

SUMMARY: Differential Equations

	Concept	Example
1	Type I: $\frac{dy}{dx} = f(x)$ Method: Integrate both sides with respect to x directly. $\frac{dy}{dx} = f(x)$ $\int dy = \int f(x) dx$ $y = \int f(x) dx$	Example 1: Solve the differential equation $\frac{dy}{dx} = \frac{1}{\sqrt{9-4x^2}}$, where $-\frac{3}{2} < x < \frac{3}{2}$ and hence find the equation of the curve for which $y = 1$ when $x = 0$. Solution: $\frac{dy}{dx} = \frac{1}{\sqrt{9-4x^2}}$ Integrating both sides w.r.t. x: $y = \int \frac{1}{\sqrt{9-4x^2}} dx$ $= \int \frac{1}{2\sqrt{\left(\frac{3}{2}\right)^2 - x^2}} dx$ This is a general solution. It is expressed in terms of c, an arbitrary constant. $= \frac{1}{2} \sin^{-1} \frac{2x}{3} + C \quad \text{----- (1)}$ Sub $y = 1$ and $x = 0$ into (1), $1 = \frac{1}{2} \sin^{-1}(0) + C \Rightarrow C = 1$ $\therefore y = \frac{1}{2} \sin^{-1} \frac{2x}{3} + 1 \quad \text{----- (2)}$ This is a particular solution. We substitute initial conditions into the general equation to find c.
	Type II: $\frac{dy}{dx} = g(y)$ Method: Separate variables and integrate both sides respectively. $\frac{dy}{dx} = g(y)$ $\int \frac{1}{g(y)} dy = \int 1 dx$ $\int \frac{1}{g(y)} dy = x$	Example 2: Find the general solution of the differential equation $\frac{dy}{dx} = 2y$, $y \neq 0$, expressing y in terms of x. Solution: $\frac{dy}{dx} = 2y$ $\int \frac{1}{y} dy = \int 2 dx$ $\ln y = 2x + C$ $ y = e^{2x+C}$ $y = \pm e^{2x+C}$ $y = Ae^{2x}, \quad A = \pm e^C$

Type III: $\frac{dy}{dx} = f(x)g(y)$ Method: Separate variables and integrate both sides respectively. $\frac{dy}{dx} = f(x)g(y)$ $\int \frac{1}{g(y)} dy = \int f(x) dx$	Example 3: Find the general solution of the differential equation $\frac{dy}{dx} - x = xy^2$, expressing y in terms of x. Solution: $\frac{dy}{dx} = x + xy^2$ $\frac{dy}{dx} = x(1 + y^2)$ $\frac{1}{1 + y^2} \frac{dy}{dx} = x$ $\int \frac{1}{1 + y^2} dy = \int x dx$ $\tan^{-1} y = \frac{1}{2} x^2 + C$ $y = \tan\left(\frac{1}{2} x^2 + C\right)$
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	<u>Concept</u>	<u>Example</u>
2	Solving DE by Substitution	
	Use given substitution to simplify the DE to either Type I, Type II or Type III first and solve the new DE	Example 4: Use the substitution $y = vx$ to solve $x \frac{dy}{dx} = 3x + y, x > 0.$
	Step 1: Differentiate the given substitution to express $\frac{dy}{dx}$ in terms of $\frac{dv}{dx}$.	Solution: Differentiate $y = vx$ with respect to x on both sides, $\frac{dy}{dx} = v + x \frac{dv}{dx} \text{ ----- (1)}$
	Step 2: Substitute into DE, and simplify to obtain DE in two variables v and x .	Sub (1) and $y = vx$ into the original D.E. : $x \left(v + x \frac{dv}{dx} \right) = 3x + vx$ $\frac{dv}{dx} = \frac{3}{x}$
	Step 3: Solve the reduced DE.	$v = \int \frac{3}{x} dx$ $= 3 \ln x + C$
	Step 4: Substitute the original variables back into the final answer.	Replace v with $\frac{y}{x}$: $\frac{y}{x} = 3 \ln x + C$ $y = 3x \ln x + Cx$

3	Modelling Questions	Question	DE formulation
	$\frac{dx}{dt}$ is proportional to x $\Rightarrow \frac{dx}{dt} = kx$, where k is a constant $\frac{dx}{dt}$ is inversely proportional to x $\Rightarrow \frac{dx}{dt} = \frac{k}{x}$, where k is a constant - In general, Rate of change = Rate of increase – Rate of decrease E.g. (a) Rate of change of population size = Birth rate – Death rate (b) Rate of change of volume = Rate of In-flow – Rate of Out-flow	[2013 MI/I/9] When a cake is removed from the oven, its temperature decreases at a <u>rate proportional to the positive difference between its temperature and the temperature of the room</u>. The temperature of the room is constant at 25°C and T is the temperature of the cake t hours after removing from the oven. Form the differential equation. [2010 Promo MI/PU2/I/7] It is given that the population of an insect colony is x (in thousands) at time t (in months). <u>The population increases at a rate that is proportional to its population x at time t. In any month, 3000 insects will die of natural causes</u>. Form the differential equation. [2012 MI 12/I/11 modified] <u>A rectangular tank has a horizontal square base.</u> <u>Water is flowing into the tank at a constant rate. Water is flowing out of the tank at a rate which is proportional to the depth of water in the tank.</u> At time t seconds, the depth of water in the tank is x metres. Form the differential equation. [TJC 2014/I/6] A contagious disease was found to infect a village with a population of 10000 people. Let x, in thousands, be the number of infected people t days after the start of the outbreak. The disease spread at a <u>rate that was proportional to the product of the number of infected people and the number of non-infected people</u>. Form the differential equation.	$\frac{dT}{dt} = k(T - 25)$ where k is a constant $\frac{dx}{dt} = \text{Birth rate} - \text{Death rate}$ $\frac{dx}{dt} = kx - 3$ where k is a constant $\frac{dV}{dt} = \text{Rate of water inflow} - \text{Rate of water outflow}$ $= a - bx$ Since $\frac{dV}{dt} = \text{Area} \times \frac{dx}{dt}$, $\text{Area} \times \frac{dx}{dt} = a - bx$ $\therefore \frac{dx}{dt} = c - dx$ where c and d are constants $\frac{dx}{dt} = kx(10 - x)$ where k is a constant