2025 Year 6 H2 Math Prelim Exam P1
Suggested Solutions

Qn	Suggested Solutions	Comments
1	$u=\frac{y}{x}\Rightarrow y=ux\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}x+u$ ... (1) $2xy\frac{\mathrm{d}y}{\mathrm{d}x}=x^2-y^2$ $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^2-y^2}{2xy}=\frac{x^2-u^2x^2}{2ux^2}=\frac{1-u^2}{2u}$ ... (2) Equate (1) with (2): $\frac{\mathrm{d}u}{\mathrm{d}x}x+u=\frac{1-u^2}{2u}$ $\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}x=\frac{1-u^2}{2u}-u=\frac{1-3u^2}{2u}$ $\frac{2u}{1-3u^2}\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$ (shown)	Use $y=ux$ or $u=\frac{y}{x}$ to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}u}{\mathrm{d}x}$ (by implicit differentiation) before showing $\frac{2u}{1-3u^2}\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$ **Note: *u*, *x*, *y* are variables**
	$\int\frac{2u}{1-3u^2}\,\mathrm{d}u=\int\frac{1}{x}\,\mathrm{d}x$ $-\frac{1}{3}\int\frac{-6u}{1-3u^2}\,\mathrm{d}u=\int\frac{1}{x}\,\mathrm{d}x$ $-\frac{1}{3}\ln\left\|1-3u^2\right\|=\ln x+c \quad (\because x>0)$ $\ln\left\|1-3u^2\right\|=-3\ln x-3c$ $1-3u^2=\pm e^{-3c}\cdot e^{\ln\frac{1}{x^3}}=\frac{A}{x^3}$ (where $A=\pm e^{-3c}$) $3u^2=1-\frac{A}{x^3}$ $\left(\frac{y}{x}\right)^2=\frac{1}{3}\left(1-\frac{A}{x^3}\right)$ $y=-\sqrt{\frac{x^2}{3}\left(1-\frac{A}{x^3}\right)}=-x\sqrt{\frac{1}{3}\left(1-\frac{A}{x^3}\right)} \quad (\because y<0)$	Separate variables *u* & *x* to find the general solution of *y* in terms of *x* by integration Take note that $x>0$ & $y<0$
		Total marks: 6

Qn	Suggested Solutions	Comments
2(a)	$u_n = 2(4-3x)^n$ $u_{n+1} = 2(4-3x)^{n+1}$ $\frac{u_{n+1}}{u_n} = 4-3x \ (\text{constant})$ Since $4-3x$ is a constant common ratio, the series is a geometric series. For the sum to infinity to exist, $\lvert 4-3x\rvert < 1$ $-1 < 4-3x < 1$ $-5 < -3x < -3$ $1 < x < \frac{5}{3}$	As it is given that the series is geometric, the general form for common ratio $\frac{u_{n+1}}{u_n} =$ constant must be used to justify it. For the sum to infinity to exist, $\lvert\text{common ratio}\rvert < 1$.
(b)	$\sum_{r=0}^{n-1}\left[2(4-3x)^r + (r+1)(2r+5)\right]$ $= \sum_{r=0}^{n-1} 2(4-3x)^r + \sum_{r=0}^{n-1}(r+1)(2r+5)$ $= \sum_{r=0}^{n-1} 2 + \sum_{r=0}^{n-1}(r+1)(2r+5) \quad (\text{since } x=1)$ $= 2n + \sum_{r=1}^{n}\left((r-1)+1\right)\left(2(r-1)+5\right)$ $= 2n + \sum_{r=1}^{n} r(2r+3)$ $= 2n + 2\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$ $= 2n + 2\left(\frac{n}{6}\right)(n+1)(2n+1) + 3\left(\frac{n}{2}\right)(1+n)$ $= n\left(\frac{2}{3}n^2 + \frac{5}{2}n + \frac{23}{6}\right)$ $a = \frac{2}{3}, b = \frac{5}{2}, c = \frac{23}{6}$	Note that $\sum_{r=0}^{n-1} \mathrm{f}(r)$ has n terms Replace r as $(r-1)$ to change the limits for $\sum_{r=0}^{r=n-1} \mathrm{f}(r)$ $\sum_{r=1}^{n} \mathrm{c} = \mathrm{c}n$ where c is a constant $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (AP) $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ is applicable for lower limit '1', & the upper limit 'n'
		Total marks: 7

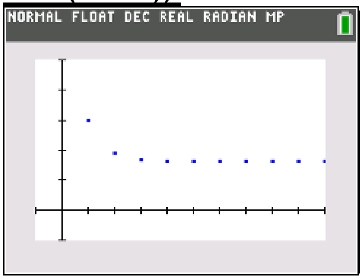
Qn	Suggested Solutions	Comments
3(a)	$e^x \sin(x+\pi) = e^x[\sin x\cos\pi + \cos x \sin\pi]$ $= -e^x \sin x$ $= -\left(1+x+\frac{x^2}{2!}+...\right)\left(x-\frac{x^3}{3!}+...\right)$ $= -x - x^2 - \frac{x^3}{3} + \cdots$ **Alternative (using repeated differentiation)** $f(x) = e^x \sin(x+\pi)$ $f'(x) = e^x\sin(x+\pi) + e^x\cos(x+\pi)$ $= y + e^x\cos(x+\pi)$ $f''(x) = \frac{dy}{dx} - e^x\sin(x+\pi) + e^x\cos(x+\pi)$ $= \frac{dy}{dx} - y + e^x\cos(x+\pi)$ OR $2e^x\cos(x+\pi)$ $f'''(x) = \frac{d^2y}{dx^2} - \frac{dy}{dx} - e^x\sin(x+\pi) + e^x\cos(x+\pi)$ $= \frac{d^2y}{dx^2} - \frac{dy}{dx} - y + e^x\cos(x+\pi)$ OR $2e^x[\cos(x+\pi) - \sin(x+\pi)]$ When x = 0, $f(0) = 0$ $f'(0) = -1$ $f''(0) = -2$ $f'''(0) = -2$ $f(x) = -x + \frac{-2}{2!}x^2 + \frac{-2}{3!}x^3 + ...$ $= -x - x^2 - \frac{x^3}{3} + \cdots$	**Common mistake:** When expanding $\sin(x+\pi)$ using Maclaurin series, some students wrongly substitute $(x+\pi)$ directly into the sine expansion: $\sin(x+\pi) =$ $(x+\pi) - \frac{(x+\pi)^3}{3!} + \frac{(x+\pi)^5}{5!} - ...$ Note that expansion of $(x+\pi)^3$ and $(x+\pi)^5$ both contain constant term, x term and x^2 terms. This means every term in the infinite expansion contributes to the first few powers of x. You cannot cleanly stop at the x^2 term, so this defeats the purpose of using Maclaurin series. ➔ **Do not substitute constants (like π) directly into Maclaurin standard series expansions.** Always simplify first. Students should use addition formula for $\sin(a+b)$ or trigonometric identity $\sin(x+\pi) = -\sin x$ before applying standard Maclaurin series
(b)	$ax(1+bx)^c = ax\left(1 + cbx + \frac{c(c-1)}{2!}(bx)^2 + ...\right)$ $= ax + abcx^2 + \frac{ab^2c(c-1)}{2}x^3 + ...$ Comparing with answer in **(a)**, $a = -1,\ abc = -1,\ \frac{ab^2c(c-1)}{2} = -\frac{1}{3}$ Solving,	Students should apply Maclaurin standard series expansion of $(1+x)^n$ which is more efficient than using repeated differentiation Students should apply result from comparing coefficient of x and x^2 term $a = -1$ and $bc = 1$ to simplify the equation for the x^3 coefficient $\frac{ab^2c(c-1)}{2} = \frac{-b(c-1)}{2}$ before

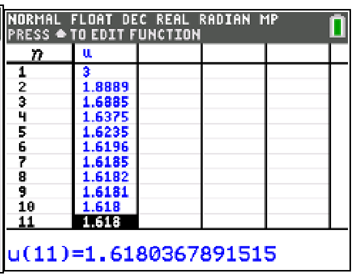
	$bc=1 \Rightarrow b=\frac{1}{c}$ and $\frac{b(c-1)}{2}=\frac{1}{3}$ $\frac{c-1}{2c}=\frac{1}{3}$ $3c-3=2c$ $c=3$ Therefore, $b=\frac{1}{3}$ $ax(1+bx)^c=-x\left(1+\frac{1}{3}x\right)^3$ The x^4 term $=-x\left[\binom{3}{3}(1)^{3-3}\left(\frac{1}{3}x\right)^3\right]=-\frac{1}{27}x^4$ The coefficient of x^4 is $-\frac{1}{27}$.	solving. This will avoid many careless mistakes made.
		Total marks: 8

Qn	Suggested Solutions	Comments
4(a)	$x_{n+1}=\frac{5x_n+2}{2x_n+3}$ As $n\to\infty$, $x_n\to l$, $x_{n+1}\to l$. $x_{n+1}=\frac{5x_n+2}{2x_n+3}$ $l=\frac{5l+2}{2l+3}$ $2l^2+3l=5l+2$ $2l^2-2l-2=0$ $l^2-l-1=0$ $l=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}=\frac{1\pm\sqrt{5}}{2}$ $\therefore l=\frac{1+\sqrt{5}}{2}$ or $l=\frac{1-\sqrt{5}}{2}$	Given that a sequence (generated by a recurrence relation) converges, letting $x_n\to l$ and $x_{n+1}\to l$ as $n\to\infty$ is the standard method to find the limit l of the sequence – do familiarise yourself with this technique. Some students assumed that $n\to\infty$ implies that $x_n\to\infty$ and got confused with the method of finding the limit of a sequence defined by a formula for the nth term (in terms of n, i.e. $u_n=\mathrm{f}(n)$ (e.g. $u_n=\frac{3n-5}{n-2}$) – do note the difference. A few students erroneously assumed that the sequence is an AP/GP from the start – not every sequence given is an AP/GP.

(b)

The sequence is **decreasing** and **converges to** $\frac{1+\sqrt{5}}{2}$ **(or 1.62 (to 3sf)).**





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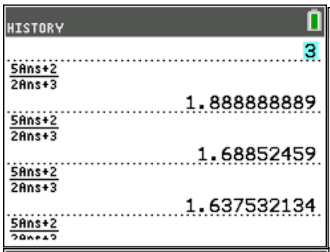
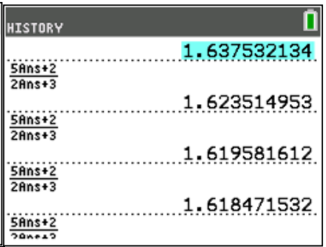
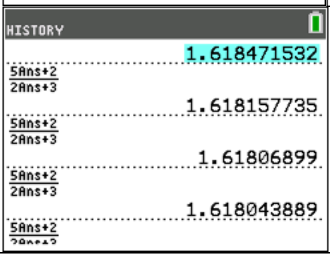






This part could have been done without **(a)** – students can use the GC to observe the behaviour of the sequence. In general, there are two parts to describing the behaviour of a sequence:

- Trends (increasing / decreasing, constant, alternating)
- Long-run behaviour of an infinite sequence (convergent or divergent)

For example: The sequence is a ______________ sequence and diverges/converges to ______.

(c)

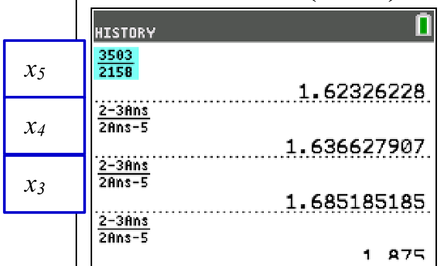
$$x_{n+1}=\frac{5x_n+2}{2x_n+3}$$

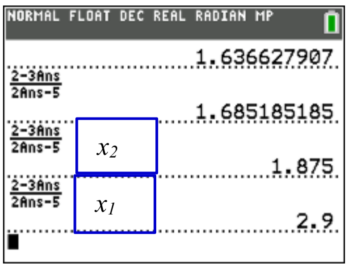
By making x_n the subject,

$$x_{n+1}\left(2x_n+3\right)=5x_n+2$$
$$\left(2x_{n+1}-5\right)x_n=2-3x_{n+1}$$
$$x_n=\frac{2-3x_{n+1}}{2x_{n+1}-5}$$

Starting with $x_5=\frac{3503}{2158}$, we can use the GC to compute

$$x_4=\frac{2-3x_5}{2x_5-5}=\frac{2-3\left(\frac{3503}{2158}\right)}{2\left(\frac{3503}{2158}\right)-5}=\ldots=\frac{6193}{3784}=\frac{563}{344}=1.63663$$





$\therefore x_1=2.9$

Students can try to use the recurrence relation (subbing in $n=4$) to first find x_4 using $x_5=\frac{5x_4+2}{2x_4+3}$; from this, it can be observed that we need to make x_4 the subject (and hence in general, x_n the subject). This will allow us to use the calculator to help compute x_3, x_2 and finally x_1 as seen from the solution. Quite a number of students did this on their own, using the recurrence relation again and again (repeating the process and making x_3 the subject then x_2 the subject, etc) and managed to obtain x_1 successfully – this is ok too for this question. But do note that this method may not be feasible if say x_{50} was given instead of x_5 and we need to obtain x_1 - we will need the help of the calculator to do this then.

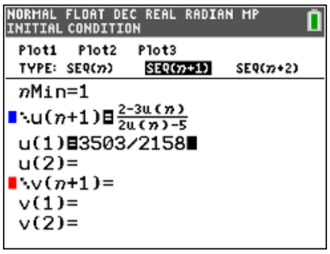
<u>Alternative</u>

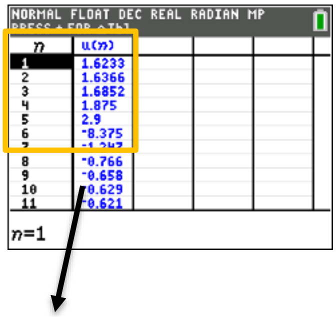
Make use of the SEQ function in the GC, using the recurrence relation,

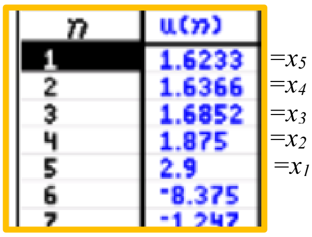
$$x_n = \frac{2-3x_{n+1}}{2x_{n+1}-5} \text{ and } x_5 = \frac{3503}{2158}$$

we can key in x_n as u_n, x_{n+1} as u_{n-1} and $x_5 = \frac{3503}{2158}$ as u_1

(we are just making use of the GC to help us compute "backwards", the values using the recurrence relationship – in the GC, we cannot input x_n in terms of x_{n+1}.







$\therefore x_1 = 2.9$

Total marks: 7

<table>
<tr><th>Qn</th><th>Suggested Solution</th><th>Comments</th></tr>
<tr><td>5(a)</td><td>
Max pt at $x=-2$, Min pt at $x=1.5$.

(some explanation below; no need to show)
By looking at the graph of $y=\mathrm{f}'(x)$, we can use the usual first or second derivative test to determine the nature of the stationary points on the graph of $y=\mathrm{f}(x)$:

First Derivative Test
<table>
<tr><td>x</td><td>-2^-</td><td>-2</td><td>-2^+</td></tr>
<tr><td>$\frac{\mathrm{d}y}{\mathrm{d}x}$ (value of $\mathrm{f}'(x)$)</td><td>+ve (f' graph above x-axis)</td><td>0</td><td>-ve (f' graph below x-axis)</td></tr>
<tr><td>Shape</td><td>/</td><td>—</td><td>\</td></tr>
</table>
$\therefore$ Maximum point at $x=-2$

<table>
<tr><td>x</td><td>1.5^-</td><td>1.5</td><td>1.5^+</td></tr>
<tr><td>$\frac{\mathrm{d}y}{\mathrm{d}x}$ (value of $\mathrm{f}'(x)$)</td><td>-ve (f' graph below x-axis)</td><td>0</td><td>+ve (f' graph above x-axis)</td></tr>
<tr><td>Shape</td><td>\</td><td>—</td><td>/</td></tr>
</table>
$\therefore$ Minimum point at $x=1.5$

Second Derivative Test
At $x=-2$, gradient of f' graph is negative $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}<0$
$\therefore$ Maximum point at $x=-2$

At $x=1.5$, gradient of f' graph is positive $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}>0$
$\therefore$ Minimum point at $x=1.5$
</td><td>Students are expected to identify clearly which are the maximum/minimum points.</td></tr>
<tr><td>(b)</td><td>f is decreasing: $-2\le x<0$ or $0<x\le 1.5$</td><td>Many students forgot to exclude the point at which $x=0$ (where the graph is undefined).</td></tr>
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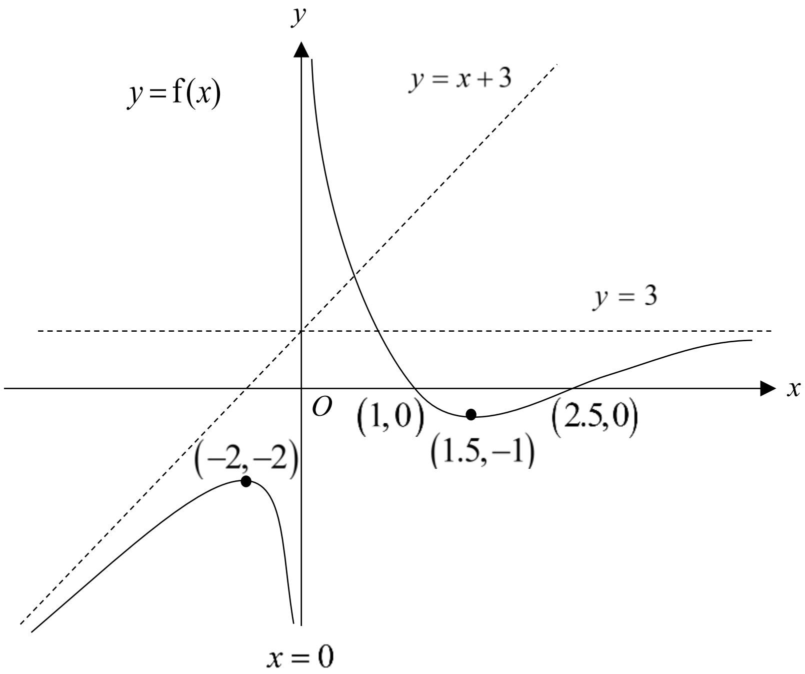
<tr>
<td>(c)</td>
<td>
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<td>Few students managed to get full marks for this question. There are a few important points to note:

The minimum point (with y-coord -1) should be "above"/ "higher" than the maximum point (with y-coord -2)
The oblique asymptote $y = x + 3$ and horizontal asymptote $y = 3$ should both intersect at the y-axis; both will pass through the point (0,3)
Some students still left the line $y = -x - 3$ but it should no longer be drawn since it is not an asymptote to the graph of $y = \mathrm{f}(x)$
Some students were careless in labelling the coordinates of the turning points/axial intercepts
Some graphs were not well drawn, especially how they tend to the asymptotes – do take note
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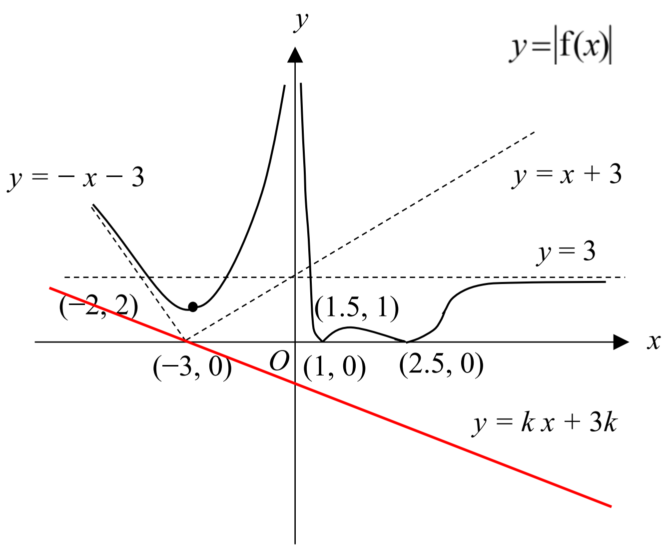
<td>(d)</td>
<td>

Range of values of x: $-1 \le k < 0$.</td>
<td>The line $y = kx + 3k$ should pass through the point (-3,0); it is the point independent of k on the line $y = kx + 3k$. We can "pivot" the line about this point in order to vary the gradient of the line, k, to observe and hence determine the range of values of k for which there is no intersection between the line $y = kx + 3k$ and the graph of $y = |\mathrm{f}(x)|$. This is equivalent to having no real solution to the equation $|\mathrm{f}(x)| = kx + 3k$, as required by the question.

Few students could get the range of values of k accurately, either missing</td>
</tr>
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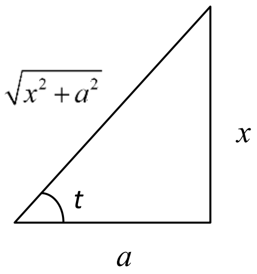
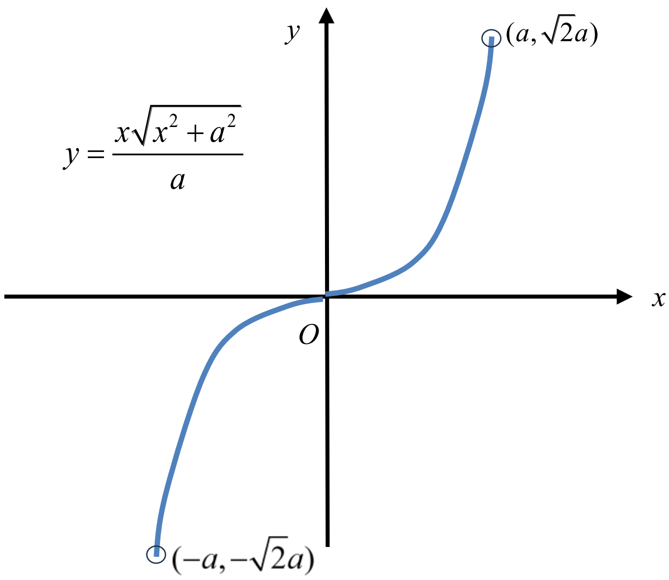
		out the value of 1, or the upper bound of 0, or carelessly "reversing" the bounds.
		Total marks: 9

Qn	Suggested Solutions	Comments
6(a)	$\text{LHS} = \dfrac{\tan t}{\sqrt{1+\tan^2 t}} = \left(\dfrac{\sin t}{\cos t}\right)\left(\dfrac{1}{\sec t}\right)$ $= \left(\dfrac{\sin t}{\cos t}\right)(\cos t) = \sin t = \text{RHS (shown)}$	As this is a show question, all steps should be shown correctly. That is, $\tan t = \dfrac{\sin t}{\cos t}$ $1+\tan^2 t = \sec^2 t$ $\sec t = \dfrac{1}{\cos t}$ Do note that $\sqrt{\sec^2 t} = \lvert\sec t\rvert = \sec t$ $\because \sec t > 0$ for $\frac{-\pi}{4} < t < \frac{\pi}{4}$
(b)	$x = a\tan t \Rightarrow \tan t = \dfrac{x}{a}$ $y = a\sec^2 t \sin t$ $= a\left(\tan^2 t + 1\right)\left(\sin t\right)$ $= a\left(\tan^2 t + 1\right)\left(\dfrac{\tan t}{\sqrt{1+\tan^2 t}}\right)$ $= a\left(\left(\dfrac{x}{a}\right)^2 + 1\right)\left(\dfrac{\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}}\right)$ $= x\sqrt{1+\left(\dfrac{x}{a}\right)^2}$ $= \dfrac{x\sqrt{x^2+a^2}}{a}$, where $-a < x < a$	Cartesian equation → only has x and y. Parametric has trigo, likely trigo identities have to be used. We should simplify this as we are given in the question that $a > 0$. $\sqrt{a^2} = \lvert a\rvert = a$ The cartesian equation comes with the domain since $-\dfrac{\pi}{4} < t < \dfrac{\pi}{4} \Rightarrow -1 < \tan t < 1$

	<u>Otherwise</u> $x = a\tan t \Rightarrow \tan t = \frac{x}{a} \Rightarrow \cos t = \frac{a}{\sqrt{x^2+a^2}}$ $y = a\sec^2 t \sin t$ $= a\left(\frac{1}{\cos^2 t}\right)(\sin t)$ $= a\left(\frac{1}{\cos t}\right)(\tan t)$ $= a\left(\frac{\sqrt{x^2+a^2}}{a}\right)\left(\frac{x}{a}\right)$ $= \frac{x\sqrt{x^2+a^2}}{a}$, where $-a < x < a$  	See this with a right-angled triangle.
(c)	$f(-x) = \frac{-x\sqrt{(-x)^2+a^2}}{a} = -\frac{x\sqrt{x^2+a^2}}{a} = -f(x)$ (shown)  	As $-a < x < a$, there should be end-points for this graph. Since this is an odd function, the graph should also be symmetrical about the origin. Can see the shape from the GC too.
(d)	Area $= 2\int_0^A \frac{x\sqrt{x^2+a^2}}{a}\,dx$ (by symmetry) $= \frac{1}{a}\int_0^A 2x\sqrt{x^2+a^2}\,dx$ $\left(\text{using } \int f'(x)[f(x)]^n\,dx\right)$ $= \left(\frac{1}{a}\right)\left[\frac{2}{3}\left(x^2+a^2\right)^{\frac{3}{2}}\right]_0^A$ $= \frac{2}{3a}\left(\left(A^2+a^2\right)^{\frac{3}{2}} - a^3\right)$	Recall: $\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$
		Total marks: 8

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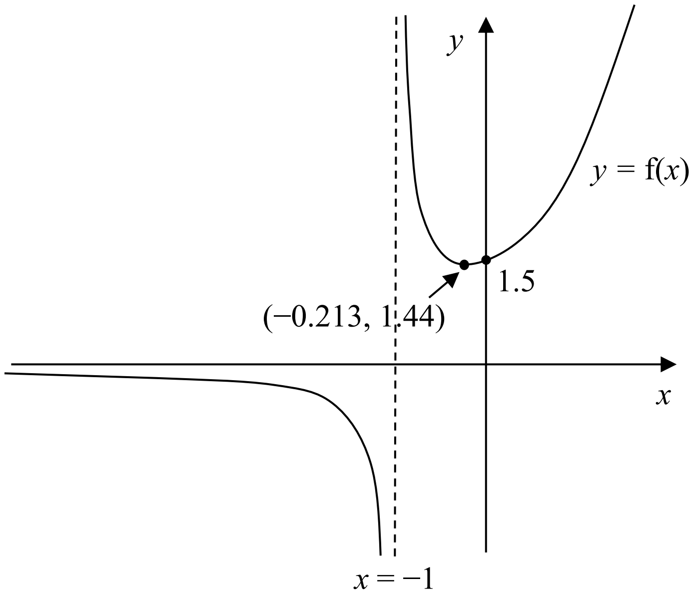
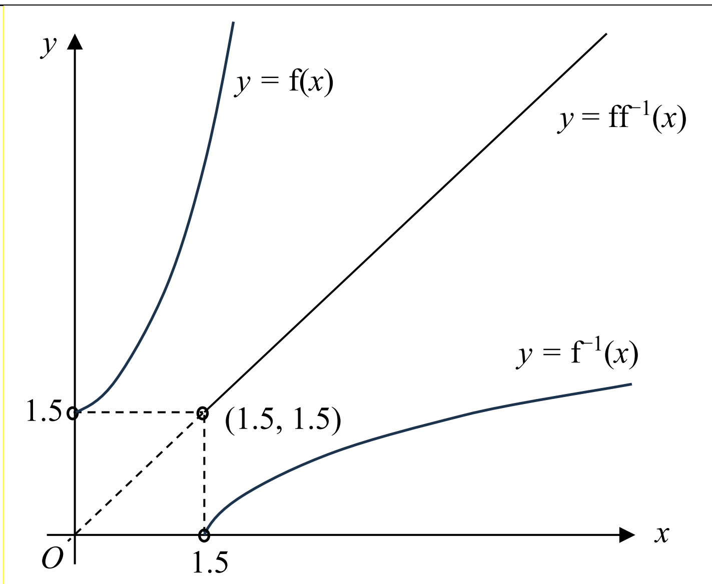
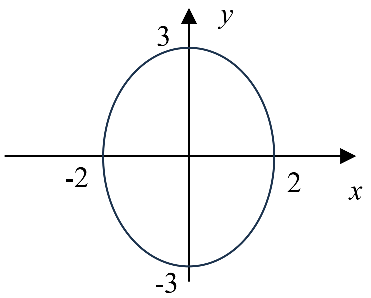
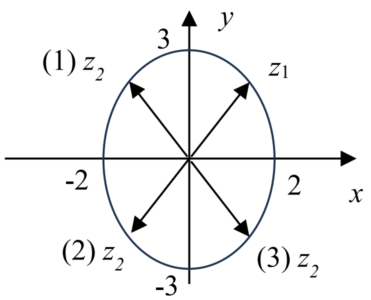
7		While it is not necessary to present the graph of f, it will be useful in visualising part of the problem.
(a)	Since $g'(x) > 0$ for all real x, and $g(1) = -0.5$, the min. value of $g(x)$ is -0.5 and it increases as x increases. Depending on the function, g either increases to $+\infty$ or an asymptotic value greater than -0.5, i.e. $R_g = [-0.5, \infty)$ or $[-0.5, a)$, where $a \in \mathbb{R}$, $a > -0.5$. Either way, $R_g \subseteq D_f$, where $D_f = \mathbb{R} \setminus \{-1\}$. Hence, fg exists. *The second part of this question is not assessed due to its unintended complexity – depending on the function g, the answer to this part will be quite different. ** Everyone was awarded the 1 mark originally assigned to this part.*	Writing D_f in set notation. • $\mathbb{R} \setminus \{-1\}$ • $(-\infty, -1) \cup (-1, \infty)$
(b)	Given $f(x) = e^x + \frac{1}{2x+2}$, $$fg(x) = f[g(x)] = e^{g(x)} + \frac{1}{2g(x)+2}.$$ Hence, $e^{g(x)} + \frac{1}{2g(x)+2} = \frac{x}{\sqrt{e}} + \frac{1}{2\ln x + 1}$. Comparing ln terms, $$2g(x) + 2 = 2\ln x + 1$$ $$g(x) = \ln x - \frac{1}{2}$$ Check: $e^{g(x)} = e^{\ln x - \frac{1}{2}} = \frac{x}{\sqrt{e}}$	

	$\therefore\ \mathrm{g}(x)=\ln x-\frac{1}{2}$. (Equivalently $\mathrm{g}(x)=\ln\frac{x}{\sqrt{\mathrm{e}}}$.) **<u>Alternative</u>** $\mathrm{fg}(x)=\frac{x}{\sqrt{\mathrm{e}}}+\frac{1}{2\ln x+1}=x\mathrm{e}^{-\frac{1}{2}}+\frac{1}{2\ln x+1}$ $=\mathrm{e}^{\ln x}\cdot\mathrm{e}^{-\frac{1}{2}}+\frac{1}{2\left(\ln x-\frac{1}{2}\right)+2}$ $=\mathrm{e}^{\left(\ln x-\frac{1}{2}\right)}+\frac{1}{2\left(\ln x-\frac{1}{2}\right)+2}$ Thus, $\mathrm{g}(x)=\ln x-\frac{1}{2}$. Note: It is possible to do this part by observation.	
(c)	For f^{-1} to exist, the function must be 1-1. From graph, min. point on f is (−0.231, 1.44). Since k needs to be an integer, the least integer value of k is 0.	
(d)(i)		• $\mathrm{ff}^{-1}(x)=x$ • $\mathrm{D}_{\mathrm{ff}^{-1}}=\mathrm{D}_{\mathrm{f}^{-1}}$ $=\mathrm{R}_{\mathrm{f}}$ $=(1.5,\ \infty)$
(d)(ii)	Observe that $\mathrm{e}+\frac{1}{4}=\mathrm{e}^{1}+\frac{1}{2(1)+2}$. This corresponds. to the point $\left(1,\ \mathrm{e}+\frac{1}{4}\right)$ on f. Hence, on the graph of f^{-1}, at $x=\mathrm{e}+\frac{1}{4}$, $y=1$. i.e. $\left(\mathrm{e}+\frac{1}{4},\ 1\right)$. Let $y=\mathrm{e}^{x}+\frac{1}{2x+2}$ (expression for f).	

	An expression for f^{-1} is $x = e^y + \dfrac{1}{2y+2}$. (Note: From here, y refers to the graph of f^{-1}.) Differentiate wrt x, $1 = e^y \cdot \dfrac{dy}{dx} + (-1)(2y+2)^{-2}(2)\dfrac{dy}{dx}$ $\dfrac{dy}{dx} = \dfrac{1}{e^y - \dfrac{2}{(2y+2)^2}}$ At point $\left(e+\dfrac{1}{4},\ 1\right)$, $\dfrac{dy}{dx} = \dfrac{1}{e^1 - \dfrac{2}{(2[1]+2)^2}} = \dfrac{1}{e-\dfrac{1}{8}} = \dfrac{8}{8e-1} = 0.386$ (to 3 s.f.) **<u>Alternative (Quicker if you know the relationship)</u>** Gradient of $y = f^{-1}(x)$ at $x = e + \dfrac{1}{4}$ $= \dfrac{1}{\text{Gradient of f at } x = 1}$ $= \dfrac{1}{f'(1)} = \dfrac{1}{2.5933} = 0.386$ (to 3 s.f.)	Notice that this is similar to $f(x) = e^x + \dfrac{1}{2x+2}$ $f'(x) = e^x - \dfrac{1}{(2x+2)^2}$ This is the same as evaluating $f'(1)$.
		Total marks: 10

<table>
<tr><th>Qn</th><th>Suggested solutions</th><th>Comments</th></tr>
<tr><td>8(a)</td><td>Since $x = \text{Re}(z)$ and $y = \text{Im}(z)$,


$x = 2\cos(t), \quad y = 3\sin(t)$
$\cos(t) = \frac{x}{2}, \quad \sin(t) = \frac{y}{3}$
$\cos^2(t) + \sin^2(t) = 1$
$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$</td><td>Concepts:
1. Relate real and imaginary component of a complex number to a set of parametric eqns.
2. Forming cartesian eqn from parametric (trigo identity)
3. Recognition of standard curves (in this case, ellipse). For this qn, be aware that the possible positions of points representing the complex number z is an ellipse. Do not draw a circle.</td></tr>
<tr><td>(b)</td><td>

From the argand diagram, we know that there are 3 possible positions for z_2.
(1) $\arg(z_1 + z_2) = \frac{\pi}{2}$
(2) $\arg(z_1 + z_2)$ is undefined as $z_1 = -z_2$
(3) $\arg(z_1 + z_2) = 0$

$\therefore \arg(z_1 + z_2)$ is either undefined, 0 or $\frac{\pi}{2}$.</td><td>Concepts:
1. $|z|$ means the distance of the point representing z from the origin.
2. Since $|z_1| = |z_2|$ and both points representing the complex numbers must be on the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$,
If (a,b) represents z_1, then z_2 must be either of these 3 options:
• (a, -b)
• (-a, b)
• (-a,-b)

3. View $z_1 + z_2$ as a vector addition, then find its argument.

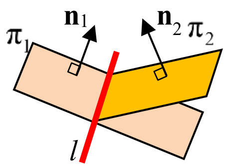
4. arg(0) is undefined.</td></tr>
<tr><td>(c)</td><td>From the diagram, if z_2 is at position (1), α will not be real.
$(z - z_1)(z - z_2) = 0 \Rightarrow z^2 - z(z_1 + z_2) + z_1 z_2$
Compare with $z^2 + \alpha z + \beta = 0$,
coefficient of the z term is $\alpha = -(z_1 + z_2)$.</td><td>Concepts:
1. Write linear factors to get the coefficient of z.
2. Check the coefficient for each of the 3 options to support your explanation.</td></tr>
</table>

	If $z_1 = a + \mathrm{i}b$, then $z_2 = -a + \mathrm{i}b$ $\alpha = -(z_1 + z_2) = -\mathrm{i}2b$ Hence, α need not be real.	
(d)	$|z_1| = \dfrac{\sqrt{26}}{2}$ $(2\cos t)^2 + (3\sin t)^2 = \left(\dfrac{\sqrt{26}}{2}\right)^2 = \dfrac{13}{2}$ $4\cos^2 t + 9\sin^2 t = \dfrac{13}{2}$ $4(1-\sin^2 t) + 9\sin^2 t = \dfrac{13}{2}$ $\sin^2 t = \dfrac{1}{2}$ $t = \dfrac{\pi}{4}$ ($\because z_1$ is in 1st quad.) Alternatively, $|z_1| = \sqrt{a^2 + b^2} = \dfrac{\sqrt{26}}{2} \Rightarrow a^2 + b^2 = \dfrac{13}{2}$ Since pt representing z_1 is on the ellipse. $\left(\dfrac{a}{2}\right)^2 + \left(\dfrac{b}{3}\right)^2 = 1$ Solving the 2 equations, $a = \sqrt{2}, \quad b = \dfrac{3}{2}\sqrt{2}$ (z_1 is in 1st quadrant, so *a* and *b* must be +ve) $z_1 = \sqrt{2} + \left(\dfrac{3}{2}\sqrt{2}\right)\mathrm{i}, \quad z_2 = -\sqrt{2} + \left(\dfrac{3}{2}\sqrt{2}\right)\mathrm{i}$ From (c), coefficients : $\alpha = -(z_1 + z_2)$ and $\beta = z_1 z_2$ $\alpha = -3\sqrt{2}\,\mathrm{i}$ $\beta = \left(\sqrt{2} + \left(\dfrac{3}{2}\sqrt{2}\right)\mathrm{i}\right)\left(-\sqrt{2} + \left(\dfrac{3}{2}\sqrt{2}\right)\mathrm{i}\right)$ $= 2\left(1 + \dfrac{3}{2}\mathrm{i}\right)\left(-1 + \dfrac{3}{2}\mathrm{i}\right)$ $= 2\left(-1 - \dfrac{9}{4}\right) = -\dfrac{13}{2}$	Concepts: 1. $|z| = \sqrt{x^2 + y^2}$ 2. Form 1 equation using parametric to solve for *t*. 3. From (b), you have an idea that z_2 must come from the 2nd quadrant. Use symmetry of ellipse to get z_2 from z_1. 4. Use z_2 and z_1 to form linear factors to get the value of α and β.
		Total marks: 11

Qn	Suggested Solution	Comments
9(a)	Shortest distance $=\dfrac{\left\|5-\begin{pmatrix}1\\-2\\4\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}\right\|}{\sqrt{2^2+(-1)^2+2^2}}=\dfrac{7}{3}$ units	Formula
	Alternative By observation, (0,–5,0) is a point on the plane. Shortest distance $=\dfrac{\left\|\left[\begin{pmatrix}0\\-5\\0\end{pmatrix}-\begin{pmatrix}1\\-2\\4\end{pmatrix}\right]\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}\right\|}{\sqrt{2^2+(-1)^2+2^2}}=\dfrac{7}{3}$ units	Deduce any point on the plane (which satisfy $2x-y+2z=5$, then use length of projection onto the normal vector of π_1.
	Alternative Let foot of perpendicular by F. Since F is on the line passing through (1,–2,4) and perpendicular to the π_1, $\overrightarrow{OF}=\begin{pmatrix}1\\-2\\4\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\2\end{pmatrix}$ Solving simultaneously, $\left[\begin{pmatrix}1\\-2\\4\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\2\end{pmatrix}\right]\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}=5$ $\begin{pmatrix}1\\-2\\4\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}=5$ $(2+2+8)+\mu(4+1+4)=5$ $\mu=-\dfrac{7}{9}$	This method is not efficient in finding shortest distance between a point and a line. However, if in the case you need to find the coordinates of the foot of perpendicular, you should learn how to do this more efficiently by grouping the terms.
	Shortest distance $=\left\|\left[\begin{pmatrix}1\\-2\\4\end{pmatrix}-\dfrac{7}{9}\begin{pmatrix}2\\-1\\2\end{pmatrix}\right]-\begin{pmatrix}1\\-2\\4\end{pmatrix}\right\|$ $=\left\|-\dfrac{7}{9}\begin{pmatrix}2\\-1\\2\end{pmatrix}\right\|$ $=\dfrac{7}{9}\sqrt{9}$ $=\dfrac{7}{3}$ units	Note that there is no need to evaluate $\left[\begin{pmatrix}1\\-2\\4\end{pmatrix}-\dfrac{7}{9}\begin{pmatrix}2\\-1\\2\end{pmatrix}\right]$.

<table>
<tr>
<td>(b)</td>
<td>
$2x - y + 2z = 5$
$x + 3y - az = 3$
Direction vector of l:
$\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}1\\3\\-a\end{pmatrix}=\begin{pmatrix}a-6\\2a+2\\7\end{pmatrix}$

Let $z = 0$,
$2x - y = 5 --- (1)$
$x + 3y = 3 --- (2)$
$x = \frac{18}{7}, y = \frac{1}{7}$ (Using GC)
$\therefore l : \mathbf{r} = \begin{pmatrix}\frac{18}{7}\\\frac{1}{7}\\0\end{pmatrix} + \lambda\begin{pmatrix}a-6\\2a+2\\7\end{pmatrix}, \lambda \in \mathbb{R}$

Alternative
$2x - y + 2z = 5 --- (1)$
$x + 3y - az = 3 --- (2)$
$2\times(2) - 1 : 7y - (2a+2)z = 1$
$y = \frac{1}{7} + \frac{2a+2}{7}z$
$(2) + 3\times(1) : 7x + (-a+6)z = 18$
$x = \frac{18}{7} + \frac{a-6}{7}z$
Let $z = \lambda'$,
$l : \mathbf{r} = \begin{pmatrix}\frac{18}{7}\\\frac{1}{7}\\0\end{pmatrix} + \lambda'\begin{pmatrix}\frac{a-6}{7}\\\frac{2a+2}{7}\\1\end{pmatrix}, \lambda' \in \mathbb{R}$
$\therefore l : \mathbf{r} = \begin{pmatrix}\frac{18}{7}\\\frac{1}{7}\\0\end{pmatrix} + \lambda\begin{pmatrix}a-6\\2a+2\\7\end{pmatrix}, \lambda \in \mathbb{R} \quad (\lambda = \frac{1}{7}\lambda')$
</td>
<td>


l is a common line on both planes. This means that $l \perp \mathbf{n}_1$ and $l \perp \mathbf{n}_2$.
Thus l is parallel to $(\mathbf{n}_1 \times \mathbf{n}_2)$.
To find a point on the line, the easiest way is to let one of the coordinates be zero so that we can solve 2 unknowns with 2 equations. Since the unknown a is the coefficient of z, we will find the common point of the two planes where its z-coordinate = 0.

To solve this algebraically, we need to make x and y in terms of z respectively.

Then the numbers independent of z will give the point on the plane, and the coefficients of z will give the direction vectors of the line of intersection.

You may also make y and z in terms of x, or make x and z in terms of y.
</td>
</tr>
<tr>
<td>(c)</td>
<td>
For l to be parallel to π_3,
$\begin{pmatrix}a-6\\2a+2\\7\end{pmatrix}\bullet\begin{pmatrix}b\\-2\\4\end{pmatrix} = 0$
$ab - 6b - 4a - 4 + 28 = 0$
$ab - 6b - 4a + 24 = 0$
$b(a-6) - 4(a-6) = 0$
$(a-6)(b-4) = 0$ (shown)
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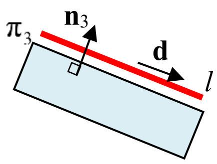
<td>
When a line is parallel to a plane, its direction vector is perpendicular to the normal vector of the plane.


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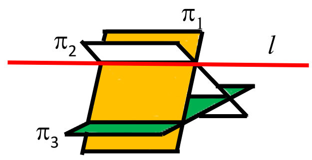
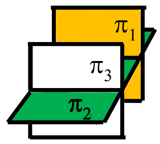
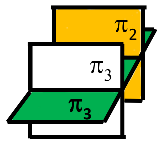
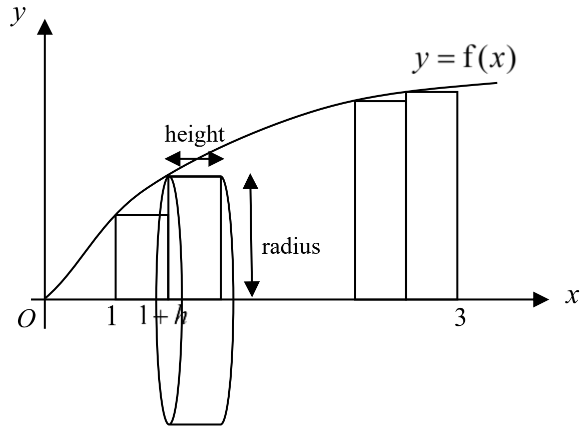
(d)	$\pi_1 : \mathbf{r}\bullet\begin{pmatrix}2\\-1\\2\end{pmatrix}=5\,;\quad \pi_2 : \mathbf{r}\bullet\begin{pmatrix}1\\3\\-a\end{pmatrix}=3\quad;\quad \pi_3 : \mathbf{r}\bullet\begin{pmatrix}b\\-2\\4\end{pmatrix}=3$ For the three planes form a triangular prism, **(1)** The line of intersection between π_1 and π_2, l, is parallel to $\pi_3 \Rightarrow a=6$ **or** $b=4$ (from **(c)**) **(2)** l is not on π_3 regardless of b. $\begin{pmatrix}\frac{18}{7}\\\frac{1}{7}\\0\end{pmatrix}\bullet\begin{pmatrix}b\\-2\\4\end{pmatrix}\neq 3$ $b\left(\frac{18}{7}\right)-2\left(\frac{1}{7}\right)+4(0)\neq 3\Rightarrow b\neq\frac{23}{18}$ **(3)** π_1 is not parallel to π_3 $\begin{pmatrix}2\\-1\\2\end{pmatrix}\neq m\begin{pmatrix}b\\-2\\4\end{pmatrix}\Rightarrow m\neq\frac{1}{2}\Rightarrow b\neq 4$ **(4)** π_2 is not parallel to π_3 $\begin{pmatrix}1\\3\\-a\end{pmatrix}\neq k\begin{pmatrix}b\\-2\\4\end{pmatrix}\Rightarrow k\neq-\frac{3}{2}\Rightarrow b\neq-\frac{2}{3}$ **or** $a\neq 6$ $a=6$ and $b\in\mathbb{R}\setminus\left\{-\frac{2}{3},\ \frac{23}{8},\ 4\right\}$	 The condition "or" means that we only need to make sure **either** $a=6$ **or** $b=4$, **not both.** To ensure this, we must make sure any point on l is not a point on π_3, i.e. does not satisfy the equation of π_3. To **exclude** this case:  To **exclude** this case:  The condition "or" means that we only need to make sure **either** $b\neq-\frac{2}{3}$ **or** $a\neq 6$, **not both.**
		Total marks: 12

<table>
<tr><th>Qn</th><th>Suggested Solution</th><th>Comments</th></tr>
<tr><td>10(a)</td><td>



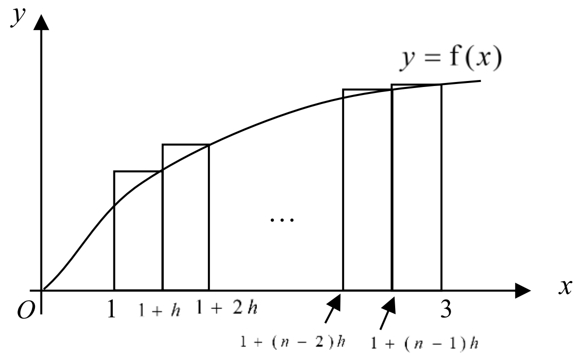
$$V=\pi h\left[\mathrm{f}(1)\right]^2+\pi h\left[\mathrm{f}(1+h)\right]^2+\ldots+\pi h\left[\mathrm{f}(1+(n-1)h)\right]^2$$
$$=\pi h\left(\left[\mathrm{f}(1)\right]^2+\left[\mathrm{f}(1+h)\right]^2+\ldots+\left[\mathrm{f}(1+(n-1)h)\right]^2\right)$$
$$=\pi h\sum_{r=0}^{n-1}\left[\mathrm{f}(1+rh)\right]^2 \quad \text{(shown)}$$
$$h=\frac{3-1}{n}=\frac{2}{n}$$

</td><td>

- Total volume of n cylindrical discs $=\sum[\pi(\text{radius})^2(\text{height})]$. In this qn, radius $=\mathrm{f}(x)$, height=h. It's incorrect to use sum area of n rectangles to become sum volume of n cylindrical discs.
- As this is a "show" question, working must be clearly presented to get full credit.
- Read the question carefully, as many students seem to have forgotten to find h.

</td></tr>
<tr><td>(b)</td><td>

$$V_2=\pi h\left[\mathrm{f}(1+h)\right]^2+\pi h\left[\mathrm{f}(1+2h)\right]^2+\ldots+\pi h\left[\mathrm{f}(1+nh)\right]^2$$
$$=\pi h\left(\left[\mathrm{f}(1+h)\right]^2+\left[\mathrm{f}(1+2h)\right]^2+\ldots+\left[\mathrm{f}(1+nh)\right]^2\right)$$
$$=\pi h\sum_{r=1}^{n}\left[\mathrm{f}(1+rh)\right]^2$$



</td><td></td></tr>
<tr><td>(c)</td><td>

From GC:

$$\lim_{n\to\infty}V_1=V=\pi\int_1^3\left(\frac{x^2}{\sqrt{1+\mathrm{e}^x}}\right)^2\mathrm{d}x=12.291=12.3\ \text{unit}^3\ \ (3\text{ sf})$$

</td><td>

- In general, volume of revolution = limiting sum of volume of cylindrical discs, $\lim_{n\to\infty}V_1=\pi\int y^2\ \mathrm{d}x$.
- Use GC to evaluate the numerical volume. The question didn't ask for "exact" volume, so no need to perform tedious integration. The low marks given indicate the effort/time needed.

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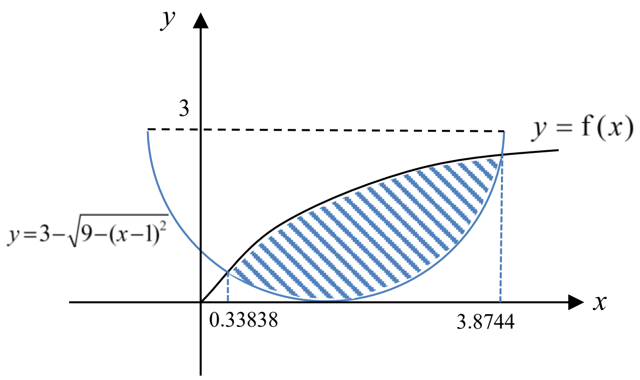
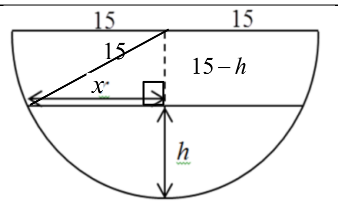
(d)	 $(x-1)^2+(y-3)^2=9.$ $y=3\pm\sqrt{9-(x-1)^2}$ The required eqn is $y=3-\sqrt{9-(x-1)^2}$ From the GC, points of intersection are $x=0.33838$ or 3.8744 Area $=\int_{0.33838}^{3.8744}\left(\frac{x^2}{\sqrt{1+e^x}}\right)-\left(3-\sqrt{9-(x-1)^2}\right)dx$ $=3.01\text{ unit}^2$ (3 sf)	• In general, when doing area/volume questions, **sketch a diagram first** to see the relationship between the curves involved. Once the required region is identified, proceed to find other relevant info e.g. intersection points in this case. Use 5sf intermediate answers for better accuracy before presenting in 3sf for the final answer. • $f(x)$ is defined for $x\geq 0$ so we should focus on the correct region. The question asked for the "area bounded" by the two curves only. Hence other "non-relevant" areas should not be included e.g. $\int_0^{0.33838}\left(3-\sqrt{9-(x-1)^2}\right)-(f(x))dx$ is bounded by the two curves + y-axis.
		Total marks: 10

Qn	Suggested Solutions	Comments
11(a)	$k=2\pi r^2(30)+2\pi rL(10)+\pi r^2(20)$ $k=80\pi r^2+20\pi rL$ $k=20\pi r(4r+L)$ $\Rightarrow L=\frac{k}{20\pi r}-4r$ $V=\frac{2}{3}\pi r^3+\pi r^2 L$ $=\frac{2}{3}\pi r^3+\pi r^2\left(\frac{k}{20\pi r}-4r\right)$ $=\frac{2}{3}\pi r^3+\frac{kr}{20}-4\pi r^3$ $=\frac{k}{20}r-\frac{10}{3}\pi r^3$ (shown)	• This is a 'show' question so clear steps must be shown as the answer is already given. • Students need to expand and simplify $\pi r^2\left(\frac{k}{20\pi r}-4r\right)$ by expanding to get $\frac{kr}{20}-4\pi r^3$. • Note that k is a fixed constant.

(b)	For maximum V as r varies, $\frac{dV}{dr}=0$ $\frac{dV}{dr}=\frac{k}{20}-10\pi r^2$ $r=\sqrt{\frac{k}{200\pi}}$ (reject $r<0$) $\frac{d^2V}{dr^2}=-20\pi r<0$ Therefore, V is maximum. Cost of material for the hemispherical base $=2\pi r^2(30)=2\pi\left(\sqrt{\frac{k}{200\pi}}\right)^2(30)=\frac{3k}{10}$ dollars	• Writing 'it is a maximum' is vague, students need to write ***V*** is a maximum instead. • **Cost** of the material is requested so we need to multiply 30 to $2\pi\left(\sqrt{\frac{k}{200\pi}}\right)^2$.
(c)(i)	 By Pythagoras' Theorem, $x^2=15^2-(15-h)^2$ $x^2=(15-(15-h))(15+(15-h))$ $x^2=h(30-h)$ $W=\frac{\pi h(3x^2+h^2)}{6}=\frac{\pi h(3(h(30-h))+h^2)}{6}$ $W=\frac{\pi h(90h-2h^2)}{6}=\frac{\pi(45h^2-h^3)}{3}$ $\frac{dW}{dh}=\frac{\pi(90h-3h^2)}{3}$ $\frac{dW}{dt}=\frac{dW}{dh}\times\frac{dh}{dt}=\frac{\pi(90h-3h^2)}{3}\times\frac{dh}{dt}$ At $h=3$, $\frac{dh}{dt}=\frac{\frac{dW}{dt}}{\frac{dW}{dh}}=\frac{-20}{\frac{\pi(90(3)-3(3)^2)}{3}}$ $=-0.078595=-0.0786$ cm/min (3 s.f.)	• Key idea is to form a right-angle triangle by finding the height, 15-h and apply pythagoras theorem to link up x and h. • We are finding the rate of change related to the **water in the tank** and the question has already denoted it by W. Don't use another notation. • As a general guide: what is given and what do you need to find? Thereafter, form the connected rate of change equation: $\frac{dh}{dt}$ (find) $=\frac{dW}{dt}$ (given) $\times$? (to form) $\therefore$? $=\frac{dh}{dW}$ Also, $\frac{dW}{dt}$ is **negative** 20 as the question states water is **leaking** $\Rightarrow$ volume of water is **decreasing** as time increases. • It is easier to differentiate W wrt to h than wrt x.

(c)(ii)	$\frac{dx}{dt}=\frac{dx}{dh}\times\frac{dh}{dt}$ $x^2=h(30-h)\Rightarrow x=\sqrt{h(30-h)}=\sqrt{30h-h^2}\quad (x>0)$ $\frac{dx}{dh}=\frac{30-2h}{2\sqrt{30h-h^2}}=\frac{15-h}{\sqrt{30h-h^2}}$ At $h=3$, $\frac{dx}{dt}=\left(\frac{15-3}{\sqrt{30(3)-(3)^2}}\right)\times(-0.078595)$ $=-0.10479=-0.105$ cm/min (3sf)	• Some students use an alternative method to differentiate W instead but it is more tedious. We cannot sub. in $h = 3$ into $W=\frac{\pi h(3x^2+h^2)}{6}$ then differentiate $W=\frac{3\pi(3x^2+9)}{6}$ wrt x. Doing so, we are treating h as a constant which is not true. ❖ Always **differentiate first** then sub. the given value.
		Total marks: 12