ANDERSON SERANGOON JUNIOR COLLEGE
JC2 Preliminary Examinations 2024
Higher 3

MATHEMATICS — 9820/01

Paper 1

17 September 2024

3 hours

Additional Materials: Answer Booklets
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages and **0** blank page.

Answer **all** the questions.

1 **(a)** A function is concave on an interval if its second derivative is negative within the interval. Show that $\sin x$ is concave for $0 < x < \pi$ and $\ln x$ is concave for $x > 0$. [1]

If f is concave in an interval and $x_1, x_2, \ldots, x_n$ are elements in the interval, *Jensen's inequality* states that

$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{f}(x_i) \le \mathrm{f}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right),$$

and equality holds if and only if $x_1 = x_2 = \ldots = x_n$. You may use this result without proof.

(i) Given that A, B and C are angles of a triangle, prove that

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}.$$ [2]

(ii) By selecting a suitable function f, use Jensen's inequality to derive the AM-GM inequality

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n}$$

for any positive real numbers $a_1, a_2, \ldots, a_n$ and any positive integer *n*. [2]

(b) The product $\mathrm{P}(n)$ is defined, for any positive integer *n*, by

$$P(n) = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{9}{8}\cdot\ldots\ldots\cdot\frac{2^{n-1}+1}{2^{n-1}}.$$

(i) Use the AM-GM inequality to show that $\mathrm{P}(n) < \left(1 + \frac{2}{n}\right)^n$. [4]

(ii) Use binomial theorem to show that $\mathrm{P}(n) < \mathrm{e}^2$ for all positive integers *n*. [3]

2 Given a sequence $w_0, w_1, w_2, \ldots\ldots$, the sequence $F_1, F_2, F_3, \ldots\ldots$ is defined by

$$F_n = w_n^2 + w_{n-1}^2 - 4w_n w_{n-1}.$$

Show that $F_n - F_{n-1} = \left(w_n - w_{n-2}\right)\left(w_n + w_{n-2} - 4w_{n-1}\right)$ for $n \geq 2$. [2]

(a) The sequence $u_0, u_1, u_2, \ldots\ldots$ has $u_0 = 1$ and $u_1 = 2$ and satisfies

$$u_n = 4u_{n-1} - u_{n-2} \quad (n \geq 2).$$

Prove that $u_n^2 + u_{n-1}^2 = 4u_n u_{n-1} - 3$ for $n \geq 1$. [3]

(b) A sequence $v_0, v_1, v_2, \ldots\ldots$ has $v_0 = 1$ and satisfies

$$v_n^2 + v_{n-1}^2 = 4v_n v_{n-1} - 3 \quad (n \geq 1). \qquad (*)$$

(i) Find v_1 and prove that, for each $n \geq 2$, either $v_n = 4v_{n-1} - v_{n-2}$ or $v_n = v_{n-2}$. [3]

(ii) Show that the sequence, with period 2, defined by

$$v_n = \begin{cases} 1 & \text{for } n \text{ even} \\ 2 & \text{for } n \text{ odd} \end{cases}$$

satisfies (*). [2]

(iii) Construct a sequence v_n with period 4 which has $v_0 = 1$, and satisfies (*), justifying your answer. [3]

3 Let f and g be continuous functions on the interval $[a, b]$. By considering $\int_a^b \left(t\mathrm{f}(x) + \mathrm{g}(x)\right)^2 \, \mathrm{d}x$ for $t \in \mathbb{R}$, show the Schwarz inequality:

$$\left(\int_a^b \mathrm{f}(x)\mathrm{g}(x)dx\right)^2 \leq \left(\int_a^b \left(\mathrm{f}(x)\right)^2 dx\right)\left(\int_a^b \left(\mathrm{g}(x)\right)^2 dx\right) \qquad (*)$$

and determine when equality holds. [4]

(a) By setting $\mathrm{f}(x) = 1$ and $\mathrm{g}(x) = \mathrm{e}^x$ in (*) and values of *a* and *b* suitably, show that for $t > 0$,

$$\frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} < \frac{t}{2}.$$ [4]

(b) Use (*) to show that

$$\frac{64}{25\pi} < \int_0^{\frac{1}{2}\pi} \sqrt{\sin x} \, dx < \sqrt{\frac{\pi}{2}}.$$ [7]

4 **(a)** Given that the variables x, y and u are connected by the differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}x}+\mathrm{f}(x)u=\mathrm{h}(x) \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{g}(x)y=u,$$

show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\big(\mathrm{g}(x)+\mathrm{f}(x)\big)\frac{\mathrm{d}y}{\mathrm{d}x}+\big(\mathrm{g}'(x)+\mathrm{f}(x)\mathrm{g}(x)\big)y=\mathrm{h}(x). \qquad (1)$$ [2]

(b) Given that the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(2+\frac{2}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{4}{x}y=\frac{6}{x} \qquad (2)$$

can be written in the same form as (1), find a first order differential equation which is satisfied by $\mathrm{g}(x)$. [2]

If $\mathrm{f}(x)=kx^n$, find a possible value for n and the corresponding value of k. [4]

Hence find a solution of (2) with $y=4$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ at $x=1$. [6]

5 Show that, if $r=(2)(3)(5).....(p)$ is the product of all primes less than or equal to a given odd prime p, then $2r-1$ has a prime factor q such that $q>p$ and $q\equiv 3\pmod 4$. [6]

Deduce that there is an infinite number of primes of the form $4n+3$, where n is an integer. [2]

Describe an infinite sequence of values of the integer n for which the values of $4n+3$ are the odd powers of 3. [3]

6 Let a be an integer. Let p and q be distinct prime numbers and α and β are positive integers.

Show that y is a solution of the congruence

$$y \equiv a \left(\text{mod } p^{\alpha} q^{\beta}\right)$$

if and only if y is a solution of both the congruences of

$$y \equiv a \left(\text{mod } p^{\alpha}\right)$$

and

$$y \equiv a \left(\text{mod } q^{\beta}\right).$$ [6]

Hence, or otherwise, find all the solutions of the congruence

$$x^3 + 10x + 9 \equiv 0 \left(\text{mod } 24\right).$$ [8]

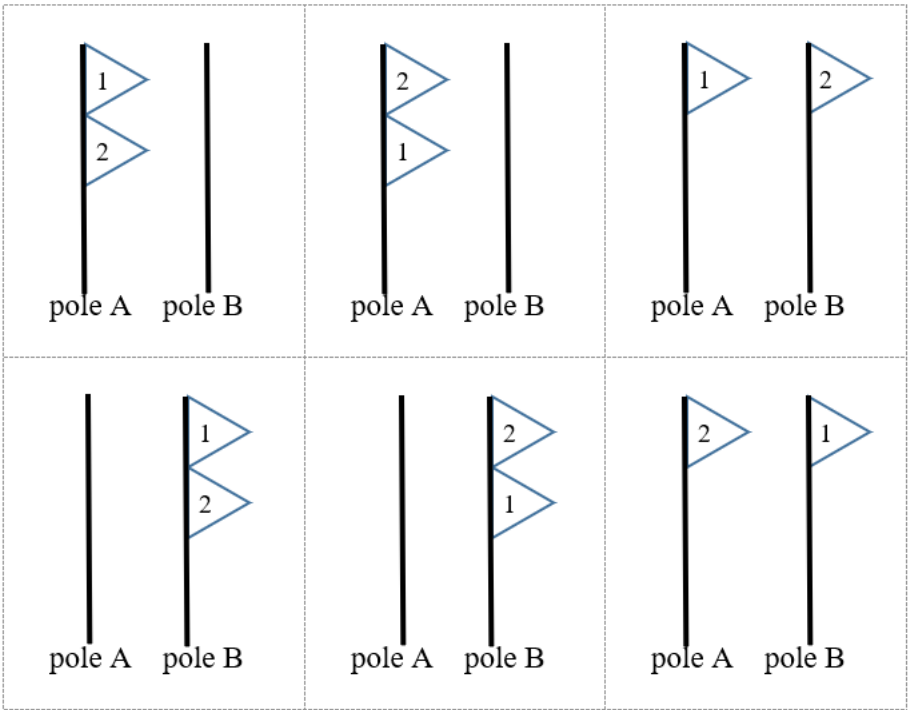
7 **(a)** Explain why the number of ways to distribute r distinct objects, where $r \geq 4$, into $(r-2)$ identical boxes such that exactly 2 boxes each contains exactly 2 objects and that no box is empty is $3\binom{r}{4}$. [3]

(b) Let $S(r,n)$ denote the total number of ways to distribute r distinct objects into n identical boxes such that no box is empty. Use the result in part **(a)** to show that for $r \geq 4$,

$$S(r, r-2) = \frac{r(r-1)(r-2)(3r-5)}{4!}.$$ [4]

(c) Mr Tay bought 10 cookies, each of a different flavour, before distributing them randomly into 8 identical boxes such that no box is empty. He asks one of his student, Xuan Kai, to open up one of these 8 identical boxes. Find the probability that the box contains exactly 2 cookies. [3]

8 In a flag display exhibition, there are m flags and k poles. In the morning, the flags are to be hung on the poles, and poles are allowed to be left empty. The order of which the flags are hung is important. The following diagram shows an illustration of all the six different ways to hang these flags for the case where $m=2$ and $k=2$.



(a) How many ways can

(i) m identical flags

(ii) m distinct flags

be hung on k distinct poles? [2]

At the end of the exhibition, the organiser decides to the take down the flags and keep them in n distinct boxes, such that no box is empty. The order of the flags placed in any box does not matter. The number of ways to keep m distinct flags in n distinct boxes is denoted by $T(m,n)$, where $m \geq n \geq 1$.

(b) State $T(m,1)$ and $T(m,m)$ for any $m \geq 1$. [2]

(c) Show that $T(m,n) = n \times T(m-1,n-1) + T(m-1,n)$ for $m > n > 1$. [4]

(d) By using principle of inclusion and exclusion, show that

$$T(m,n) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)^m .$$ [3]