TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ____

CIVICS GROUP: ____

H3 MATHEMATICS 9820/01

Paper 1

19 SEPTEMBER 2025

3 hours

Additional materials: Printed Answer Booklet

List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.

Write your answers in the spaces provided in the Printed Answer Booklet. Follow the instructions on the front cover of the Printed Answer Booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 5 printed pages and 1 blank pages.

[Turn Over

1 **(i)** Show that $\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$, where $x_1, x_2, \ldots, x_n$ are real numbers. [2]

(ii) Show that $\sum_{i=1}^{n}(1+a)^{i-1} \geq n(1+a)^{\frac{n-1}{2}}$ for $a \geq 0$. Determine the value of a for the equality to hold. [5]

(iii) Hence show $\sum_{i=1}^{n}(1+a)^{2i} \geq n(1+a)^{n+1}$ for $a \geq 0$. [3]

2 **(i)** Prove that for any even integer $n > 1$, $n^4 + 4^n$ is even. [2]

(ii) Given that $\left(x^4 + 4y^4\right) = \left(x^2 + ay^2 + bxy\right)\left(x^2 + ay^2 - bxy\right)$, determine the values of a and b where $a, b > 0$. [3]

(iii) Hence prove that for any integer $n > 1$, $n^4 + 4^n$ is not prime. [5]

3 There are k identical marbles to be distributed among six children.

(a) For this part of the question, let $k = 15$.

(i) Find the number of ways to distribute the marbles among the six children if there are no restrictions. [2]

(ii) How many ways can the marbles be distributed among six children if each child receives at least one marble? [2]

(iii) Find the probability that Tom, one of the children, receives exactly two marbles. [2]

(b) The marbles are now numbered from 1 to k. Find, in terms of k, the number of ways to distribute the marbles among the six children if

(i) there are no restrictions, [1]

(ii) each child must receive at least one marble. [5]

4 **(a)** The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = x_2 = \dfrac{3}{2}$ and $x_n = \frac{x_{n-1}x_{n-2}+1}{x_{n-1}+x_{n-2}}$, for $n \geq 3$.

Another sequence $y_1, y_2, y_3, \ldots$ is defined by $y_n = x_n - 1$, for $n \geq 1$.

(i) Show that $y_n = \frac{y_{n-1}y_{n-2}}{y_{n-1}+y_{n-2}+2}$ for $n \geq 3$. [1]

(ii) Prove by mathematical induction that $0 < y_n < 1$ for $n \geq 1$. [5]

(iii) Using parts **(a)(i)** and **(a)(ii)**, show that $y_n < \dfrac{1}{2} y_{n-1}$ for $n \geq 3$. [1]

(iv) Hence show that the series $y_1 + y_2 + y_3 + \ldots$ converges. [3]

(b) **(i)** [You may use the result $\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$.]

Given that $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(2n+1)x}{\sin x}\,dx$, show that, for any positive integer n,

$$I_n - I_{n-1} = \frac{1}{n}\sin\left(\frac{1}{2}n\pi\right).$$ [2]

(ii) Hence or otherwise find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(11x)}{\sin x}\,dx$. [2]

5 **(a)** [In this question, a sequence $\{a_n\}_{n=1,2,\ldots}$ is said to be bounded if there exists a positive real number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.]

Let $\{a_n\}_{n=1,2,\ldots}$ be a bounded sequence satisfying the condition

$$a_{n+1} + p^{-n} > a_n \quad \text{where } p \geq 2.$$

Prove that the sequence $\{a_n - p^{-n+1}\}_{n=1,2,\ldots}$ is

(i) strictly increasing, [3]

(ii) bounded. [2]

Hence show that $\lim_{n\to\infty} a_n$ exists, justifying each step of your argument. [2]

(b) **(i)** A complex number $z = 1 - e^{i\theta}\cos\theta$ where $0 \leq \theta \leq \pi$. Find the modulus and argument of z in terms of θ. [4]

(ii) Show that $\displaystyle\sum_{r=0}^{n} (-1)^r \binom{n}{r} \cos^r\theta\cos r\theta = \sin^n\theta\cos n\left(\theta - \frac{\pi}{2}\right)$. [3]

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

Derangement

A derangement is a permutation of a set in which none of the elements appear in their original positions. It is also known as a permutation with no fixed points. For a set of *n* elements, a derangement is an arrangement where no element is in the position it started in.

For example, consider the set $\{1,2,3\}$. The derangements of this set (permutations where no element is in its original position) are $\{2,3,1\}$ and $\{3,1,2\}$.

Note that $\{3,2,1\}$ is not a derangement of $\{1,2,3\}$ because 2 is a fixed point, that is, it appears in its original position.

The number of derangements of an n-element set is called the n^{th} derangement number and is commonly denoted by D_n.

In the above example for the derangement of the set $\{1,2,3\}$, we can verify that $D_3 = 2$ by listing out all possible cases.

It is known that the n^{th} derangement number satisfies the following 2nd and 1st order recurrence relations:

a) $D_n = (n-1)\left(D_{n-1} + D_{n-2}\right)$ for $n \geq 3$, and

b) $D_n = n\,D_{n-1} + \left(-1\right)^n$ for $n \geq 2$

with initial conditions $D_1 = 0$ and $D_2 = 1$.

An explicit formula for D_n is given by:

$$D_n = n!\sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \text{for} \quad n \geq 1.$$

This formula can be found using the Principle of Inclusion and Exclusion and is applied in various fields, including combinatorics, probability, cryptography, and even real-life scenarios.

For example, in a gift exchange, each participant in a group is randomly assigned another person in the group to give a gift to. However, one essential rule is that no one should be assigned to himself/herself — in other words, each participant should get someone else's gift, not his/her own. This problem is a classic example of a derangement, as we are looking for a way to arrange the participants' assignments so that no one's name matches his/her own.

6 **(a)** Prove the 2nd order recurrence relation $D_n = (n-1)\left(D_{n-1} + D_{n-2}\right)$ for $n \geq 3$. [3]

(b) By using the 2nd order recurrence relation in part **(a)** and considering a substitution of $x_n = D_n - nD_{n-1}$, prove the 1st order recurrence relation

$$D_n = n\,D_{n-1} + \left(-1\right)^n \text{ for } n \geq 2\,.$$ [4]

(c) Hence, by considering a substitution of $y_n = \dfrac{D_n}{n!}$, prove that

$$D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} \text{ for } n \geq 1.$$ [5]

(d) Use the Principle of Inclusion and Exclusion to prove that

$$D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} \text{ for } n \geq 1.$$ [4]

(e) Hence, in a gift exchange among a group of 7 people, find the number of ways to assign each person someone else in the group to give a gift to, ensuring that no one is assigned to himself/herself. [1]

(f) For large values of *n*, find, without using a calculator, the ratio of the n^{th} derangement number to the number of permutations without restriction. [3]

End of Paper