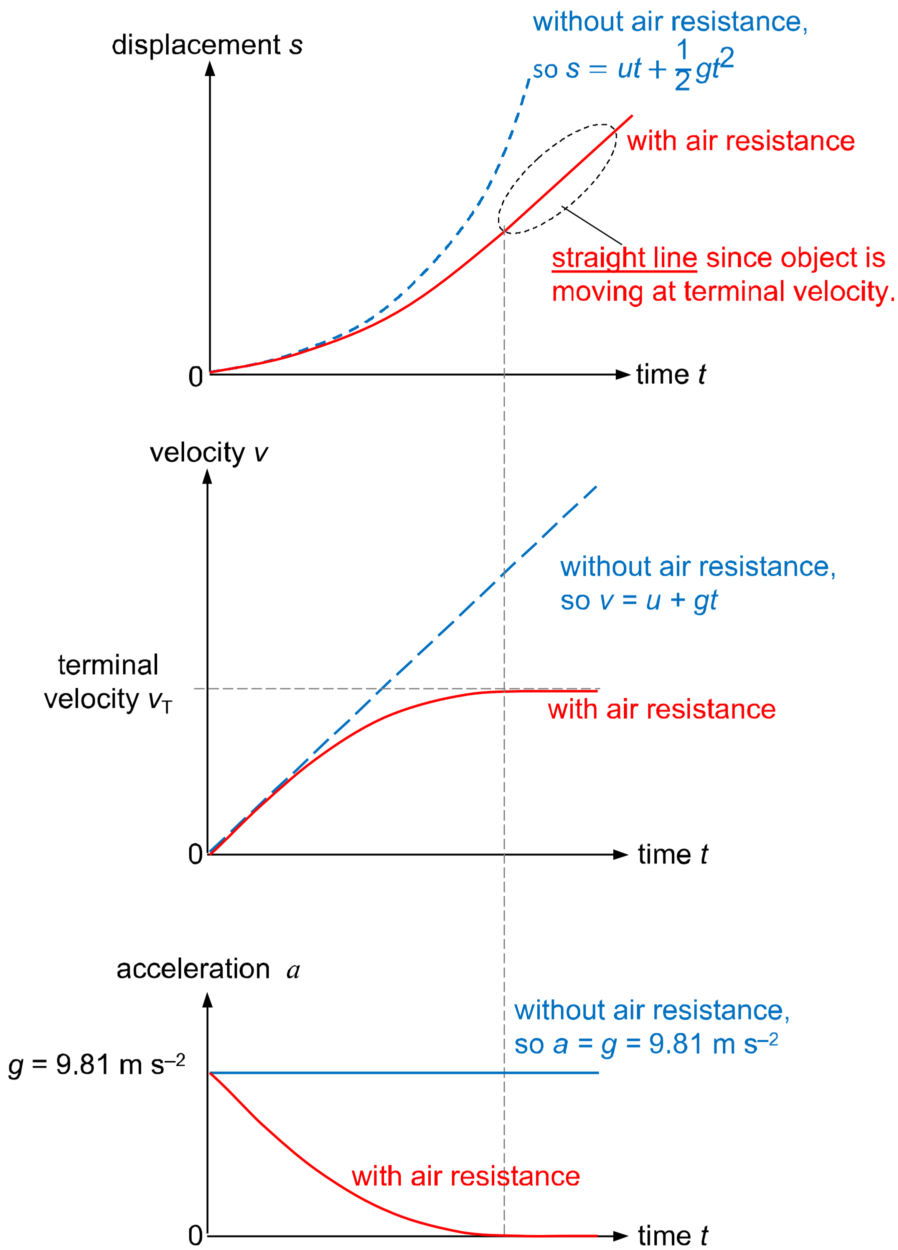


Fig. 5.5

Example 5.3 (Calculating terminal velocity)

An object of mass 3.0 kg is falling vertically from rest. Air resistance R, in newtons, is given by

$$R = 0.6v$$

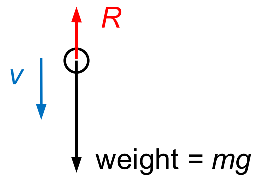
where v is the velocity in metres per second.

Determine

(a) the maximum velocity of the object,

(b) the acceleration of the object when its velocity is 12 m s^{-1}. (2017 P1 Q4)

Solution



(a) maximum velocity (or terminal velocity) occurs when the air resistance equals to weight of object, so

$0.6v = 3.0 \times 9.81$

v = 49 m s^{-1} (2 s.f.)

(b) when v = 12 m s^{-1}, $R = 0.6 \times 12 = 7.2$ N

acceleration $a = \dfrac{\text{resultant force}}{\text{mass}} = \dfrac{3.0 \times 9.81 - 7.2}{3.0}$ = 7.4 m s^{-2} (2 s.f.)

Example 5.4 (Effect of air resistance on an object thrown vertically upwards)

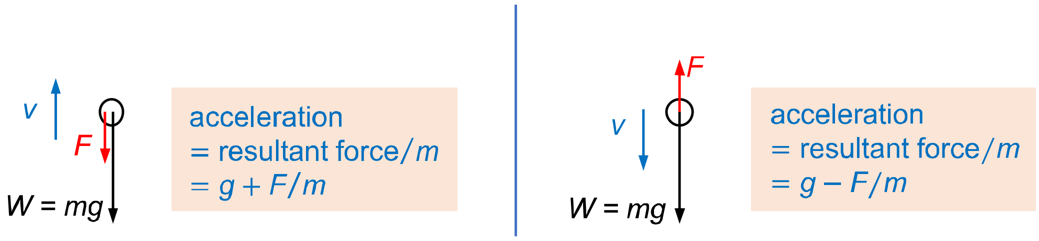
A ball is thrown vertically upwards from ground level with speed u. The times of flight for the upward motion (moving away from the ground) is t_u and the downward motion (returning to the ground) is t_d. Air resistance is **not** negligible.

(a) Explain whether $t_d < t_u$, $t_d = t_u$ or $t_d > t_u$.

(b) Sketch the variation with time t of the velocity v of the ball.

Solution

(a) – At a given speed, the net acceleration force (see below) when the ball is moving downwards is less than the retarding force when it is moving up. Therefore, when the ball is moving downwards, there is a smaller average acceleration.



– The smaller average acceleration leads to a lower velocity when it returns to the ground (since $v^2 = u^2 + 2as \Rightarrow a = \frac{v^2 - u^2}{2s}$).

So, <u>average speed $\bar{v}$ moving up is greater than average speed $\bar{v}'$ moving downwards</u> ($\bar{v} > \bar{v}'$).

– Since the distance s moving up and moving downwards are equal,

$$s = \bar{v} \times t_u = \bar{v}' \times t_d$$

Therefore, $t_d > t_u$.

(b) – Curved graph since changes to drag force is incremental.

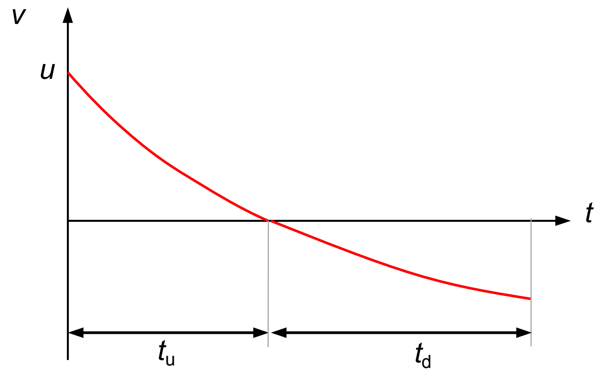
– Numerical value of gradient is always decreasing:

(Up) Resultant downward force decreases with increasing height as ball slows.

(Down)Resultant downward force decreases with decreasing height as ball speeds up.

– Graph ends with lower speed (average speed upwards > average speed downwards).

– Areas left and right of $v = 0$ are equal (since same distance travelled).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