**MOE H3 Math     Mathematical Proofs and Reasoning**

**Problem Set 3**

1. Use Mathematical Induction to prove the following:

   (a) For each positive integer $n$, $1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n + 1)! - 1$.

   **Solution:** Let $P(n) : 1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n + 1)! - 1$.
   Basis step: Let $n = 1$. Left side has just one term, namely $1(1!) = 1$.
   Right side is $(1 + 1)! - 1 = 1$. So $P(1)$ is true.
   Inductive step: Assume $P(k)$ is true. So

   $$1(1!) + 2(2!) + 3(3!) + \cdots + k(k!) = (k + 1)! - 1.$$

   We now add $(k + 1)[(k + 1)!]$ to both sides of this equation. This gives

   $$\begin{aligned} 1(1!) + 2(2!) + 3(3!) + \cdots + k(k!) + (k+1)[(k+1)!] &= (k+1)! - 1 + (k+1)[(k+1)!] \\ &= [(k+1)!](k+2) - 1 \\ &= (k+2)! - 1 \end{aligned}$$

   This proves that if $P(k)$ is true, then $P(k + 1)$ is true.
   Hence by Mathematical induction, we have proven: For each positive integer $n$, $1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n + 1)! - 1$.

   (b) For each positive integer $n$, $(-7)^n - 9^n$ is divisible by 16.

   **Solution:** Let $P(n) : (-7)^n - 9^n$ is divisible by 16.
   Basis step: Let $n = 1$. $(-7)^1 - 9^1 = -16$ is divisible by 16.
   So $P(1)$ is true.
   Inductive step: Assume $P(k)$ is true.
   So $(-7)^k - 9^k$ is divisible by 16, which means $(-7)^k - 9^k = 16a$ for some integer $a$.
   We now multiply 9 to both sides of this equation. This gives

   $$\begin{aligned} 9[(-7)^k - 9^k] &= 9(16a) \\ 9(-7)^k - 9^{k+1} &= 9(16a) \\ (16 - 7)(-7)^k - 9^{k+1} &= 9(16a) \\ (-7)(-7)^k - 9^{k+1} &= 9(16a) - (16)(-7)^k \\ (-7)^{k+1} - 9^{k+1} &= 16[9a - (-7)^k] \end{aligned}$$

   i.e. $(-7)^{k+1} - 9^{k+1}$ is divisible by 16.
   This proves that if $P(k)$ is true, then $P(k + 1)$ is true.
   Hence by Mathematical induction, we have proven: For each positive integer $n$, $(-7)^n - 9^n$ is divisible by 16.

(c) For each positive integer $n$ with $n \geq 3$, $\left(1+\frac{1}{n}\right)^n < n$.

**Solution:** Let $P(n) : \left(1+\frac{1}{n}\right)^n < n$.

Basis step: Let $n = 3$.

Since $\left(1+\frac{1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27} < 3$, so $P(3)$ is true.

Inductive step: Assume $P(k)$ is true (with $k \geq 3$).

This means $\left(1+\frac{1}{k}\right)^k < k \quad (*)$.

Multiplying $\left(1+\frac{1}{k}\right)$ on both sides of $(*)$, we have

$$\left(1+\frac{1}{k}\right)^{k+1} < k\left(1+\frac{1}{k}\right) = k+1 \qquad (**)$$

On the other hand, since $\frac{1}{k+1} < \frac{1}{k}$, then $1+\frac{1}{k+1} < 1+\frac{1}{k}$ and hence

$$\left(1+\frac{1}{k+1}\right)^{k+1} < \left(1+\frac{1}{k}\right)^{k+1} \qquad (***).$$

Combining $(**)$ and $(***)$, we have $\left(1+\frac{1}{k+1}\right)^{k+1} < k+1$.

This proves that if $P(k)$ is true, then $P(k+1)$ is true.

Hence by Mathematical induction, we have proven: For each positive integer $n$ with $n \geq 3$, $\left(1+\frac{1}{n}\right)^n < n$.

(d) For each positive integer $n$ with $n \geq 6$, $n^3 < n!$.

**Solution:** Let $P(n) : n^3 < n!$.

Basis step: Let $n = 6$.

$6^3 = 216$ and $6! = 720 > 6^3$.

So $P(6)$ is true.

Inductive step: Suppose $P(k)$ is true where $k \geq 6$ (i.e. $k! > k^3$).

To show $P(k+1)$ is true.

First of all, note that $(k+1)^2 = k^2 + 2k + 1 < k^2 + k^2 + k^2 = 3k^2 \leq k^3 \quad (*)$ when $k \geq 3$.

Now

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &> (k+1)k^3 \quad \text{induction hypothesis} \\ &> (k+1)(k+1)^2 \quad \text{by (*)} \\ &= (k+1)^3\end{aligned}$$

This proves that if $P(k)$ is true, then $P(k+1)$ is true.
Hence by Mathematical induction, we have proven: $n! > n^3$ for all $n \geq 6$.

2. Let $f_1, f_2, \ldots, f_n, \ldots$ be the Fibonacci sequence.
   i.e. The sequence is defined recursively by

$$f_1 = 1 \text{ and } f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3$$

   Prove each of the following:

   (a) For each positive integer $n$, $f_{5n}$ is a multiple of 5.

   **Solution:** Let $P(n) : f_{5n}$ is a multiple of 5.
   Basis step: Let $n = 1$. We see that $f_5 = 5$ which is divisible by 5.
   So $P(1)$ is true.
   Inductive step: Assume $P(k)$ is true. This means $f_{5k}$ is a multiple of 5.

$$\begin{aligned}\text{Now } f_{5(k+1)} = f_{5k+5} &= f_{5k+4} + f_{5k+3} \\ &= (f_{5k+3} + f_{5k+2}) + (f_{5k+2} + f_{5k+1}) \\ &= f_{5k+3} + 2f_{5k+2} + f_{5k+1} \\ &= (f_{5k+2} + f_{5k+1}) + 2(f_{5k+1} + f_{5k}) + f_{5k+1} \\ &= f_{5k+2} + 4f_{5k+1} + 2f_{5k} \\ &= (f_{5k+1} + f_{5k}) + 4f_{5k+1} + 2f_{5k} \\ &= 5f_{5k+1} + 3f_{5k}\end{aligned}$$

   Since $5f_{5k+1}$ is a multiple of 5 (by definition) and $f_{5k}$ is a multiple of 5 (by hypothesis), so the right side is a multiple of 5.
   This proves that if $P(k)$ is true, then $P(k+1)$ is true.
   Hence by Mathematical induction, we have proven: For each positive integer $n$, $f_{5n}$ is a multiple of 5.

   (b) For each positive integer $n$, $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.

   **Solution:** Let $P(n) : f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$
   Basis step: Let $n = 1$. Left side is just the term $f_1$ and right side is $f_2$. Since both $f_1$ and $f_2$ are 1, $P(1)$ is true.

Inductive step: Assume $P(k)$ is true. This means $f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}$. Now add $f_{2k+1}$ on both sides:

$$\begin{aligned} f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} &= f_{2k} + f_{2k+1} \\ f_1 + f_3 + \cdots + f_{2k-1} + f_{2(k+1)-1} &= f_{2k+2} = f_{2(k+1)} \end{aligned}$$

This proves that if $P(k)$ is true, then $P(k+1)$ is true.

Hence by Mathematical induction, we have proven: For each positive integer $n$, $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.

(c) For each positive integer $n$, $2f_n + 3f_{n+1} = f_{n+4}$.

**Solution:** Let $P(n) : 2f_n + 3f_{n+1} = f_{n+4}$.

Basis step: $P(1)$ and $P(2)$

$2f_1 + 3f_2 = 2 + 3 = 5 = f_5$. So $P(1)$ is true.

$2f_2 + 3f_3 = 2 + 6 = 8 = f_6$. So $P(2)$ is true.

Inductive step: Assume $P(k-1)$ and $P(k)$ are true.

To show $P(k+1)$ is true.

$$\begin{aligned} & 2f_{k+1} + 3f_{k+2} \\ = \; & 2(f_{k-1} + f_k) + 3(f_k + f_{k+1}) \\ = \; & (2f_{k-1} + 3f_k) + (2f_k + 3f_{k+1}) \\ = \; & f_{k+3} + f_{k+4} \quad \text{by } P(k-1) \text{ and } P(k) \text{ of the induction hypothesis} \\ = \; & f_{k+5} \end{aligned}$$

This proves that if $P(k-1)$ and $P(k)$ are true, then $P(k+1)$ is true.

Hence by Mathematical induction, we have proven: $2f_n + 3f_{n+1} = f_{n+4}$ for all $n \in \mathbb{Z}^+$.

(d) For each positive integer $n$, $f_n$ is even if and only if $3 \mid n$.

**Solution:** The biconditional statement that can be rephrased as:
(i) if $3 \mid n$, then $f_n$ is even;
(ii) if $3 \nmid n$, then $f_n$ is odd.

For (i), we let $P(n) : f_n$ is even.
We shall use a variation of induction to prove $P(n)$ for all $n = 3h$ with $h \in \mathbb{Z}^+$.
Basis step: $P(3)$ is true, as $f_3 = 2$ which is even.
Inductive step: $P(k) \to P(k+3)$.
Assume $f_k$ is even. Then

$$\begin{aligned} f_{k+3} &= f_{k+2} + f_{k+1} \\ &= (f_{k+1} + f_k) + f_{k+1} \\ &= 2f_{k+1} + f_k \end{aligned}$$

Since both $2f_{k+1}$ and $f_k$ are even, so $f_{k+3}$ is even.

Hence, by PMI, $f_n$ is even for all $n = 3h$ with $h \in \mathbb{Z}^+$.

For (ii), we let $P(n) : f_n$ is odd.
We shall use a variation of induction to prove $P(n)$ for all $n = 3h+1$ and $n = 3h+2$ with $h \in \mathbb{Z}^+ \cup \{0\}$.
Basis step: $P(1)$ and $P(2)$ are true, as $f_1 = f_2 = 1$ which are odd.
Inductive step: $P(k) \to P(k+3)$.
Assume $f_k$ is odd. Then

$$\begin{aligned} f_{k+3} &= f_{k+2} + f_{k+1} \\ &= (f_{k+1} + f_k) + f_{k+1} \\ &= 2f_{k+1} + f_k \end{aligned}$$

Since $2f_{k+1}$ is even and $f_k$ is odd, so $f_{k+3}$ is odd.
Hence, by PMI, $f_n$ is odd for all $n = 3h+1$ and $n = 3h+2$ with $h \in \mathbb{Z}^+ \cup \{0\}$.

3. Use mathematical induction together with the following result:

For any prime $p$ and any integers $a, b$, if $p \mid ab$ and $p \nmid a$, then $p \mid b$.

prove that, for any prime $p$ and any integers $q_1, q_2, \cdots, q_n$, if $p \mid q_1 q_2 \cdots q_n$, then $p \mid q_i$ for some $i$.

**Solution:** Let $P(n)$: If $p \mid q_1 q_2 \cdots q_n$, then $p \mid q_i$ for some $i$.

Basis step: $P(1)$ is true, as the statement becomes:
If $p \mid q_1$, then $p \mid q_i$ for some $i$, and we can just take $i = 1$.

Inductive step: $P(k) \to P(k+1)$.

Suppose $p \mid q_1 q_2 \cdots q_{k+1}$. If $p \mid q_{k+1}$, then we are done.
If $p \nmid q_{k+1}$, by (*), we have $p \mid q_1 \cdots q_k$. By induction hypothesis, $p \mid q_i$ for some $i$.

Hence we have proven the inductive step.

By mathematical induction, the statement is proven for all integer $n$.

4. Suppose you want to prove $P(n)$ is true (only) for all integers $n \geq 7$ that are <u>not divisible by 4</u> using a version of mathematical induction as follow:

   (i) Basis step: $P(a), P(b), P(c)$ are true; and

   (ii) Inductive step: $(\forall\, k \in \mathbb{Z}^+)\ \ P(k) \to P(k+d)$ is true.

   What should be the values for $a, b, c, d$?

**Solution:** Explicitly, we want to prove $P(n)$ for $n = 7, 9, 10, 11, 13, 14, \ldots$

Since we only want integers that are not divisible by 4, we only consider those integers that have remainders 1, 2, 3 modulo 4.

So in the inductive step, we should have $P(k) \to P(k+4)$, together with the basis step with three terms corresponding to remainders 1, 2 and 3. Since we require $n \geq 7$, the three terms in the basis step should be $7, 9, 10$.

Therefore, we have $a = 7, b = 9, c = 10, d = 4$.

5. Find the mistake in the following "proof" that purports to show that:
Every nonnegative integer power of every nonzero real number is 1.

*Proof.* Let $r$ be any nonzero real number and let the predicate $P(n)$ be

$$P(n) : r^n = 1.$$

Basis step: $P(0)$ is true because $r^0 = 1$ by definition of 0-th power.
Inductive step: $P(k-1)$ and $P(k) \to P(k+1)$

Suppose that $r^{k-1} = 1$ and $r^k = 1$. This is the induction hypothesis. We must show that $r^{k+1} = 1$. Now

$$\begin{aligned} r^{k+1} &= r^{k+k-(k-1)} \\ &= \frac{r^k \cdot r^k}{r^{k-1}} \\ &= \frac{1 \cdot 1}{1} \quad \text{(by induction hypothesis)} \\ &= 1 \end{aligned}$$

Thus $r^{k+1} = 1$.
Hence the inductive step is proven. □

**Solution:** The mistake is in the inductive step when $k = 0$. In this case, $r^{k-1} = r^{-1}$ which is not equal to 1 unless $r = 1$. Therefore, we are not able to prove $P(0) \to P(1)$. Hence $P(1)$ and the rest of the $P(n)$'s cannot be proven to be true.

6. Prove that for integers $n \geq 2$,

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}.$$

**Solution:** Let $P(n):\quad \prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$.

Base case $n=2$:

$$\prod_{i=2}^{2}\left(1-\frac{1}{i^2}\right)=1-\frac{1}{2^2}=\frac{3}{4}\quad\text{and}\quad\frac{2+1}{2\cdot 2}=\frac{3}{4}.$$

Thus $P(2)$ holds.

Inductive step: Assume $P(n)$ holds for some $n\geq 2$, i.e.

$$\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}.$$

Then

$$\prod_{i=2}^{n+1}\left(1-\frac{1}{i^2}\right)=\left(\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)\right)\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+1}{2n}\cdot\left(\frac{(n+1)^2-1}{(n+1)^2}\right).$$

Since $(n+1)^2-1=n(n+2)$, we get

$$\frac{n+1}{2n}\cdot\frac{n(n+2)}{(n+1)^2}=\frac{n+2}{2(n+1)}.$$

This is exactly $P(n+1)$.

By mathematical induction, $P(n)$ holds for all integers $n\geq 2$.

7. Suppose $x$ is a real number with $x\neq 0$ and $x+\frac{1}{x}\in\mathbb{Z}$. Prove by induction on $n$ that

$$x^n+\frac{1}{x^n}\in\mathbb{Z}\quad\text{for all } n\in\mathbb{Z}_{>0}.$$

**Solution:** Let $S_n:=x^n+x^{-n}$.

Consider, as hinted,

$$\left(x^k+\frac{1}{x^k}\right)\left(x+\frac{1}{x}\right)=x^{k+1}+x^{k-1}+x^{-(k-1)}+x^{-(k+1)}=\left(x^{k+1}+x^{-(k+1)}\right)+\left(x^{k-1}+x^{-(k-1)}\right).$$

That is

$$S_{k+1}=\left(x+\frac{1}{x}\right)S_k-S_{k-1}\qquad(k\geq 1),$$

Base cases. $S_1=x+x^{-1}\in\mathbb{Z}$ by hypothesis. Also,

$$S_2=(x+x^{-1})^2-2\in\mathbb{Z},$$

since $x + x^{-1} \in \mathbb{Z}$.

Inductive step (strong). Assume $S_m \in \mathbb{Z}$ for all $1 \le m \le k$ with $k \ge 2$. Then by the recurrence,

$$S_{k+1} = \left(x + \frac{1}{x}\right) S_k - S_{k-1},$$

and each factor on the right-hand side is an integer by the hypothesis (and the assumption that $x + x^{-1} \in \mathbb{Z}$). Hence $S_{k+1} \in \mathbb{Z}$.

By strong induction, $S_n = x^n + x^{-n} \in \mathbb{Z}$ for all $n \ge 1$.

8. Prove that for positive integers $n$ and positive real numbers $x_1, \dots, x_n$,

$$\frac{1}{n}\sum_{i=1}^{n} x_i \;\ge\; \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

First prove the statement for $n = 2^m$ for $m \ge 0$ by induction on $m$. The general result now follows by proving the converse of the usual inductive step: if the result holds for $n = k + 1$, where $k$ is a positive integer, then it holds for $n = k$.

**Solution:** we will prove the statement for $n = 2^m$ by induction on $m \ge 0$.

Base cases: When $m = 0$, $n = 1$ is trivial. When $m = 1$, $n = 2$. Then

$$\left(\frac{x_1 + x_2}{2}\right)^2 - x_1x_2 = \frac{(x_1 - x_2)^2}{4} \ge 0 \implies \frac{1}{2}(x_1 + x_2) \ge (x_1x_2)^{\frac{1}{2}}.$$

Inductive step. Assume the inequality holds for $n = 2^m$. Given $2^{m+1}$ positive numbers $x_1, \dots, x_{2^{m+1}}$, form the pairwise means

$$y_j = \frac{x_{2j-1} + x_{2j}}{2} \qquad (1 \le j \le 2^m).$$

By the $n = 2$ case, $y_j \ge \sqrt{x_{2j-1}x_{2j}}$, hence

$$\prod_{j=1}^{2^m} y_j \;\ge\; \left(\prod_{i=1}^{2^{m+1}} x_i\right)^{1/2}.$$

Apply the induction hypothesis to $y_1, \dots, y_{2^m}$:

$$\frac{1}{2^m}\sum_{j=1}^{2^m} y_j \;\ge\; \left(\prod_{j=1}^{2^m} y_j\right)^{1/2^m}.$$

The left-hand side equals $\dfrac{1}{2^{m+1}}\displaystyle\sum_{i=1}^{2^{m+1}} x_i$. Combining the two displayed inequalities gives

$$\frac{1}{2^{m+1}}\sum_{i=1}^{2^{m+1}} x_i \;\ge\; \left(\prod_{i=1}^{2^{m+1}} x_i\right)^{1/2^{m+1}}.$$

Thus AM≥GM holds for $n = 2^{m+1}$.

By induction, AM≥GM holds for all $n = 2^m$.

Now we will prove the converse step: $k + 1 \Rightarrow k$.

Assume the inequality holds for some $k + 1 \geq 2$. Let $x_1, \ldots, x_k > 0$ and set their arithmetic mean $A = \frac{1}{k}\sum_{i=1}^{k} x_i$. Apply the $k + 1$ case to the numbers $x_1, \ldots, x_k, A$:

$$\frac{1}{k+1}\Big(\sum_{i=1}^{k} x_i + A\Big) \;\geq\; \Big(A\prod_{i=1}^{k} x_i\Big)^{1/(k+1)}.$$

Since $\sum_{i=1}^{k} x_i = kA$, the left-hand side is $A$. Raise both sides to the power $k + 1$ and divide by $A\,(> 0)$:

$$A^k \;\geq\; \prod_{i=1}^{k} x_i \quad\Longleftrightarrow\quad A \;\geq\; \Big(\prod_{i=1}^{k} x_i\Big)^{1/k}.$$

Thus, if AM≥GM holds for $k + 1$, it holds for $k$.

Conclusion for all $n$.

For any $n \in \mathbb{Z}_{>0}$, choose $m$ with $2^m \geq n$. By Step 1 the inequality holds for $2^m$. Applying Step 2 repeatedly (from $2^m$ down to $n$) yields the result for $n$.

Hence, for all positive integers $n$ and all $x_1, \ldots, x_n > 0$,

$$\frac{1}{n}\sum_{i=1}^{n} x_i \;\geq\; \Big(\prod_{i=1}^{n} x_i\Big)^{1/n}.$$

## Hints

1. *Hint for Question 1.* (a) In the inductive step, add an additional term to both sides of $P(k)$.
(b) In the inductive step, multiply 9 to both sides of $P(k)$.
(c) Basis step is $n = 3$. You need to use the inequality $\left(1+\dfrac{1}{k+1}\right)^{k+1} < \left(1+\dfrac{1}{k}\right)^{k+1}$ in the inductive step.
(d) Basis step is $n = 3$. You need to use the inequality $(k+1)^2 \leq k^3$ for $k \geq 3$ in the inductive step.

2. *Hint for Question 2.* (a) Let $P(n)$ be the predicate '$f_{5n}$ is a multiple of 5'. In the inductive step, try to show $f_{5(k+1)} = 5f_{5k+1} + 3f_{5k}$. (Use the recursive relation of Fibonacci sequence.)
(b) In the inductive step, $P(k) : f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}$. Add $f_{2k+1}$ to both sides of $P(k)$.
(c) Prove $P(1)$ and P(2) in the basis step; and use $P(k-1)$ and $P(k)$ in the induction hypothesis.
(d) Note that this is a biconditional statement that can be rephrased as:
(i) if $3 \mid n$, then $f_n$ is even; (ii) if $3 \nmid n$, then $f_n$ is odd.
For each case, use a variation of PMI to prove it. The inductive step should be $P(k) \to P(k+3)$.

3. *Hint for Question 3.* Use PMI on the number of terms in the product.

4. *Hint for Question 7.* Use Principle of Strong Induction.
For the inductive step consider $\left(x^k + \dfrac{1}{x^k}\right)\left(x + \dfrac{1}{x}\right)$.

5. *Hint for Question 8.* Assuming statement holds for $n = 2^m$, to show that statement holds for $2^{m+1}$, define
$$y_j = \frac{x_{2j-1} + x_{2j}}{2} \qquad (1 \leq j \leq 2^m).$$
Use the $n = 2$ case to show that $y_j \geq (x_{2j-1}x_{2j})^{1/2}$.
To extend beyond powers of two, assume AM$\geq$GM holds for some $k+1 \geq 2$. For $x_1, \ldots, x_k > 0$, let $A = \dfrac{1}{k}\displaystyle\sum_{i=1}^{k} x_i$. Apply the $k+1$ case to $x_1, \ldots, x_k, A$.