

**ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION**

Higher 2

FURTHER MATHEMATICS

9649/01

Paper 1

15 September 2026

3 hours

Additional Materials: Printed Answer Booklet
 List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.

Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [] at the end of each question or part question.

This document consists of **9** printed pages and **1** blank page.



Anglo-Chinese Junior College

1 Consider $f(x) = (\sin(\pi x) + 0.1)^7$ for $0 \leq x \leq 2$.

(a) Use your calculator to evaluate $\int_0^2 f(x) dx$ to 6 decimal places. [1]

(b) Estimate $\int_0^2 f(x) dx$ using Simpson's Rule with four strips. Give your answer to 6 decimal places. [2]

A student estimates $\int_0^2 f(x) dx$ using the trapezium rule with four strips, and finds that the percentage error of the estimate is 58.5%.

(c) Calculate the percentage error when Simpson's Rule is used in part **(b)**. [1]

(d) Sketch the graph of $y = f(x)$ and explain why the percentage error is greater when using Simpson's Rule than the trapezium rule. [3]

2 A family of polar curves has equations

$$C_n : r = ak^{n-1}(1 + \cos \theta), \text{ for } 0 \leq \theta \leq \pi$$

where n is a positive integer, $a > 0$ and $0 < k < 1$.

(a) Without using a calculator, find the arc length of C_1 , leaving your answer in terms of a . [4]

The area of the region bounded by C_n and the horizontal axis is denoted by A_n .

(b) Show that $\sum_{n=1}^{\infty} A_n$ is finite. [4]

3 It is given that the complex number $z = e^{i\alpha} + e^{i\beta}$, where $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$ and $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$.

(a) Show that $\arg(z) = \frac{1}{2}(\alpha + \beta)$ and write down $|z|$ as a single trigonometric expression in terms of α and β . [2]

(b) By considering $(e^{i\alpha} + e^{i\beta})^2$, prove that the identity

$$\tan(\alpha + \beta) = \frac{\sin 2\alpha + 2\sin(\alpha + \beta) + \sin 2\beta}{\cos 2\alpha + 2\cos(\alpha + \beta) + \cos 2\beta}.$$

holds if $\alpha + \beta \neq \pm\frac{1}{2}\pi$. [2]

The complex numbers p , q and r have modulus 1 and positive real part.

(c) Deduce that $\frac{(p+q)(q+r)(r+p)}{pqr}$ is a real number. [3]

4 Consider the differential equation

$$\frac{dy}{dx} = \frac{\sin(\frac{13}{4}\pi x)}{2y},$$

with $y=1$ when $x=1$.

(a) Find the value of y when $x = \frac{5}{4}$, giving your answer correct to 4 decimal places. [3]

(b) Use one step of both the Euler method and the improved Euler method to calculate approximations to the value of y when $x = \frac{5}{4}$, giving each answer correct to 4 decimal places. [4]

(c) Determine which method is better and explain why the method with the worse approximation did not perform as well. [3]

- 5 Let $\mathbf{A}$ be a 3×3 real matrix with eigenvalues 1, 2, 3 and corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

- (a) Explain why $\mathbf{A}$ is diagonalisable. Hence find $\mathbf{A}$. [3]

- (b) Find all vectors $\mathbf{y} \in \mathbb{R}^3$ satisfying $\mathbf{A}^{100}\mathbf{y} = \mathbf{A}^{101}\mathbf{y}$. [3]

- (c) Find all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $\lim_{n \rightarrow \infty} \frac{1}{3^n} \mathbf{A}^n \mathbf{x} = \mathbf{0}$. [3]

- 6 In an Argand diagram, the point P represents the complex number z satisfying

$$|z - 9 + 4i| = 3|z - 1 - 4i|.$$

Let the locus of P be the curve C .

- (a) By letting $z = x + iy$ and finding the Cartesian equation of C , show that C is a circle, and state the centre and radius of C . [3]

- (b) Find the exact minimum value of $\arg(z - 9 + 4i)$. [2]

A given complex number a with $0 < \arg(a) < \frac{1}{2}\pi$ has the property that there is exactly one complex number w that satisfies

$$|w - 9 + 4i| = 3|w - 1 - 4i| \quad \text{and} \quad \left| \frac{w - a^*}{w + a^*} \right| = 1.$$

- (c) Find all possible value(s) of a with this property, and state the corresponding value(s) of w in each case. [5]

- 7 A surface S is defined implicitly by the equation $z^3 + 12z = x^3 + 3xy^2 - 15x - 12y - 4$ for all real values of x, y and z .

- (a) Show that the x -coordinates of the stationary points of the surface S satisfy the equation

$$x^4 - 5x^2 + 4 = 0.$$

Hence, find the coordinates of all four stationary points of S . [5]

- (b) Using the second partial derivative test, determine the nature of each of these four stationary points. [5]

- (c) Let (x_0, y_0, z_0) be the coordinates of the stationary point where z has a minimum value. Find the quadratic approximation of z in the region of (x_0, y_0) . [3]

Show that this quadratic approximation can be expressed in the form $z \approx \mathbf{X}^T \mathbf{M} \mathbf{X} + c$ where

$\mathbf{X} = \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$ for real constants c, p, q and r , and $\mathbf{X}^T \mathbf{M} \mathbf{X}$ is a scalar and

solve the matrix $\mathbf{M}$. [2]

- 8 (a) Suppose S_1 and S_2 are the column spaces of matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ where

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 2 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & k \end{pmatrix}.$$

Determine the possible values of a, b, c and k such that S_1 is a subspace of S_2 . [5]

- (b) The matrix $\mathbf{M}$ is defined as

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The matrix $\mathbf{M}$ defines a linear transformation from $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Find a basis for

- (i) the row space of $\mathbf{M}$; [2]
 (ii) the null space of $\mathbf{M}$. [2]

- 9 A curve has equation

$$y = x \sin^{-1} x \text{ for } 0 \leq x \leq \frac{1}{2}\sqrt{3}.$$

The length of the curve is denoted by L .

- (a) Explain, with the aid of a diagram, why L must satisfy the inequality

$$L > \sqrt{\frac{3}{4} + \frac{\pi^2}{12}}. \quad [2]$$

- (b) Write down the integral that represents L and find its numerical value. [2]

The region R is bounded by the curve $y = x \sin^{-1} x$, the y -axis and the line $y = \frac{\sqrt{3}}{6}\pi$. The region R is rotated completely about the y -axis to form a solid of revolution S .

- (c) Show that the volume of S can be expressed as

$$\frac{\sqrt{3}\pi^2}{24} + \frac{2\pi}{3} \int_0^k f(x) \, dx,$$

where $f(x)$ is to be determined. State the exact value of the constant k .

Hence, use the substitution $t = 1 - x^2$ to find the exact volume of S . [8]

- 10 A physicist is studying steady-state heat conduction on thin rectangular plates with non-uniform heating. In one particular experiment, the physicist uses a thin rectangular plate occupying the region $0 \leq x \leq 1$, $0 \leq y \leq \frac{1}{2}\pi$. The steady-state temperature $u(x, y)$, in $^{\circ}\text{C}$, at any point (x, y) on the plate is governed by Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (3 - 2x^2) \cos y.$$

The physicist uses an ice-water bath to maintain the temperature of the top edge at 0°C , and uses programmable heating strips to control the temperature profiles of the remaining edges of the plate such that

- $u(0, y) = \cos y$,
- $u(1, y) = 4 \cos y$, and
- $u(x, 0) = v(x)$, where v is a function dependent only on x ,

as shown in Figure 1 below.

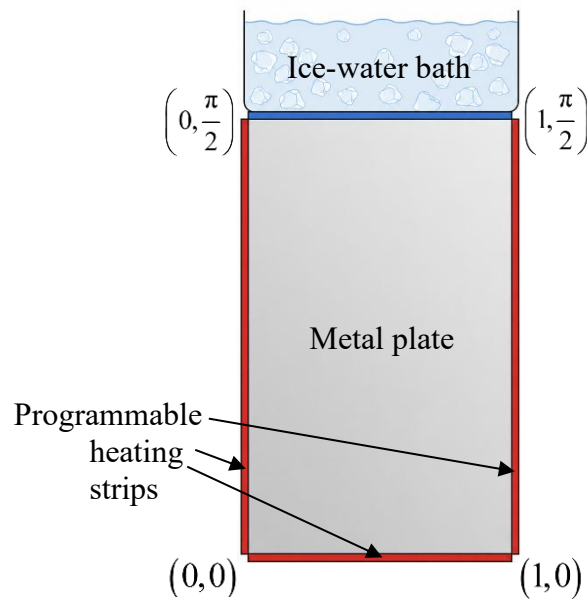


Figure 1

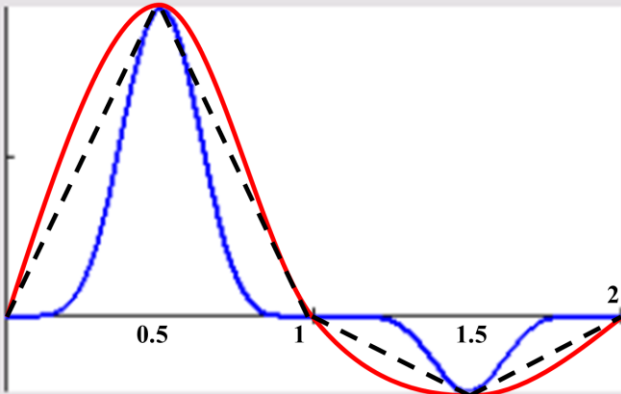
The physicist models the steady-state temperature of the plate using a separable solution of the form $u = v \cos y$.

- (a) Show that Poisson's equation can be reduced to

$$\frac{d^2 v}{dx^2} - v = 3 - 2x^2. \quad [2]$$

- (b) Find the values of $v(0)$ and $v(1)$. Hence, find u in terms of x and y exactly. [8]
- (c) Without the use of a calculator, use your answer from part (b) to determine, with explanation, the coordinates of the point on the plate with the highest steady-state temperature. [3]

Prelim P1 Solutions

1(a)	$\int_0^2 f(x) dx = 0.463960$ (to 6 dp)	
1(b)	$\int_0^2 f(x) dx \approx \frac{1}{6} [f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)]$ $= 0.980280 \text{ (to 6 dp)}$	
1(c)	$\% \text{ Error} = \frac{0.980280 - 0.463960}{0.463960} \times 100\%$ $= 111\%$	
1(d)	The graph of $y = f(x)$ is sketched in blue.  For each pair of strips, Simpson's Rule approximates the actual curve by using a <u>quadratic curve</u> passing through the three points, as shown in red. However, for the same strip, the trapezium rule uses a <u>straight line</u> between two points to approximate the actual curve, as shown in black. From the sketch, there is a greater error when approximating the actual curve using Simpson's Rule than the trapezium rule.	
2(a)	$C_1 : r = a(1 + \cos \theta), \text{ for } 0 \leq \theta \leq \pi$ $\frac{dr}{d\theta} = a(-\sin \theta)$ Total arc length of C_1 $= \int_0^\pi \sqrt{a^2(1 + \cos \theta)^2 + (-a \sin \theta)^2} d\theta$ $= a \int_0^\pi \sqrt{1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta$ $= a \int_0^\pi \sqrt{2 + 2\cos \theta} d\theta$ $= a \int_0^\pi \sqrt{2 + 2\left(2\cos^2 \frac{\theta}{2} - 1\right)} d\theta$ $= 2a \int_0^\pi \cos \frac{\theta}{2} d\theta$ $= 2a \left[2 \sin \frac{\theta}{2} \right]_0^\pi$ $= 4a$	

2(b)	$A_n = \frac{1}{2} \int_0^\pi a^2 k^{2n-2} (1 + \cos \theta)^2 d\theta$ $= \frac{a^2 k^{2n-2}}{2} \int_0^\pi (1 + \cos \theta)^2 d\theta$ The integral $\int_0^\pi (1 + \cos \theta)^2 d\theta$ is finite, positive and independent of n. Alternatively, $A_n = \frac{a^2 k^{2n-2}}{2} \int_0^\pi 1 + 2 \cos \theta + \cos^2 \theta d\theta$ $= \frac{a^2 k^{2n-2}}{2} \int_0^\pi 1 + 2 \cos \theta + \frac{1 + \cos 2\theta}{2} d\theta$ $= \frac{a^2 k^{2n-2}}{2} \int_0^\pi \frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta d\theta$ $= \frac{a^2 k^{2n-2}}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^\pi$ $= \frac{a^2 k^{2n-2}}{2} \left[\frac{3}{2} \pi - 0 \right]$ $= \frac{3a^2 k^{2n-2}}{4} \pi$ $\frac{A_{n+1}}{A_n} = \frac{\frac{a^2 k^{2n}}{2} \int_0^\pi (1 + \cos \theta)^2 d\theta}{\frac{a^2 k^{2n-2}}{2} \int_0^\pi (1 + \cos \theta)^2 d\theta} = k^2$ So A_n is a geometric progression. Since $0 < k < 1 \Rightarrow 0 < k^2 < 1$, this geometric progression is convergent and hence $\sum_{n=1}^{\infty} A_n$ is finite.	
3(a)	$e^{i\alpha} + e^{i\beta} = e^{i\left(\frac{\alpha+\beta}{2}\right)} \left[e^{i\left(\frac{\alpha-\beta}{2}\right)} + e^{-i\left(\frac{\alpha-\beta}{2}\right)} \right]$ $= e^{i\left(\frac{\alpha+\beta}{2}\right)} \times 2 \operatorname{Re} \left(e^{i\left(\frac{\alpha-\beta}{2}\right)} \right)$ $= 2 \cos \left(\frac{\alpha-\beta}{2} \right) e^{i\left(\frac{\alpha+\beta}{2}\right)}$ Hence, $\arg(z) = \frac{\alpha+\beta}{2}$ and $z = \left 2 \cos \left(\frac{\alpha-\beta}{2} \right) \right$. Note: $\arg(z)$ can be found geometrically by considering the appropriate rhombus in an Argand diagram and using the fact that	

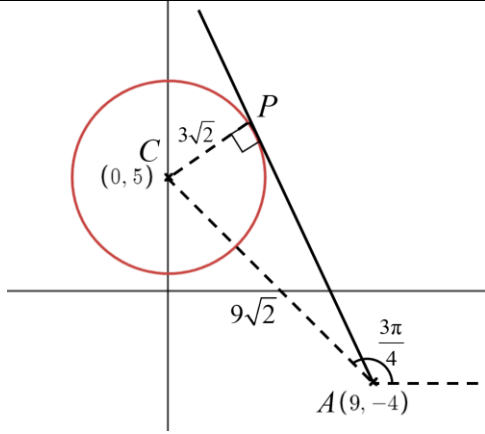
	diagonals bisect interior angles. Then, z can be found by applying the cosine rule. Note that since $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$ and $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$, we have $\frac{1}{2}\pi < \frac{\alpha - \beta}{2} < \frac{1}{2}\pi$ so $\cos\left(\frac{\alpha - \beta}{2}\right)$ is positive.	
3(b)	From (i), $\arg(z^2) = 2\arg(z) = \alpha + \beta$ On the other hand, $z^2 = (e^{i\alpha} + e^{i\beta})^2$ $= e^{i2\alpha} + 2e^{i(\alpha+\beta)} + e^{i2\beta}$ $= [\cos 2\alpha + 2\cos(\alpha + \beta) + \cos 2\beta]$ $+ i[\sin 2\alpha + 2\sin(\alpha + \beta) + \sin 2\beta]$ Since $\tan[\arg(z^2)] = \frac{\operatorname{Im}(z^2)}{\operatorname{Re}(z^2)}$, $\tan(\alpha + \beta) = \frac{\sin 2\alpha + 2\sin(\alpha + \beta) + \sin 2\beta}{\cos 2\alpha + 2\cos(\alpha + \beta) + \cos 2\beta} \quad (\text{proven})$	
3(c)	Let $p = e^{i\alpha}, q = e^{i\beta}, r = e^{i\gamma}$, where $-\frac{1}{2}\pi < \alpha, \beta, \gamma < \frac{1}{2}\pi$. Applying (a) and properties of argument: $\arg\left[\frac{(p+q)(q+r)(r+p)}{pqr}\right]$ $= \arg(e^{i\alpha} + e^{i\beta}) + \arg(e^{i\beta} + e^{i\gamma}) + \arg(e^{i\gamma} + e^{i\alpha})$ $- \arg(e^{i\alpha}) - \arg(e^{i\beta}) - \arg(e^{i\gamma})$ $= \frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} + \frac{\gamma + \alpha}{2} - \alpha - \beta - \gamma$ $= 0$ Thus, $\frac{(p+q)(q+r)(r+p)}{pqr}$ is a real number. Alternative Method: $\frac{(p+q)(q+r)(r+p)}{pqr} = \left(\frac{p+q}{r}\right)\left(\frac{q+r}{p}\right)\left(\frac{r+p}{q}\right)$ $= \left(\frac{p}{r} + \frac{q}{r}\right)\left(\frac{q}{p} + \frac{r}{p}\right)\left(\frac{r}{q} + \frac{p}{q}\right)$ $= 2 + \frac{p}{r} + \frac{r}{p} + \frac{q}{r} + \frac{r}{q} + \frac{q}{p} + \frac{p}{q}$ Note that $\frac{p}{q} = e^{i(\alpha-\beta)}$ and $\frac{q}{p} = e^{i(\beta-\alpha)} = \left(\frac{p}{q}\right)^*$ so $\frac{p}{q} + \frac{q}{p}$ is real. Similarly, $\frac{p}{r} + \frac{r}{p}$ and $\frac{q}{r} + \frac{r}{q}$ are real so the entire sum is real.	

4(a)	$\frac{dy}{dx} = \frac{\sin\left(\frac{13\pi}{4}x\right)}{2y}$ $2y \frac{dy}{dx} = \sin\left(\frac{13\pi}{4}x\right)$ $\int 2y \, dy = \int \sin\left(\frac{13\pi}{4}x\right) \, dx$ $y^2 = -\frac{4}{13\pi} \cos\left(\frac{13\pi}{4}x\right) + c.$ Sub (1,1): $1^2 = -\frac{4}{13\pi} \cos\left(\frac{13\pi}{4}\right) + c$ $c = \frac{4}{13\pi} \cos\left(\frac{13\pi}{4}\right) + 1.$ Thus, $y^2 = -\frac{4}{13\pi} \cos\left(\frac{13\pi}{4}x\right) + \frac{4}{13\pi} \cos\left(\frac{13\pi}{4}\right) + 1$ $y = \sqrt{\frac{4}{13\pi} \left(\cos\left(\frac{13\pi}{4}\right) - \cos\left(\frac{13\pi}{4}x\right) \right) + 1} \quad (\because y(1) = 1 > 0)$ When $x = \frac{5}{4}$, $y = 0.913611 \text{ (6 d.p.)}$ $= 0.9136 \text{ (4 d.p.)}$	
4(b)	Euler method: $y(1.25) \approx 1 + \frac{1}{4} \left(\frac{\sin\left(\frac{13\pi}{4}\right)}{2} \right)$ $= 0.911612 \text{ (6 d.p.)}$ $= 0.9116 \text{ (4 d.p.)}$ Improved Euler method: $y\left(\frac{5}{4}\right) \approx 1 + \frac{1}{4} \left(\frac{1}{2} \right) \left(\frac{\sin\left(\frac{13\pi}{4}\right)}{2} + \frac{\sin\left(\frac{13\pi}{4} \times \frac{5}{4}\right)}{2(0.911612)} \right)$ $= 0.969181 \text{ (6 d.p.)}$ $= 0.9692 \text{ (4 d.p.)}$	
4(c)	<u>Comparing absolute errors:</u> Euler: $0.9136 - 0.9116 = 0.002$ Improved Euler: $0.9136 - 0.9692 = 0.0556 > 0.002$ The Euler method provided the better approximation. $\left. \frac{dy}{dx} \right _{(1,1)} = \frac{\sin\left(\frac{13\pi}{4}\right)}{2} = -0.3536 < 0$ $\left. \frac{dy}{dx} \right _{\left(\frac{5}{4}, 0.911612\right)} = \frac{\sin\left(\frac{13\pi}{4} \times \frac{5}{4}\right)}{2(0.911612)} = 0.1070 > 0$	

	which is a change in sign of the gradient values. Since the improved Euler uses the average of these two gradient values (which partially cancel each other out) in computing an approximation, it produced a worse approximation compared to the Euler method. OR Stationary points occur when $\frac{dy}{dx} = 0$: $\frac{\sin\left(\frac{13\pi}{4}x\right)}{2y} = 0$ $\frac{13\pi}{4}x = k\pi \text{ where } k \in \mathbb{Z}$ $x = \frac{4k}{13}.$ <table><tr><th>k</th><th>x</th></tr><tr><td>3</td><td>$\frac{4}{13} \times 3 = 0.92308$</td></tr><tr><td>4</td><td>$\frac{4}{13} \times 4 = 1.2308$</td></tr><tr><td>5</td><td>$\frac{4}{13} \times 5 = 1.5385$</td></tr></table> Since there is exactly one stationary point in the interval $(1, 1.25)$ at $x = 1.2308$ which is very close to 1.25, the two gradient values used in the improved Euler method have different signs, partially cancelling each other out. This led to the Euler method producing a better approximation than the improved Euler method.	k	x	3	$\frac{4}{13} \times 3 = 0.92308$	4	$\frac{4}{13} \times 4 = 1.2308$	5	$\frac{4}{13} \times 5 = 1.5385$	
k	x									
3	$\frac{4}{13} \times 3 = 0.92308$									
4	$\frac{4}{13} \times 4 = 1.2308$									
5	$\frac{4}{13} \times 5 = 1.5385$									
5(a)	The three given eigenvectors are linearly independent. Hence $\mathbf{P}$ is invertible. $\mathbf{A} = \mathbf{PDP}^{-1} \text{ where } \mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} & \frac{1}{2} \\ -1 & 1 & 2 \end{pmatrix} \text{ (from GC)}$									
5(b)	Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are three linearly independent vectors in $\mathbb{R}^3$, they span $\mathbb{R}^3$. Hence any vector $\mathbf{y} \in \mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.									

	$\mathbf{y} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$ $\mathbf{A}^{100} \mathbf{y} = \alpha_1 \mathbf{A}^{100} \mathbf{v}_1 + \alpha_2 \mathbf{A}^{100} \mathbf{v}_2 + \alpha_3 \mathbf{A}^{100} \mathbf{v}_3$ $= \alpha_1 1^{100} \mathbf{v}_1 + \alpha_2 2^{100} \mathbf{v}_2 + \alpha_3 3^{100} \mathbf{v}_3$ $\mathbf{A}^{101} \mathbf{y} = \alpha_1 1^{101} \mathbf{v}_1 + \alpha_2 2^{101} \mathbf{v}_2 + \alpha_3 3^{101} \mathbf{v}_3$ $\mathbf{A}^{100} \mathbf{y} = \mathbf{A}^{101} \mathbf{y}$ $\Rightarrow 0 \cdot \alpha_1 \mathbf{v}_1 - 2^{100} \alpha_2 \mathbf{v}_2 - 2 \cdot 3^{100} \alpha_3 \mathbf{v}_3 = \mathbf{0}$ Since $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, $\alpha_1 \in \mathbb{R}, \quad \alpha_2 = 0, \quad \alpha_3 = 0$ $\mathbf{y} = t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, t \in \mathbb{R}$ Alternative method: Since $\mathbf{A}$ is invertible, $\mathbf{A}^{100}$ is invertible, $\mathbf{A}^{100} \mathbf{y} = \mathbf{A}^{101} \mathbf{y}$ $\mathbf{y} = \mathbf{A} \mathbf{y}$ $\mathbf{A} \mathbf{y} = 1 \cdot \mathbf{y}$ Hence $\mathbf{y}$ is an eigenvector of $\mathbf{A}$ with eigenvalue of 1, so $\mathbf{y}$ is a scalar multiple of $\mathbf{v}_1$. $\therefore \mathbf{y} = t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, t \in \mathbb{R}$	
5(c)	Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are three linearly independent vectors in $\mathbb{R}^3$, they span $\mathbb{R}^3$. Hence any vector $\mathbf{x} \in \mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Let $\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$ $\mathbf{A}^n \mathbf{x} = c_1 \mathbf{A}^n \mathbf{v}_1 + c_2 \mathbf{A}^n \mathbf{v}_2 + c_3 \mathbf{A}^n \mathbf{v}_3 = c_1 (1^n) \mathbf{v}_1 + c_2 (2^n) \mathbf{v}_2 + c_3 (3^n) \mathbf{v}_3$ $\frac{\mathbf{A}^n \mathbf{x}}{3^n} = c_1 \left(\frac{1}{3} \right)^n \mathbf{v}_1 + c_2 \left(\frac{2}{3} \right)^n \mathbf{v}_2 + c_3 \mathbf{v}_3$ $\lim_{n \rightarrow \infty} \frac{\mathbf{A}^n \mathbf{x}}{3^n} = c_3 \mathbf{v}_3 = \mathbf{0}$ $\Rightarrow c_3 = 0 \text{ since } \mathbf{v}_3 \neq \mathbf{0}$ $\mathbf{x} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}$ Alternative method:	

	$\lim_{n \rightarrow \infty} \frac{1}{3^n} \mathbf{A}^n = \lim_{n \rightarrow \infty} \frac{1}{3^n} \mathbf{P} \mathbf{D}^n \mathbf{P}^{-1}$ $= \lim_{n \rightarrow \infty} \frac{1}{3^n} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}$ $= \lim_{n \rightarrow \infty} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{3}\right)^n & 0 & 0 \\ 0 & \left(\frac{2}{3}\right)^n & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ To solve $\lim_{n \rightarrow \infty} \frac{1}{3^n} \mathbf{A}^n \mathbf{x} = \mathbf{0}$: $\left(\begin{array}{ccc c} 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$ If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ then $x_1 - x_2 - x_3 = 0$ so $\mathbf{x} = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \lambda, \mu \in \mathbb{R}.$	
6(a) AO3	Let $z = x + iy$, where $x, y \in \mathbb{R}$, $ (x-9) + (y+4)i = 3 (x-1) + (y-4)i $ $\sqrt{(x-9)^2 + (y+4)^2} = 3\sqrt{(x-1)^2 + (y-4)^2}$ $(x-9)^2 + (y+4)^2 = 9(x-1)^2 + 9(y-4)^2$ $x^2 - 18x + 81 + y^2 + 8y + 16 = 9x^2 - 18x + 9 + 9y^2 - 72y + 144$ $8x^2 + 8y^2 - 80y + 56 = 0$ $x^2 + y^2 - 10y + 7 = 0$ $x^2 + (y-5)^2 = 18$ $x^2 + (y-5)^2 = (3\sqrt{2})^2$ This is a circle centred at $5i$ with radius $3\sqrt{2}$.	
6(b) AO2	Let the centre of the circle be C, the point $9 - 4i$ be A, and the point for which the required argument is minimised be P. $5i - (9 - 4i) = -9 + 9i$	



Thus, $AC = |-9 + 9i| = 9\sqrt{2}$, and $\arg(-9 + 9i) = \frac{3\pi}{4}$.

$$\sin \angle CAP = \frac{3\sqrt{2}}{9\sqrt{2}} \Rightarrow \angle CAP = \sin^{-1}\left(\frac{1}{3}\right)$$

$$\text{Minimum } \arg(z - 9 + 4i) = \frac{3\pi}{4} - \sin^{-1}\left(\frac{1}{3}\right)$$

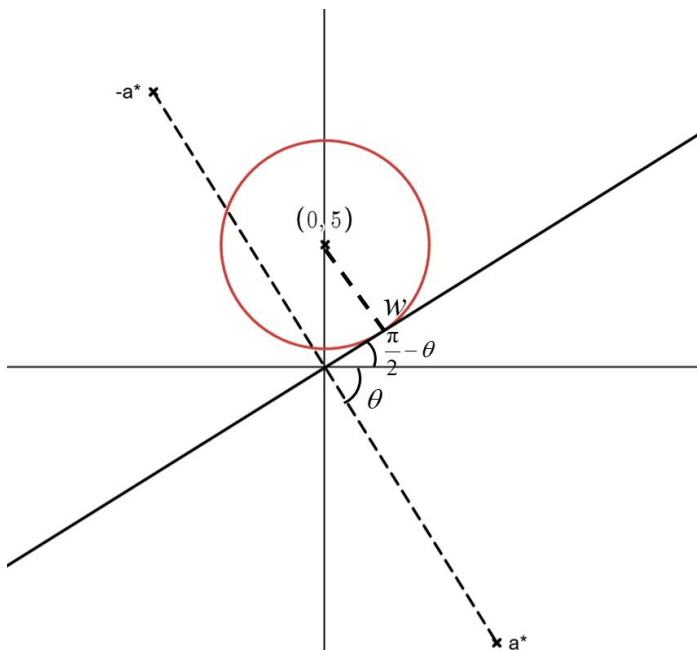
6(c)
AO2

If $\arg(a) = \theta$, then $\arg(a^*) = -\theta$ and $\arg(-a^*) = \pi - \theta$.

$\left| \frac{w - a^*}{w + a^*} \right| = 1 \Rightarrow |w - a^*| = |w + a^*|$, i.e. w lies on the perpendicular bisector of the line segment joining a^* and $-a^*$.

This perpendicular bisector passes through the origin at an angle of $\frac{\pi}{2} - \theta$ with respect to the positive real axis.

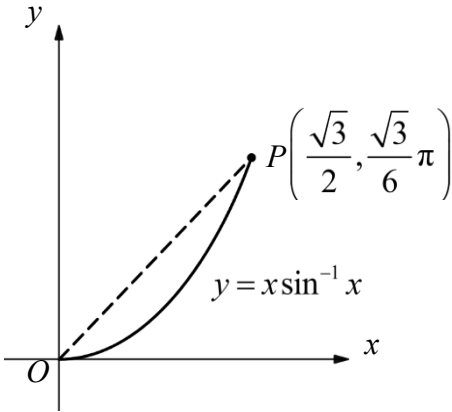
The given property implies that the perpendicular bisector is tangent to the locus of P , with w as the point of tangency.



	So, $\arg(w) = \frac{\pi}{2} - \sin^{-1}\left(\frac{3\sqrt{2}}{5}\right) \Rightarrow \theta = \sin^{-1}\left(\frac{3\sqrt{2}}{5}\right)$, And, $w = \sqrt{5^2 - (3\sqrt{2})^2} = \sqrt{7}$. Therefore, $a = re^{i\theta}$ for all $r > 0$ and $\theta = \sin^{-1}\left(\frac{3\sqrt{2}}{5}\right)$, and the corresponding $w = \sqrt{7}e^{i\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{3\sqrt{2}}{5}\right)\right)}$ or $\sqrt{7}e^{i\left(\sin^{-1}\left(\frac{\sqrt{7}}{5}\right)\right)}$.	
7(a)	$z^3 + 12z = x^3 + 3xy^2 - 15x - 12y - 4$ Differentiate partially with respect to x, $(3z^2 + 12)\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 15 \quad \dots (1) \Rightarrow \frac{\partial z}{\partial x} = \frac{x^2 + y^2 - 5}{z^2 + 4}$ Differentiate partially with respect to y, $(3z^2 + 12)\frac{\partial z}{\partial y} = 6xy - 12 \quad \dots (2) \Rightarrow \frac{\partial z}{\partial y} = \frac{2(xy - 2)}{z^2 + 4}$ For stationary points of S, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ Since $z^2 + 4 > 0$ for all $z \in \mathbb{R}$, $xy - 2 = 0$ and $x^2 + y^2 - 5 = 0$ Substitute $y = \frac{2}{x}$ into $x^2 + y^2 - 5 = 0$, $x^2 + \frac{4}{x^2} - 5 = 0$ $x^4 - 5x^2 + 4 = 0 \text{ (shown)}$ $(x^2 - 4)(x^2 - 1) = 0$ $(x + 2)(x - 2)(x + 1)(x - 1) = 0$ $x = -2$ or 2 or -1 or 1 $y = -1$ or 1 or -2 or 2 At $(-2, -1)$, $z^3 + 12z - 24 = 0 \Rightarrow z = 1.64 \therefore (-2, -1, 1.64)$ At $(2, 1)$, $z^3 + 12z + 32 = 0 \Rightarrow z = -2 \therefore (2, 1, -2)$ At $(-1, -2)$, $z^3 + 12z - 22 = 0 \Rightarrow z = 1.5331 \therefore (-1, -2, 1.53)$ At $(1, 2)$, $z^3 + 12z + 30 = 0 \Rightarrow z = -1.9149 \therefore (1, 2, -1.91)$	
7(b)	Differentiate (1) partially with respect to x, $6z\left(\frac{\partial z}{\partial x}\right)^2 + (3z^2 + 12)\frac{\partial^2 z}{\partial x^2} = 6x$ Differentiate (1) partially with respect to y, $6z\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + (3z^2 + 12)\frac{\partial^2 z}{\partial y \partial x} = 6y$ Differentiate (2) partially with respect to y,	

	$6z\left(\frac{\partial z}{\partial y}\right)^2 + (3z^2 + 12)\frac{\partial^2 z}{\partial y^2} = 6x$ For stationary points of S, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ $\therefore \frac{\partial^2 z}{\partial x^2} = \frac{2x}{z^2 + 4}, \frac{\partial^2 z}{\partial y \partial x} = \frac{2y}{z^2 + 4}, \frac{\partial^2 z}{\partial y^2} = \frac{2x}{z^2 + 4}$ $D(x, y) = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial y \partial x}\right)^2 = \frac{4(x^2 - y^2)}{(z^2 + 4)^2}$ Since denominator of $D(x, y)$ is a square, the sign of D is the same as the sign of $x^2 - y^2$. Likewise, the sign of $\frac{\partial^2 z}{\partial x^2}$ is the same as the sign of x. For $(-2, -1, 1.64)$, $D > 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$. Hence maximum point. For $(2, 1, -2)$, $D > 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$. Hence minimum point. For $(-1, -2, 1.53)$ and $(1, 2, -1.91)$, $D < 0$. Hence saddle points.	
7(c)	At $(x_0, y_0, z_0) = (2, 1, -2)$, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0, \frac{\partial^2 z}{\partial x^2} = \frac{1}{2}, \frac{\partial^2 z}{\partial y \partial x} = \frac{2(1)}{2^2 + 4} = \frac{1}{4}, \frac{\partial^2 z}{\partial y^2} = \frac{1}{2}$ $z \approx Q(x, y) = -2 + 0 + 0 + \frac{1}{2}\left(\frac{1}{2}\right)(x-2)^2 + \left(\frac{1}{4}\right)(x-2)(y-1) + \frac{1}{2}\left(\frac{1}{2}\right)(y-1)^2$ $z \approx -2 + \frac{1}{4}(x-2)^2 + \frac{1}{4}(x-2)(y-1) + \frac{1}{4}(y-1)^2$ $z \approx \mathbf{X}^T \mathbf{M} \mathbf{X} + c$ $= \begin{pmatrix} x-2 & y-1 \end{pmatrix} \begin{pmatrix} p & q \\ q & r \end{pmatrix} \begin{pmatrix} x-2 \\ y-1 \end{pmatrix} - 2$ $= \begin{pmatrix} p(x-2) + q(y-1) & q(x-2) + r(y-1) \end{pmatrix} \begin{pmatrix} x-2 \\ y-1 \end{pmatrix} - 2$ $= p(x-2)^2 + q(x-2)(y-1) + q(x-2)(y-1) + r(y-1)^2 - 2$ $= p(x-2)^2 + 2q(x-2)(y-1) + r(y-1)^2 - 2$ By comparing coefficients, $\mathbf{M} = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} \end{pmatrix}$	
8(a)	Note: For S_1 to be a subspace of S_2 , we need $S_1 \subseteq S_2$.	

	The columns of $\mathbf{A}_1$ are $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Since $u_2 = 2u_1$, $S_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$. The columns of $\mathbf{A}_2$ are $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ 0 \\ k \end{pmatrix}$. $\det(A_2) = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & k \end{vmatrix} = 2k.$ Case 1: $k \neq 0$. Then $\det(A_2) \neq 0$ and the columns of $\mathbf{A}$ are linearly independent and $S_2 = \mathbb{R}^3$. Hence, $S_1 \subseteq S_2$, $\forall a, b, c \in \mathbb{R}$. Case 2: $k = 0$ If $k = 0$, $\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Since $v_3 = -v_1$, $S_2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$. For $S_1 \subseteq S_2$, we need $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in S_2$, e.g. $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$. Solving, $a = k_1$, $b = 2k_2$, $c = k_2$ Thus, $b = 2c$. Conclusion: If $k \neq 0$, $a, b, c \in \mathbb{R}$ If $k = 0$, $a \in \mathbb{R}, b = 2c$.	
8(b)	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2 - R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ A basis for the row space is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ Solve $\mathbf{M}(\mathbf{x}) = \mathbf{0}$	

	Let $\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\mathbf{M}(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $a + c = 0$ $b + c = 0$ Thus $\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} k \\ k \\ -k \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ A basis for the null space is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$	
9(a)	When $x = \frac{\sqrt{3}}{2}$, $y = \frac{\sqrt{3}}{2} \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{6} \pi$  $OP = \sqrt{\left(\frac{\sqrt{3}}{2} \right)^2 + \left(\frac{\sqrt{3}}{6} \pi \right)^2} = \sqrt{\frac{3}{4} + \frac{\pi^2}{12}}$ Since OP is the shortest distance between O and P, $L > OP$. Hence $L > \sqrt{\frac{3}{4} + \frac{\pi^2}{12}}$.	
9(b)	$\frac{dy}{dx} = x \left(\frac{1}{\sqrt{1-x^2}} \right) + \sin^{-1} x$ $L = \int_0^{\frac{\sqrt{3}}{2}} \sqrt{1 + \left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1} x \right)^2} dx$ $= 1.33 \text{ (3 s.f.)}$	

9(c)	When $x = \frac{\sqrt{3}}{2}$, $y = \frac{\sqrt{3}}{2} \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{6} \pi$ Hence, $k = \frac{\sqrt{3}}{2}$. Volume of S $= \pi \left(\frac{\sqrt{3}}{2}\right)^2 \left(\frac{\sqrt{3}}{6} \pi\right) - \int_0^{\frac{\sqrt{3}}{2}} 2\pi x (x \sin^{-1} x) dx$ $= \frac{\sqrt{3}}{8} \pi^2 - 2\pi \int_0^{\frac{\sqrt{3}}{2}} x^2 \sin^{-1} x dx$ $= \frac{\sqrt{3}}{8} \pi^2 - 2\pi \left(\left[\frac{x^3}{3} \sin^{-1} x \right]_0^{\frac{\sqrt{3}}{2}} - \int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{3\sqrt{1-x^2}} dx \right)$ $= \frac{\sqrt{3}}{8} \pi^2 - 2\pi \left(\left[\frac{\frac{3\sqrt{3}}{8} \pi}{3 \cdot 3} - 0 \right] - \frac{1}{3} \int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}} dx \right)$ $= \frac{\sqrt{3}}{8} \pi^2 - \frac{\sqrt{3}}{12} \pi^2 + \frac{2\pi}{3} \int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}} dx$ $= \frac{\sqrt{3}\pi^2}{24} + \frac{2\pi}{3} \int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}} dx$ $t = 1 - x^2$ $\frac{dt}{dx} = -2x$ $\frac{dx}{dt} = -\frac{1}{2x} \left(= -\frac{1}{2\sqrt{1-t}} \because 0 \leq x \leq \frac{\sqrt{3}}{2} \right)$ $\frac{\sqrt{3}\pi^2}{24} + \frac{2\pi}{3} \int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}} dx$ $= \frac{\sqrt{3}\pi^2}{24} + \frac{2\pi}{3} \int_1^{\frac{1}{4}} \frac{x^3}{\sqrt{1-x^2}} \left(-\frac{1}{2x} \right) dt$ $= \frac{\sqrt{3}\pi^2}{24} + \frac{\pi}{3} \int_{\frac{1}{4}}^1 \frac{1-t}{\sqrt{t}} dt$ $= \frac{\sqrt{3}\pi^2}{24} + \frac{\pi}{3} \int_{\frac{1}{4}}^1 t^{-\frac{1}{2}} - t^{\frac{1}{2}} dt$	
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	$= \frac{\sqrt{3}\pi^2}{24} + \frac{\pi}{3} \left[2\sqrt{t} - \frac{2}{3}t^{\frac{3}{2}} \right]_{\frac{1}{4}}^1$ $= \frac{\sqrt{3}\pi^2}{24} + \frac{2\pi}{3} \left[\left(1 - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{24}\right) \right]$ $= \frac{\sqrt{3}\pi^2}{24} + \frac{2\pi}{3} \left(\frac{5}{24} \right)$ $= \frac{\sqrt{3}\pi^2}{24} + \frac{5\pi}{36}$	
10(a)	$u = v \cos y$ $\frac{\partial u}{\partial x} = \frac{dv}{dx} \cos y \quad \frac{\partial u}{\partial y} = -v \sin y$ $\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2} \cos y \quad \frac{\partial^2 u}{\partial y^2} = -v \cos y$ Substituting into $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (3 - 2x^2) \cos y$: $\frac{d^2 v}{dx^2} \cos y - v \cos y = (3 - 2x^2) \cos y$ $\frac{d^2 v}{dx^2} - v = 3 - 2x^2 \quad (\text{shown})$	
(b)	$v(0) = u(0, 0) = \cos 0 = 1.$ $v(1) = u(1, 0) = 4 \cos 0 = 4.$ For $\frac{d^2 v}{dx^2} - v = 3 - 2x^2$, auxiliary equation: $m^2 - 1 = 0 \Rightarrow m = \pm 1.$ Complementary function, $v_c = Ae^x + Be^{-x}.$ Particular integral: $v_p = ax^2 + bx + c,$ $v_p' = 2ax + b,$ $v_p'' = 2a.$ Substituting into the original DE: $v_p'' - v_p = 3 - 2x^2$ $2a - (ax^2 + bx + c) = 3 - 2x^2$ $-ax^2 + bx + 2a - c = 3 - 2x^2$ $\Rightarrow a = 2, b = 0, c = 1.$ $v_p = 2x^2 + 1.$ Thus, $v = Ae^x + Be^{-x} + 2x^2 + 1.$	

	Substituting $v(0) = 1$: $1 = A + B + 1 \Rightarrow A = -B.$ Substituting $v(1) = 4$: $4 = Ae + Be^{-1} + 2(1)^2 + 1$ $1 = -Be + Be^{-1}$ $1 = B(e^{-1} - e)$ $B = \frac{1}{e^{-1} - e} \Rightarrow A = \frac{1}{e - e^{-1}}.$ $v = \left(\frac{1}{e - e^{-1}} \right) e^x + \left(\frac{1}{e^{-1} - e} \right) e^{-x} + 2x^2 + 1$ $v = \frac{e^x - e^{-x}}{e - e^{-1}} + 2x^2 + 1$ $u = \left(\frac{e^x - e^{-x}}{e - e^{-1}} + 2x^2 + 1 \right) \cos y.$	
(c)	From (b), $u = \left(\frac{e^x - e^{-x}}{e - e^{-1}} + 2x^2 + 1 \right) \cos y$. For $0 \leq x \leq 1$ and $0 \leq y \leq \frac{\pi}{2}$, observe that: $\cos y$ is decreasing so maximum u will be at minimum y; e^x, $-e^{-x}$ and $2x^2 + 1$ are increasing (and $e - e^{-1}$ is positive) so v is increasing, hence maximum u will be at maximum x. Thus, the coordinates of the point on the plate with the highest steady-state temperature is the point where x is maximum, and y is minimum, $(1, 0)$.	