Work Solution: Calculus Practice

Differentiation: Equation of Tangents and Normal

1 **(i)**

$$y = 4(2x-3)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = 6(2x-3)^{\frac{1}{2}}(2)$$

$$= 12\sqrt{2x-3}$$

(ii)

When $x = 2$,

$$\frac{dy}{dx} = 12$$

$$m_{normal} = -\frac{1}{12}$$

$$y = 4$$

$$\therefore\ (y-4) = -\frac{1}{12}(x-2)$$

$$y = -\frac{1}{12}x + \frac{25}{6}$$

(iii)

For $\frac{dy}{dx} = 12\sqrt{2x-3}$,

$\sqrt{2x-3} > 0$ for $x > \frac{3}{2}$

$12\sqrt{2x-3} > 0$ B1 (allow ECF)

$\therefore\ \frac{dy}{dx} > 0$

Hence, $y = 4(\sqrt{2x-3})^3$ is an increasing function for $x > \frac{3}{2}$.

2	2(a)	$\dfrac{dy}{dx}=\dfrac{(3x-4)(-1)-(2-x)(3)}{(3x-4)^2}$ $=\dfrac{-3x+4-6+3x}{(3x-4)^2}$ $=-\dfrac{2}{(3x-4)^2}$
	2(b)	$2y-16x=3$ $y=8x+\dfrac{3}{2}$ $Gradient\ of\ normal=8$ $Gradient\ of\ tangent=-\dfrac{1}{8}$ $\Rightarrow\ -\dfrac{2}{(3x-4)^2}=-\dfrac{1}{8}$ $16=(3x-4)^2$ $3x-4=4 \quad or \quad 3x-4=-4$ $x=\dfrac{8}{3} \quad or \quad x=0$ $y=-\dfrac{1}{6} \qquad y=-\dfrac{1}{2}$ $Coordinates=\left(\dfrac{8}{3},-\dfrac{1}{6}\right) \quad or \quad \left(0,-\dfrac{1}{2}\right)$

3	(a)	$y=(x-3)\sqrt{2x+3}$ $\frac{dy}{dx}=(2x+3)^{\frac{1}{2}}+\frac{1}{2}(x-3)(2)(2x+3)^{-\frac{1}{2}}$ $=\sqrt{2x+3}+\frac{x-3}{\sqrt{2x+3}}$ $=\frac{2x+3+x-3}{\sqrt{2x+3}}$ $=\frac{3x}{\sqrt{2x+3}}$ $k=3$
	(b)	When $x=11$, $y=(11-3)\sqrt{2(11)+3}=40$ $\frac{dy}{dx}=\frac{3(11)}{\sqrt{2(11)+3}}=\frac{33}{5}$ $y=\frac{33}{5}x+c$ $40=\frac{33}{5}(11)+c$ $c=-32.6$ Equation of tangent is $y=6.6x-32.6$
	(c)	$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$ $5=\frac{33}{5}\times\frac{dx}{dt}$ $\frac{dx}{dt}=\frac{25}{33}\ units/s$
4	$f'(x) = Akxe^{kx} + Ae^{kx}$ $Akxe^{kx} + Ae^{kx} + 2ke^{kx} + 6Axe^{kx} = 0$ $A + 2k = 0$ --------------- (1) $Ak + 6A = 0$ ------------ (2) From (2) $A(k + 6) = 0$, $A = 0$ (rej) or $k = -6$ sub $k = -6$ into (1), $A = 12$	
5	**(a)**	

	$y=\ln\left(xe^{-3x}\right)=\ln x-3x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}-3$ M1 Gradient at point $P=-1\div\frac{1}{2}=-2$ M1 $-2=\frac{1}{x}-3$ $1=\frac{1}{x}$ $x=1$ M1 (equate $\frac{dy}{dx}=-2$ & attempt to solve for x) $y=\ln\left(e^{-3}\right)=-3$ Coordinates of $P=(1,-3)$ A1 **(b)** $y-(-3)=\frac{1}{2}(x-1)$ $y=\frac{1}{2}x-\frac{7}{2}$ M1 (find equation of normal) At Q, $\frac{1}{2}x-\frac{7}{2}=0$ $x=7$ M1 (find x-intercept) Area of triangle OQP $=\frac{1}{2}\times7\times3=10.5\text{ units}^2$ A1

Differentiation: Increasing and Decreasing Function

1

$\frac{dy}{dx} = 2(2x-3)(2)\sqrt{x-1} + (2x-3)^2(\frac{1}{2})(x-1)^{-\frac{1}{2}}$ — (M1)

$= 4(2x-3)\sqrt{x-1} + \frac{(2x-3)^2}{2\sqrt{x-1}}$

$= \frac{8(2x-3)(x-1) + (2x-3)^2}{2\sqrt{x-1}}$

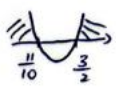
$\frac{8(2x-3)(x-1) + (2x-3)^2}{2\sqrt{x-1}} > 0$ — (M1)

$8(2x-3)(x-1) + (2x-3)^2 > 0$ and $2\sqrt{x-1} > 0$.

$(2x-3)(8x-8+2x-3) > 0$ $\quad\quad$ $\sqrt{x-1} > 0$

$(2x-3)(10x-11) > 0$ $\quad\quad$ $x-1 > 0$

$\underbrace{x > 1}_{\text{(M1)}}$



$x < \frac{11}{10}$ and $x > \frac{3}{2}$

Since $x > 1$,

$\underbrace{1 < x < \frac{11}{10}}_{\text{(A1)}}$ and $\underbrace{x > \frac{3}{2}}_{\text{(A1)}}$

only

2 (a)

$$y = 3 + \left(\frac{x}{2} - 1\right)^4$$

$$\frac{dy}{dx} = 4\left(\frac{x}{2} - 1\right)^3 \bullet \frac{1}{2} = 2\left(\frac{x}{2} - 1\right)^3$$ M1 (find 1st derivative)

Let $\frac{dy}{dx} = 0$,

$$2\left(\frac{x}{2} - 1\right)^3 = 0$$

$$\frac{x}{2} - 1 = 0$$ M1 (equate to zero and attempt to find x)

$$x = 2$$

$$\Rightarrow y = 3$$

Stationary point $= (2, 3)$ A1, A1 (correct pair of coordinates)

(b) **(i)** Justify whether y is increasing or decreasing for values of x less than p. [2]

For $x < 2$,

$$\left.\begin{aligned} &\left(\frac{x}{2} - 1\right)^3 < 0 \\ &\frac{dy}{dx} = 2\left(\frac{x}{2} - 1\right)^3 < 0 \end{aligned}\right\}$$ M1 (use $\left(\frac{x}{2} - 1\right)^3 < 0$ to show $dy/dx > 0$)

Therefore, y is **decreasing** when $x < 2$. A1

(ii) Hence infer whether y is increasing or decreasing for values of x greater than p. [1]

For $x > 2$,

$$\frac{dy}{dx} = \left(\frac{x}{2} - 1\right)^3 > 0$$

Therefore, y is **increasing** when $x > 2$. B1

(c) What do the results of **part (b)** imply about the stationary point? [1]

The stationary point is a **minimum point**. B1

Differentiation: Connected Rate of Change

1	**(i)(a)**

$$V = 15\pi h^2 - \frac{\pi}{3}h^3$$

$$\frac{dV}{dh} = 30\pi h - \pi h^2 \text{ — (M1)}$$

$$= 30\pi(12) - \pi(12)^2$$

$$= 360\pi - 144\pi$$

$$= 216\pi \ \# \text{ — (A1)}$$

Alt. M:

$$\frac{dV}{dh} = 2\pi h\left(15 - \frac{h}{3}\right) + \pi h^2\left(-\frac{1}{3}\right) \text{ — [M1]}$$

$$= 2\pi h\left(15 - \frac{h}{3}\right) - \frac{\pi h^2}{3}$$

$$= 216\pi. \text{ — [A1]}$$

(i)(b)

$$\frac{dV}{dt} = -2\pi.$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$-2\pi = 216\pi \times \frac{dh}{dt} \text{ — (M1)}$$

(award 1m even if students miss out negative sign)

$$\frac{dh}{dt} = \frac{-2\pi}{216\pi}$$

$$= -\frac{1}{108}$$

$$\therefore \text{Rate} = \frac{1}{108} \text{ cm/s.} \text{ — (A1)}$$

(ii)

$$15^2 = (15-h)^2 + r^2 \text{ — (M1)}$$

$$225 = 225 - 30h + h^2 + r^2$$

$$r^2 = 30h - h^2$$

$$r = \sqrt{30h - h^2} \quad \text{(shown)} \ \# \text{ — (A1)}$$

(iii)

$$r = (30h - h^2)^{\frac{1}{2}}$$

$$\frac{dr}{dh} = \frac{1}{2}(30h - h^2)^{-\frac{1}{2}}(30 - 2h) \quad \text{— M1}$$

$$= \frac{15-h}{\sqrt{30h - h^2}}$$

$$= \frac{15-12}{\sqrt{30(12) - 12^2}}$$

$$= 0.20412 \quad \text{— M1}$$

$$\frac{dr}{dt} = \frac{dr}{dh} \times \frac{dh}{dt}$$

$$= 0.20412 \times \left(-\frac{1}{108}\right)$$

$$= -0.00189 \text{ cm/s} \# \quad \text{— A1}$$

(iv)

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r \quad \text{— M1}$$

$$= 2\pi\left(\sqrt{30h - h^2}\right)$$

$$= 2\pi\left(\sqrt{30(12) - 12^2}\right)$$

$$= 92.343.$$

$$\frac{dA}{dt} = 92.343 \times (-0.00189)$$

$$= -0.17452$$

$$= -0.175 \text{ cm}^2/\text{s} \# \quad \text{— A1}$$

2	**(a)** Rate of water out from dispenser = Rate of increase of water in cup. $= \pi(10cm)(10cm)(0.0015\ cm/s)$ — (M1) $= \frac{3\pi}{20}\ cm^3/s$. — (A1) **(b)** $V = \frac{1}{3}\pi r^2 h$ $\frac{r}{4} = \frac{h}{10} \Rightarrow h = \frac{5r}{2}$ — (M1) $V = \frac{1}{3}\pi r^2\left(\frac{5r}{2}\right)$ $= \frac{5\pi r^3}{6}$ — (A1) **(c)** $\frac{dV}{dr} = \frac{5\pi}{6}(3r^2)$ $= \frac{5\pi r^2}{2}$. — (M1) When $V = \frac{20\pi}{3}$, $\frac{5\pi r^3}{6} = \frac{20\pi}{3}$ — (M1) $r^3 = 8$ $r = 2$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\frac{3\pi}{20} = \frac{5\pi r^2}{2} \times \frac{dr}{dt}$ — (M1) $\frac{dr}{dt} = \frac{3\pi}{20} \div \frac{5\pi r^2}{2}$ at $r = 2$, $\frac{dr}{dt} = \frac{3\pi}{20} \div \frac{5\pi(2)^2}{2}$ $= \frac{3}{200}$ — (A1)

3	**(i)** $A = 6x^2$ $\qquad\qquad$ $V = x^3$ $\frac{dA}{dx} = 12x$ $\qquad\qquad$ $\frac{dV}{dx} = 3x^2$ $\frac{dA}{dt} = \frac{dA}{dx}\cdot\frac{dx}{dt}$ $\qquad\qquad$ $\frac{dV}{dt} = \frac{dV}{dx}\cdot\frac{dx}{dt}$ $-10 = (12x)\frac{dx}{dt}$ $\qquad\qquad$ $-45 = (3x^2)\frac{dx}{dt}$ $\frac{dx}{dt} = -\frac{5}{6x}$ $\qquad\qquad$ $\frac{dx}{dt} = -\frac{15}{x^2}$ $\Rightarrow \quad -\frac{5}{6x} = -\frac{15}{x^2}$ $\Rightarrow \quad 5x^2 = 90x$ $\Rightarrow \quad x(x-18) = 0$ $\Rightarrow \quad x = 0\,(rejected) \quad or \quad x = 18\ mm$ **(ii)** $\frac{dx}{dt} = -\frac{5}{6\times 18} = -\frac{5}{108}$

4	$A = 4\pi r^2$ $\frac{dA}{dr} = 8\pi r$ $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $-2 = 8\pi(0.5) \times \frac{dr}{dt}$ $\frac{dr}{dt} = \frac{1}{8\pi(0.5)} \times -2$ $= -\frac{1}{2\pi}$ cm/s $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $= 4\pi(0.5)^2 \times -\frac{1}{2\pi}$ $= -\frac{1}{2} cm^3 / s$ The rate of change of volume at r = 0.5 cm is $-\frac{1}{2} cm^3 / s$.
5	$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ $\frac{dy}{dx} = -2$ $4 \times \left(-\frac{1}{2}\right)(x+3)^{-\frac{3}{2}} = -2$ $x + 3 = 1$ $(-2, 4)$

Differentiation: Maxima and Minima Problems

1 **(a)**

$2\pi r + 4x = 40$ — (M1)

$2\pi r = 40 - 4x$

$r = \dfrac{40 - 40x}{2\pi}$

$= \dfrac{20 - 2x}{\pi}$ — (A1)

(b)

$A = \pi r^2 + 2x(2r)$

$= \pi \left(\dfrac{20-2x}{\pi}\right)^2 + 2x(2)\left(\dfrac{20-2x}{\pi}\right)$ — (M1)

$= \pi\left(\dfrac{400 - 80x + 4x^2}{\pi^2}\right) + \dfrac{4x(20-2x)}{\pi}$

$= \dfrac{400 - 80x + 4x^2 + 80x - 8x^2}{\pi}$

$= \dfrac{400 - 4x^2}{\pi}$ — (A1)

(c)

$\dfrac{dA}{dx} = -\dfrac{4}{\pi}(2x)$

$= -\dfrac{8x}{\pi}$ — (M1)

$-\dfrac{8x}{\pi} = 0 \Rightarrow x = 0$. — (A1)

(d)

$\left.\begin{array}{l} \dfrac{d^2A}{dx^2} = -\dfrac{8}{\pi} \ (<0) \\ \text{Since } \dfrac{d^2A}{dx^2} < 0, \text{ area is a max.} \end{array}\right\}$ (M1).

$\left.\begin{array}{l} \text{When } x = 0,\ r = \dfrac{20 - 2(0)}{\pi}. \\ \qquad = \dfrac{20}{\pi} \\ \text{Yes, I agree. Area is a max when } x = 0 \\ \qquad \text{and } r = \dfrac{20}{\pi}. \end{array}\right\}$ (A1).

2	**(i)**	$\pi r(4r) + \pi r^2 + 2\pi rh = 300$ $5\pi r^2 + 2\pi rh = 300$ $h = \frac{1}{2\pi r}(300 - 5\pi r^2)$ *or accept* $\frac{150}{\pi r} - \frac{5r}{2}$
	(ii)	$V = \frac{1}{3}\pi r^2\left(r\sqrt{15}\right) + \pi r^2 h$ $= \frac{\sqrt{15}}{3}\pi r^3 + \pi r^2\left(\frac{1}{2\pi r}\right)(300 - 5\pi r^2)$ $= \frac{\sqrt{15}}{3}\pi r^3 + 150r - \frac{5}{2}\pi r^3$ $= 150r + \left(\frac{\sqrt{15}}{3} - \frac{5}{2}\right)\pi r^3 \quad (shown)$
	(iii)	$\frac{dV}{dr} = 150 + 3\pi r^2\left(\frac{\sqrt{15}}{3} - \frac{5}{2}\right)$ *For stationary value of* V, $\frac{dV}{dr} = 0.$ $\Rightarrow \quad 150 + 3\pi r^2\left(\frac{\sqrt{15}}{3} - \frac{5}{2}\right) = 0$ $\Rightarrow \quad r^2 = \frac{-150}{3\pi\left(\frac{\sqrt{15}}{3} - \frac{5}{2}\right)}$ $= 13.16412$ $\Rightarrow \quad r = 3.62824\ cm$ *When* $r = 3.62824, \quad V = 362.824\ cm^3$ $\approx 363\ cm^3$ *Now,* $\frac{d^2V}{dr^2} = 6\pi r\left(\frac{\sqrt{15}}{3} - \frac{5}{2}\right) = -82.6848 < 0$ $\Rightarrow \quad V$ *is a maximum.*

3	**(a)** $y = ax + b(2x-1)^{-1}$ $\frac{dy}{dx} = a - \frac{2b}{(2x-1)^2}$ The curve has a stationary point at $A(2, 7)$. $\Rightarrow \quad \frac{dy}{dx} = 0$ $a - \frac{2b}{(2\times 2-1)^2} = 0$ $a = \frac{2b}{9} \quad ...(1)$ Substitute (2, 7) into the curve $y = ax + b(2x-1)^{-1}$: $7 = 2a + \frac{b}{3} \quad ...(2)$ Substitute (1) into (2): $7 = \frac{4b}{9} + \frac{b}{3}$ $7 = \frac{7b}{9}$ $b = 9$ $\therefore \ a = 2, b = 9$ **(b)** $\frac{dy}{dx} = 0$ $2 - \frac{18}{(2x-1)^2} = 0$ $\frac{18}{(2x-1)^2} = 2$ $2x - 1 = \pm 3$ $x = 2, -1$ When $x = -1,\ y = -5$ <u>The coordinates of the other stationary point are (-1, -5).</u> **(c)**

	$\frac{d^2y}{dx^2} = 36(2x-1)^{-3}\cdot 2$ $= \frac{72}{(2x-1)^3}$ At $A(2,\ 7)$, $\frac{d^2y}{dx^2} > 0 \Rightarrow$ minimum point At $(-1,-5).,\frac{d^2y}{dx^2} < 0 \Rightarrow$ maximum point
4	(a) $l = 12x + 13x + 5x + y + y$ $l = 30x + 2y$ (b) $12xy + \frac{1}{2}(12x)(5x) = 96$ $12xy + 30x^2 = 96$ $12xy = 96 - 30x^2$ $y = \frac{8}{x} - \frac{5x}{2}$ $l = 30x + 2\left(\frac{8}{x} - \frac{5x}{2}\right)$ $l = 30x + \frac{16}{x} - 5x$ $l = 25x + \frac{16}{x}$ (*shown*) (c) $\frac{dl}{dx} = 25 - \frac{16}{x^2}$ $when\ \frac{dl}{dx} = 0,$ $25 - \frac{16}{x^2} = 0$ $x^2 = \frac{16}{25}$ $x = \frac{4}{5}$ $\frac{d^2l}{dx^2} = \frac{32}{x^3}$ $when\ x = \frac{4}{5},$ $\frac{d^2l}{dx^2} = 62.5 > 0$ l is a minimum value.
5	$\frac{dy}{dx} = \frac{(x+3)(2x+2)-(x^2+2x+5)(1)}{(x+3)^2}$ $\frac{2x^2+8x+6-x^2-2x-5}{(x+3)^2} = 0$ $x^2 + 6x + 1 = 0$ $x = \frac{-6\pm\sqrt{6^2-4(1)(1)}}{2}$ $= -3 \pm 2\sqrt{2}$ When $x = -3 - 2\sqrt{2}$, y is a maximum When $x = -3 + 2\sqrt{2}$, y is a minimum $P = (-0.172,\ 1.66)$

6 **(a)** Given that the volume of the slope is 240 m^3, show that the surface area of the prism, A m^2, is given by

$$A = 12x^2 + \frac{480}{x}.$$ [4]

$\frac{1}{2}(3x)(4x)L = 240$ [M1] form SA equation

$L = \frac{40}{x^2}$ [M1] L the subject

$\text{Area} = (3x)(4x) + 5xL + 3xL + 4xL$ [M1] Surface Area

$= 12x^2 + 12x(\frac{40}{x^2})$ [A1] Sub,Simplify

$= 12x^2 + \frac{480}{x}$

(b) Given that x and L can vary, find the value of x for which A has a stationary value and determine whether this value of A is maximum or minimum. [5]

$\frac{dA}{dx} = 24x - \frac{480}{x^2}$ [M1]

Solve $\frac{dA}{dx} = 0$ [M1]

$x = 2.71$ (3sf) [A1]

$\frac{d^2A}{dx^2} = 24 + \frac{960}{x^3}$ [M1]

Since $x > 0$, $\frac{d^2A}{dx^2} > 0$ and surface area is minimum. [A1]

**can substitute $x = 2.71$ to show too*