MOE H3 Math   Mathematical Proofs and Reasoning

Problem Set 4

1. Show that there exists integers x and y that satisfy
$$(2n+1)x + (9n+4)y = 1$$
for every integer n.

   Solution: By constructive proof. Take $x = 9$ and $y = -2$.

   We check: $(2n+1)\times 9 + (9n+4)\times(-2) = (18n+9) + (-18n-8) = 1$ for any n.

   Hence $x = 9$ and $y = -2$ provide a pair of solution to this equation.

2. Prove that for every pair of irrational numbers p and q such that $p < q$, there is an irrational x such that $p < x < q$.

   Solution: Consider the average of p and q: $p < \frac{p+q}{2} < q$.

   If $\frac{p+q}{2}$ is irrational, take $x = \frac{p+q}{2}$ and we are done.

   If $\frac{p+q}{2}$ is a rational number r, take the average of p and r: $p < \frac{p+r}{2} < r < q$. Since p is irrational and r is rational, so $\frac{p+r}{2}$ is irrational (this can be proven by contradiction). In this case, we take $x = \frac{p+y}{2}$.

3. Show that there is one and only one integer t such that $t, t+2, t+4$ are all prime numbers.

   Solution: Existence: By construction, take $t = 3$. Then $3, 5, 7$ are all prime numbers.

   Uniqueness: Suppose $s, s+2, s+4$ are all primes, and $s > 3$. There are two cases: (i) $s \equiv 1 \mod 3$ and (ii) $s \equiv 2 \mod 3$.

   For case (i), $s+2 \equiv 3 \equiv 0 \mod 3$ which means $3 \mid s+2$. So $s+2$ is not a prime, which is a contradiction.

   For case (ii), $s+4 \equiv 6 \equiv 0 \mod 3$ which means $3 \mid s+4$. So $s+4$ is not a prime, which is a contradiction.

   Hence, it is not possible to find another triplet $s, s+2, s+4$ which are all prime numbers.

4. Given n real numbers $a_1, a_2, \ldots, a_n$. Show that there exists an a_i $(1 \leq i \leq n)$ such that a_i is greater than or equal to the mean (average) value of the n numbers.

Solution: By contradiction.

Let $\bar{a}$ denote the mean value of the n given numbers. Suppose $a_i < \bar{a}$ for all a_i.

Then $\bar{a} = \dfrac{a_1 + a_2 + \cdots + a_n}{n} < \dfrac{\bar{a} + \bar{a} + \cdots + \bar{a}}{n} = \dfrac{n\bar{a}}{n} = \bar{a}$.

We derive $\bar{a} < \bar{a}$, which is a contradiction.

So there must be some a_i such that $a_i \geq \bar{a}$.

5. Prove that there are infinitely many prime numbers that are congruent to 3 modulo 4.

Solution: Use a proof by contradiction.

Suppose there are only finitely many primes that are congruent to 3 modulo 4.

Let $p_1, p_2, \ldots, p_m$ be the list of all the primes that are congruent to 3 modulo 4.

We construct an integer M by $M = (p_1 p_2 \ldots p_m)^2 + 2$.

We have the following observation:

(i) $M \equiv 3 \mod 4$.

(ii) Every p_i divides $M - 2$.

(iii) None of the p_i divides M. [Otherwise, together with (ii), this will imply p_i divides 2, which is impossible.]

(iv) M is not a prime number. [Otherwise, by (i), M is a prime number congruent to 3 modulo 4. But $M \neq p_i$ for all $1 \leq i \leq m$. This contradicts to the assumption that $p_1, p_2, \ldots, p_m$ are all the prime numbers congruent to 3 modulo 4.]

From the above discussion, we know that M is a composite number by (iv). So it has a prime factorization $M = q_1 q_2 \cdots q_k$.

Since M is odd, all these prime factors q_j must be odd, and hence q_j must be congruent to either 1 or 3 modulo 4.

By (iii), q_j cannot be any of the p_i. So all q_j must be congruent to 1 modulo 4. Then M, which is the product of q_j, must also be congruent to 1 modulo 4. This contradicts (i) that M is congruent to 3 modulo 4.

Hence we conclude that there must be infinitely many primes that are congruent to 3 modulo 4.

6. Prove that, for any positive integer n, there is a perfect square m^2 (m is an integer) such that $n \leq m^2 \leq 2n$.

Solution: By contradiction.

Suppose there is a positive integer n such that for all perfect squares m^2, we either have $m^2 < n$ or $2n < m^2$.

Take an integer m_0 such that $m_0^2 < n$ and $(m_0+1)^2 > 2n$. i.e. m_0^2 is the largest square smaller than n while $(m_0+1)^2$ is the smallest square larger than $2n$.

By combining these two inequalities, we have

$$m_0^2 + 2m_0 + 1 > 2n > 2m_0^2 \quad (*)$$

Comparing the leftmost term and rightmost term in $(*)$, we have

$$2 > m_0^2 - 2m_0 + 1 = (m_0 - 1)^2$$

which is impossible except for $m_0 = 0, 1$ and 2.

If $m_0 = 0$, then by $(*)$, $1 > 2n > 0$, which is impossible, as there is no integer between 0 and 1.

If $m_0 = 1$, then by $(*)$, $4 > 2n > 2$, which is impossible, as there is no even integer between 2 and 4.

If $m_0 = 2$, then by $(*)$, $9 > 2n > 8$, which is impossible, as there is no integer between 8 and 9.

So it is not possible to find a positive integer n such that for all perfect squares m^2, we either have $m^2 < n$ or $2n < m^2$.

Hence we conclude that, for any positive integer n, there is a perfect square m^2 such that $n \le m^2 \le 2n$.

7. (Y2022 Q3)
For any real number s, the greatest integer less than or equal to s is denoted by $\lfloor s \rfloor$. For example, $\lfloor 3.7 \rfloor = 3$ and $\lfloor 5 \rfloor = 5$.

   (a) For any positive integer a and real number t, it is given that t can be written as $an + p$, where n is an integer and $a > p \ge 0$. Prove that
   $$\int_0^a \left\lfloor \frac{x+t}{a} \right\rfloor dx = t.$$

   Solution: First note that if $-1 < s < 1$, then for integer n, $\lfloor n + s \rfloor = n + \lfloor s \rfloor$. Also, if $0 < c < 1$, then
   $$\int_c^{1+c} \lfloor x \rfloor dx = c$$
   as the graph is the constant 0 graph on $(c, 1)$ and the constant 1 on $(1, 1+c)$. Hence, the integral, which is the area of a rectangle with breath $((1+c)-c)$ and height 1, is equal to $((1+c)-c) \times 1 = c$. So,
   $$\begin{aligned}\int_0^a \left\lfloor \frac{x+t}{a} \right\rfloor dx &= \int_0^a \left\lfloor \frac{x+an+p}{a} \right\rfloor dx = \int_0^a \left\lfloor \frac{x+p}{a} + n \right\rfloor dx \\ &= \int_0^a \left\lfloor \frac{x+p}{a} \right\rfloor + n \, dx = na + \int_0^a \left\lfloor \frac{x+p}{a} \right\rfloor dx\end{aligned}$$

Perform change of variable $s = \dfrac{x+p}{a}$, we have

$$\int_0^a \left\lfloor \frac{x+p}{a} \right\rfloor dx = \int_{\frac{p}{a}}^{1+\frac{p}{a}} \lfloor s \rfloor a \, ds = a\left(\frac{p}{a}\right) = p.$$

Hence,

$$\int_0^a \left\lfloor \frac{x+t}{a} \right\rfloor dx = na + +p = t.$$

(b) For any positive integer a and b and real number x

(i) prove that

$$\left\lfloor \frac{\left\lfloor \frac{x}{a} \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{x}{ab} \right\rfloor,$$

Solution: Write $x = kab + r$ for some integer k and $0 \le r < ab$. Then $\left\lfloor \frac{x}{ab} \right\rfloor = \left\lfloor \frac{kab+r}{ab} \right\rfloor = k$. On the other hand,

$$\left\lfloor \frac{\left\lfloor \frac{x}{a} \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{abk+r}{a} \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{bk + \left\lfloor \frac{r}{a} \right\rfloor}{b} \right\rfloor = k + \left\lfloor \frac{\left\lfloor \frac{r}{a} \right\rfloor}{b} \right\rfloor.$$

Now since $0 \le r < ab$, $0 \le \frac{r}{a} < b$, and hence $0 \le \left\lfloor \frac{r}{a} \right\rfloor < b$, and so, $\left\lfloor \frac{\left\lfloor \frac{r}{a} \right\rfloor}{b} \right\rfloor = 0$.

Hence,

$$\left\lfloor \frac{\left\lfloor \frac{x}{a} \right\rfloor}{b} \right\rfloor = k = \left\lfloor \frac{x}{ab} \right\rfloor.$$

(ii) find

$$\int_0^{ab} (fg(x) - gf(x))dx$$

where $f(x) = \left\lfloor \frac{x+a}{b} \right\rfloor$ and $g(x) = \left\lfloor \frac{x+b}{a} \right\rfloor$.

Solution:

$$fg(x) = \left\lfloor \frac{\left\lfloor \frac{x+b}{a} \right\rfloor + a}{b} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{x+b}{a} + a \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{x+b+a^2}{ab} \right\rfloor$$

where the third equality follows from (i). Then by (a),

$$\int_0^{ab} fg(x)\,dx = \int_0^{ab} \left\lfloor \frac{x+b+a^2}{ab} \right\rfloor dx = \int_0^{ab} \left\lfloor \frac{x+(b+a^2)}{ab} \right\rfloor dx = b+a^2.$$

Similarly, $gf(x) = \left\lfloor \dfrac{x+a+b^2}{ab} \right\rfloor$ and so

$$\int_0^{ab} gf(x)\,dx = \int_0^{ab} \left\lfloor \frac{x+(a+b^2)}{ab} \right\rfloor dx = a+b^2.$$

Hence,

$$\begin{aligned}\int_0^{ab} (fg(x)-gf(x))dx &= \int_0^{ab} fg(x)\,dx - \int_0^{ab} gf(x)\,dx = b+a^2-a-b^2 \\ &= (a-b)(a+b-1).\end{aligned}$$

8. (Y2024 Q7)
Let $S = \{1, 2, ..., 50\}$ and let D be a subset of S of size 27.

(a) Show that there are 25 subsets of S of the form $\{a, a+5\}$ whose union is S. Apply the pigeonhole principle to prove that D must contain two numbers that differ by exactly 5.

Solution: Let $a_{ij} = 10(i-1)+j$, for $i = 1, 2, ..., 5$ and $j = 1, 2, .., 5$. Then it is clear that $\{a_{i_1j_1}, a_{i_1j_1}+5\} \cap \{a_{i_2j_2}, a_{i_2j_2}+5\} = \varnothing$ if $(i_1, j_1) \neq (i_2, j_2)$, and that

$$\bigcup_{i,j=1}^{n} \{a_{ij}, a_{ij}+5\} = S.$$

By pigeon hole principle, since D contains 27 numbers, it must be that there is a $i, j = 1, ..., 5$ such that $\{a_{ij}, a_{ij}+5\} \subseteq D$ (for otherwise, D only contain at most 1 from each of the set $\{a_{ij}, a_{ij}+5\}$, and thus it can at most have 25 numbers).

(b) Prove that D must contain two numbers that differ by exactly 6.
Show that D does not necessarily contain two numbers that differ by exactly 7.

Solution: Let $a_{ij} = 12(i-1)+j$, for $i = 1, 2, ..., 4$, $j = 1, 2, ..., 6$. Then

$$\bigcup_{\substack{i=1,2,...,4 \\ j=1,2,...,6}} \{a_{ij}, a_{ij}+6\} \cup \{49\} \cup \{50\} = S.$$

There are a total of 26 sets. So, by pigeon hole principle, there must be a i, j such that $\{a_{ij}, a_{ij}+6\} \subseteq D$.

The set

$$D = \{1, 2, ..., 7,\ 15, 16, ..., 21,\ 29, 30, ..., 35\ 43, 44, ..., 48\}$$

contains 27 numbers but no two of them differ by exactly 7.

(c) Determine the maximum possible size of a subset of S that contains no four consecutive numbers.

Solution: Divide S into blocks of 4,

$$\{1,2,3,4\},\{5,6,7,8\},\{9,10,11,12\},...,\{45,46,47,48\},\{49,50\}$$

and one last set. Then from each of the block of 4, we can choose up to 3 of them, and we can take the last two numbers $\{49,50\}$ in the set. There are $\left\lfloor \frac{50}{4} \right\rfloor = 12$ of blocks of 4. So, the set contains

$$12 \times 3 + 2 = 38$$

numbers.

(d) Determine the maximum possible size of a subset of S that contains no two numbers whose sum is a multiple of 10.

Solution: $a+b$ is a multiple of 10 if and only if $a+b \equiv 0 \pmod{10}$. So, for

$$\begin{array}{rl} a=1, & b=9,19,29,39,49 \\ a=2, & b=8,18,28,38,48 \\ a=3, & b=7,17,27,37,47 \\ a=4, & b=6,16,26,36,46 \\ a=5, & b=15,25,35,45 \\ a=10, & b=20,30,40,50 \\ a=11, & b=9,19,29,39 \end{array},$$

etc. Observe that if $a \equiv 1 \pmod{10}$, then the numbers to avoid in the set are all $b \equiv 9 \mod 10$. We can pick up to 5 such a. For $a \equiv 2 \pmod{10}$, the numbers to avoid in the set are all $b \equiv 8 \mod 10$. We can pick up to 5 such a too. The pattern is similar for $a \equiv 1,2,3,4 \pmod{10}$. Now for $a \equiv 5 \mod 10$, the number to avoid are also $b \equiv 5 \pmod{10}$. So, we can only pick 1 of such number. Finally, it is clear we can only pick 1 number from among those that are congruent to $0 \pmod{10}$. So, in total, we can pick up to

$$5 \times 4 + 1 + 1 = 22$$

number in S so that no two numbers whose sum is a multiple of 10.

9. A function $f : X \to Y$ is said to be <u>*injective*</u> if

$$\forall x_1, x_2 \in X, \quad f(x_1) = f(x_2) \implies x_1 = x_2.$$

A function $f : X \to Y$ is said to be <u>*surjective*</u> if

$$\forall y \in Y, \quad \exists x \in X \text{ such that } f(x) = y.$$

Use the pigeonhole principle and proof by contradiction to prove: given nonempty finite sets X and Y with $|X| = |Y|$, a function $f : X \to Y$ is injective if and only if it is surjective.

Solution: Let $f(X) = \{y \in Y \mid f(x) = y \text{ for some } x \in X\}$ denote the image of f. Let X and Y be finite nonempty sets with $|X| = |Y| = n$.

($\Rightarrow$) Suppose f is injective but not surjective. Then there exists some $y_0 \in Y$ such that $y_0 \notin f(X)$. Hence the image $f(X) \subseteq Y \setminus \{y_0\}$, so $|f(X)| \leq n - 1$. But since f is injective, each element of X maps to a distinct element of $f(X)$, so $|X| = |f(X)|$. Therefore, $|X| \leq n - 1$, contradicting $|X| = n$.

($\Leftarrow$) Suppose f is surjective but not injective. Then there exist distinct $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. Thus at least two elements of X map to the same element of Y, so $|f(X)| < |X|$. But surjectivity gives $|f(X)| = |Y| = |X|$, a contradiction.

Therefore, for finite sets X, Y with $|X| = |Y|$, f is injective if and only if it is surjective.

Hints

1. *Hint for Question 1.* Since this is must be true for all n, compare coefficients of n.
2. *Hint for Question 3.* Existence part is by construction; uniqueness part: show that one of $t, t+2, t+4$ must be divisible by 3.
3. *Hint for Question 4.* Prove by contradiction by assuming all the n numbers are less than the average value.
4. *Hint for Question 5.* The idea is similar to the proof for infinitely many primes. Use the fact that integers of the form $4k+1$ are closed under multiplication.
5. *Hint for Question 6.* Prove by contradiction.
6. *Hint for Question 7.* For part (b), write $x = kab + r$ for some integer k and $0 \le r < ab$.