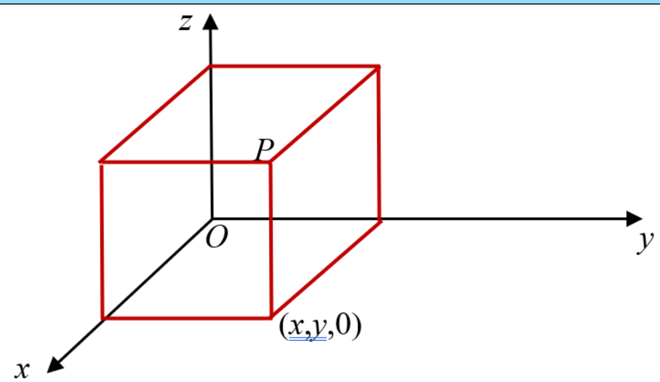
1 A rectangular box is placed in the region $R=\{(x,y,z): x,y,z\ge 0\}$ with one corner at the origin, three faces in the coordinate planes and the corner P diagonally opposite the origin on the plane $x+y+z=3$.

Use partial differentiation to find the maximum possible surface area for such a box, showing your working clearly and justifying your conclusion.

[5]

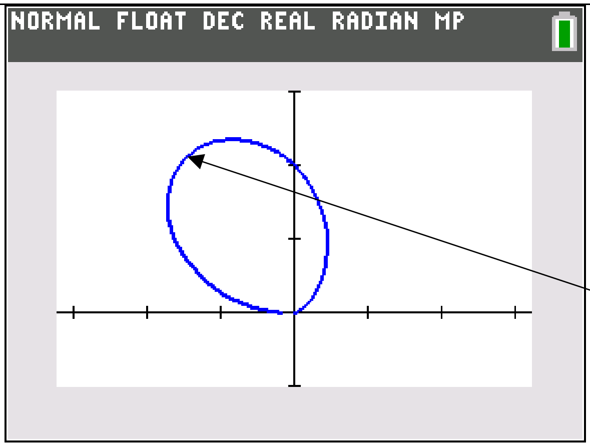
Qn	Solution
1	 Surface area of box, $S=2(xy+xz+yz)$ where $x+y+z=3$. Substitute $z=3-x-y$ into the expression for S gives $S=2(xy+xz+yz)=2\left[xy+x(3-x-y)+y(3-x-y)\right]$ $=2\left(3x-x^2+3y-y^2-xy\right)$ $\frac{\partial S}{\partial x}=2(3-2x-y)\overset{\text{set}}{=}0\Rightarrow 2x+y=3$ $\frac{\partial S}{\partial y}=2(3-2y-x)\overset{\text{set}}{=}0\Rightarrow x+2y=3$ Solving gives $x=y=1\Rightarrow z=1$. $\frac{\partial^2 S}{\partial x^2}=-4<0, \frac{\partial^2 S}{\partial y^2}=-4, \frac{\partial^2 S}{\partial y\partial x}=-2,$ $D=\frac{\partial^2 S}{\partial x^2}\cdot\frac{\partial^2 S}{\partial y^2}-\left(\frac{\partial^2 S}{\partial y\partial x}\right)=(-4)(-4)-(-2)^2=12>0$ By the second derivative test, the box which is a unit cube has maximum surface area 6 units^2.

2 The curve C has polar equation $r = 2\sin\theta(1-\cos\theta)$ for $0 \le \theta \le \pi$.

(a) Find $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ and hence find the polar coordinates of the point P of C that is furthest from the pole. [3]

(b) Sketch C, indicating the polar coordinates of the point P. [2]

(c) Find the exact area of the region bounded by C and the two lines $\theta = 0$ and $\theta = \dfrac{\pi}{4}$. [4]

Qn	Solutions
2(a)	$\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = 2\left[(1-\cos\theta)\cos\theta + \sin\theta(\sin\theta)\right] = 2(\cos\theta - \cos 2\theta)$. $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = 0 \Rightarrow 2(\cos\theta - \cos 2\theta) = 0 \Rightarrow \cos\theta = \cos 2\theta \Rightarrow \theta = \dfrac{2\pi}{3} \in [0, \pi]$. Coordinates of point furthest from O is $\left(2\sin\dfrac{2\pi}{3}\left(1-\cos\dfrac{2\pi}{3}\right), \dfrac{2\pi}{3}\right) = \left(\dfrac{3\sqrt{3}}{2}, \dfrac{2\pi}{3}\right)$.
2(b)	 $\left(\dfrac{3\sqrt{3}}{2}, \dfrac{2\pi}{3}\right)$

2(c)	Area of the region bounded by C and the two lines $\theta=0$ and $\theta=\frac{\pi}{4}$ $=\frac{1}{2}\int_0^{\frac{\pi}{4}} 4\sin^2\theta(1-\cos\theta)^2\,\mathrm{d}\theta$ $=2\int_0^{\frac{\pi}{4}} \sin^2\theta\left(1-2\cos\theta+\cos^2\theta\right)\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}} 2\sin^2\theta\,\mathrm{d}\theta-4\int_0^{\frac{\pi}{4}}\cos\theta\sin^2\theta\,\mathrm{d}\theta+\frac{1}{2}\int_0^{\frac{\pi}{4}}\sin^2 2\theta\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{4}}(1-\cos 2\theta)\,\mathrm{d}\theta-4\int_0^{\frac{\pi}{4}}\cos\theta\sin^2\theta\,\mathrm{d}\theta+\frac{1}{4}\int_0^{\frac{\pi}{4}}(1-\cos 4\theta)\,\mathrm{d}\theta$ $=\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}-\frac{4}{3}\left[\sin^3\theta\right]_0^{\frac{\pi}{4}}+\frac{1}{4}\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}}$ $=\frac{\pi}{4}-\frac{1}{2}-\frac{4}{3}\left(\frac{2\sqrt{2}}{8}\right)+\frac{1}{4}\left(\frac{\pi}{4}\right)$ $=\frac{5\pi}{16}-\frac{1}{2}-\frac{\sqrt{2}}{3}$

3 The sequence $\{u_n\}$ is given by the recurrence relation $u_{n+2}=\frac{1}{3}\left(2u_{n+1}+u_n-24\right)$ for $n\geq 0,$ together with initial terms $u_1=p$ and $u_2=q,$ where p and q are positive integers. The sequence $\{v_n\}$ is given by $v_n=u_n+6n$ for $n\geq 0.$

(a) Find an expression v_n in terms of n, p and q. [5]

(b) Given that the limit for which v_n converges to is 13, find the possible values of p and q. [2]

(c) Hence, find u_n in terms of n, p and q. State what happens to u_n for large values of n. [2]

Qn	**Solution**		
(a)	$u_{n+2}=\frac{1}{3}\left(2u_{n+1}+u_n-24\right)$ $v_{n+2}-6(n+2)=\frac{1}{3}\left(2\left(v_{n+1}-6(n+1)\right)+v_n-6n-24\right)$ $v_{n+2}-6n-12=\frac{2}{3}v_{n+1}-4n-4+\frac{1}{3}v_n-2n-8$ $3v_{n+2}-2v_{n+1}-v_n=0$ Characteristic equation: $3\lambda^2-2\lambda-\lambda=0$ $(3\lambda+1)(\lambda-1)=0$ $\lambda=-\frac{1}{3}$ or 1 General solution: $v_n=A(-\frac{1}{3})^n+B(1)^n=A(-\frac{1}{3})^n+B$ Initial conditions: $v_1=u_1+6(1)=p+6$ $v_2=u_2+6(2)=q+12$ $A(-\frac{1}{3})+B=p+6$ $\quad ---(1)$ $A(-\frac{1}{3})^2+B=q+12$ $\quad ---(2)$		

	$(2)-(1)$: $\frac{4}{9}A = q - p + 6$ $A = \frac{9}{4}(q - p + 6)$ Subt into (1): $B = \frac{1}{3}A + p + 6 = \frac{3}{4}(q - p + 6) + p + 6 = \frac{1}{4}(3q + p + 42)$ $\therefore v_n = \frac{9}{4}(q - p + 6)(-\frac{1}{3})^n + \frac{1}{4}(3q + p + 42)$
(b)	$\frac{1}{4}(3q + p + 42) = 13$ $3q + p = 10$ for positive integers p and q, possible values: $q = 1,\ p = 7$ $q = 2,\ p = 4$ $q = 3,\ p = 1$
(c)	$u_n = v_n - 6n$ $\quad = \frac{9}{4}(q - p + 6)(-\frac{1}{3})^n + \frac{1}{4}(3q + p + 42) - 6n$ As $n \to \infty$, $u_n \to -\infty$

4 The curve with equation $y = \ln x$ and the line with equation $y = x - 2$ intersects at $x = \alpha$ and $x = \beta$, where $\alpha < \beta$.

(a) It is given that β lies in the interval $(k, k+1)$, where k is an integer, state the value of k. Justify that there is indeed a root in the interval $(k, k+1)$. [2]

(b) Show that one application of linear interpolation using **(b)** will obtain a first approximation for β as 3.138439, correct to 6 decimal places. [2]

(c) Hence, use the Newton-Rhapson method to approximate β, correct up to 4 decimal places. Justify that the approximation is correct up to 4 decimal places. [3]

(d) It is known that the Newton-Rhapson method, when applied to $\mathrm{f}(x)$ can approximate the root α using, as long as the first approximation lies in the interval $(0, c)$. Find the largest value of c, leaving your answer in exact form. [3]

Qn	Solution
4(a)	$k = 3$ $\mathrm{f}(3) = 0.098612 > 0$ $\mathrm{f}(4) = -0.61370 < 0$ By the Intermediate Value Theorem, there exist a root in $(3,4)$
4(c)	$x_1 = \dfrac{3\lvert \mathrm{f}(4)\rvert + 4\lvert \mathrm{f}(3)\rvert}{\lvert \mathrm{f}(3)\rvert + \lvert \mathrm{f}(4)\rvert}$ ≈ 3.13843858 $= 3.138439$ (6dp)

4(d)	$\mathrm{f}'(x)=\frac{1}{x}-1$ $x_2=x_1-\frac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)}$ $=3.138439-\frac{\ln 3.138439-3.138439+2}{\frac{1}{3.138439}-1}$ ≈ 3.146197 $=3.1462$ (4dp) To justify $\beta=3.1462$ (correct to 4dp), $\mathrm{f}(3.14615)=0.000029483>0$, $\mathrm{f}(3.14625)=-0.000038732<0$ By the Intermediate Value Theorem, $\beta\in(3.14615,3.14625)$. Any number in this interval is rounded to 3.1462.
4(e)	Consider the tangent to $y=\mathrm{f}(x)$ at point $(c,\mathrm{f}(c))$: $y-\mathrm{f}(c)=\mathrm{f}'(c)(x-c)$ $y-(\ln c-c+2)=\left(\frac{1}{c}-1\right)(x-c)$ Sub in $(0,0)$, $0-\ln c+c-2=c-1$ $\ln c=-1$ $c=\mathrm{e}^{-1}$

5 The country of Ganyan is looking to export timbre to boost its economy. In 2025, the government of Ganyan acquired a timbre plantation and decided to let the plantation grow.

After t number of years, the amount of timbre, T thousand tonnes, can be modelled as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = kT\left(1 - \frac{T}{a}\right),$$

where a and k are positive constants.

(a) State the significance of a, in the context of the question. [1]

For the rest of the question, let $a = 1000$. It is also given that when $t = 0$, $T = 200$, $\frac{\mathrm{d}T}{\mathrm{d}t} = 16$.

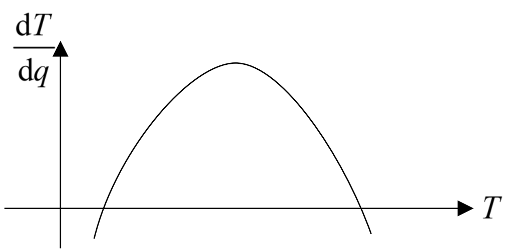
(b) Find the time taken for the timbre to double in amount. [4]

Upon reaching $T = 800$, the government of Ganyan starts harvesting a fixed amount of timbre, h thousand tonnes per year.

(c) Given that $k = 0.1$, find the range of values of h for the harvest to be sustainable in the long run. [2]

(d) For $h = 15$, find the equilibrium amount of timbre in the long run. [3]

Qn	Solution		
5(a)	a is the carrying capacity, which is the maximum possible amount of timbre the plantation can have.		
5(b)	$t=0$, $T=200$, $\frac{\mathrm{d}T}{\mathrm{d}t}=16$ $16=200k\left(1-\frac{200}{1000}\right)$ $k=0.1$ $\frac{\mathrm{d}T}{\mathrm{d}t}=0.1T\left(1-\frac{T}{1000}\right)$ $\int \frac{1}{T\left(1-\frac{T}{1000}\right)}\mathrm{d}T=\int 0.1\mathrm{d}t$ $\int \frac{1}{T}+\frac{1}{1000-T}\mathrm{d}T=\int 0.1\mathrm{d}t$ $\ln\left\lvert\frac{T}{1000-T}\right\rvert=0.1t+C$ $\frac{T}{1000-T}=A\mathrm{e}^{0.1t}$, $A=\pm\mathrm{e}^{C}$ $\frac{200}{800}=A$ $A=\frac{1}{4}$ $T=400$, $\frac{400}{600}=\frac{1}{4}\mathrm{e}^{0.1t}$ $t=10\ln\frac{8}{3}$ ≈ 9.8082 $=9.81$ years (3sf)		

5(c)	Let q years be the number of years after the harvesting starts. $\frac{\mathrm{d}T}{\mathrm{d}q}=0.1T\left(1-\frac{T}{1000}\right)-h$ $\frac{\mathrm{d}T}{\mathrm{d}q}=\frac{1}{10000}T\left(1000-T\right)-h$ $=-\frac{1}{10000}\left(T-500\right)^2-h+25$  For $\frac{\mathrm{d}T}{\mathrm{d}q}\geq 0$, $-h+25\geq 0$ $h\leq 25$ $\therefore 0\leq h\leq 25$
5(d)	$\frac{\mathrm{d}T}{\mathrm{d}q}=0.1T\left(1-\frac{T}{1000}\right)-15$ $=-\frac{1}{10000}\left(T-500\right)^2+10$ Let $\frac{\mathrm{d}T}{\mathrm{d}q}=0$ $T=500-\sqrt{100000}$ or $T=500+\sqrt{100000}$ Since $T=800>500-\sqrt{100000}$, $T\rightarrow 500+\sqrt{100000}$

6 The prestigious Nanyang D' Art Gallery was preparing for its grand reopening. The gallery's owner, Frankie, had commissioned a local artist, Stein, to create a stunning installation. Stein decided to use a linear transformation to transform a set of 4D vectors into a new set of vectors that would form the basis of this installation. The linear transformation was represented by a 4×4 matrix,

$$\mathbf{A}=\begin{pmatrix}2\alpha & 1 & 0 & 1\\ 1 & 2 & \alpha & 1\\ 0 & 1 & 2 & 2\\ 1 & 0 & 1 & 2\end{pmatrix}.$$

(a) It is given that $\alpha=1$.

(i) Find a basis for the range space of matrix **A,** that gives the set of transformed vectors. [2]

(ii) This stunning installation must pass through the point $(1,2,1,0)$. By finding a basis for the null space, determine the set of vectors that satisfy this condition. [3]

(b) Using row operations, determine if it is possible to find another value of α such that the dimension of the range space of matrix **A** is the same as that in **(a)(i)**. [5]

Qn	Solution
(a)(i)	$\alpha=1,\mathbf{A}=\begin{pmatrix}2 & 1 & 0 & 1\\ 1 & 2 & 1 & 1\\ 0 & 1 & 2 & 2\\ 1 & 0 & 1 & 2\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0 & 0.75\\ 0 & 1 & 0 & -0.5\\ 0 & 0 & 1 & 1.25\\ 0 & 0 & 0 & 0\end{pmatrix}$ using GC Basis required $=\left\{\begin{pmatrix}2\\1\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\\1\end{pmatrix}\right\}$.
(a)(ii)	Observe that $\begin{pmatrix}2 & 1 & 0 & 1\\ 1 & 2 & 1 & 1\\ 0 & 1 & 2 & 2\\ 1 & 0 & 1 & 2\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\2\\1\\1\end{pmatrix}$, this means that $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ is a particular solution.

Using $\begin{pmatrix} 1 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 1.25 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x + \dfrac{3}{4}w = 0 \\ y - \dfrac{1}{2}w = 0 \\ z + \dfrac{5}{4}w = 0 \end{cases}$

i.e. $x = -\dfrac{3}{4}w, y = \dfrac{1}{2}w, z = -\dfrac{5}{4}w \Rightarrow \mathbf{x} = \begin{pmatrix} -3 \\ 2 \\ -5 \\ 4 \end{pmatrix}$

So, the set required = $\left\{ \mathbf{x} \in \mathbb{R}^4 : \mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ -5 \\ 4 \end{pmatrix}, \text{ where } \lambda \in \mathbb{R} \right\}$

(b)

$$\begin{pmatrix} 2\alpha & 1 & 0 & 1 \\ 1 & 2 & \alpha & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_4 - R_2} \begin{pmatrix} 2\alpha & 1 & 0 & 1 \\ 1 & 2 & \alpha & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -2 & 1-\alpha & 1 \end{pmatrix} \xrightarrow[R_4 + 2R_3]{R_1 - 2\alpha R_2} \begin{pmatrix} 0 & 1-4\alpha & -2\alpha^2 & 1-2\alpha \\ 1 & 2 & \alpha & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 5-\alpha & 5 \end{pmatrix}$$

$$\xrightarrow{R_1 - (1-4\alpha)R_3} \begin{pmatrix} 0 & 0 & -2+8\alpha-2\alpha^2 & -1+6\alpha \\ 1 & 2 & \alpha & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 5-\alpha & 5 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & \alpha & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2+8\alpha-2\alpha^2 & -1+6\alpha \\ 0 & 0 & 5-\alpha & 5 \end{pmatrix}$$

	To have same dimension, we need: $-2+8\alpha-2\alpha^2=k(5-\alpha) \qquad -----(1)$ $-1+6\alpha=5k \Rightarrow \alpha=\frac{1}{6}(1+5k) \qquad -----(2)$ Sub (2) into (1): $-2+8\left(\frac{1}{6}\right)(1+5k)-2\left[\frac{1}{6}(1+5k)\right]^2=k\left(5-\frac{1}{6}(1+5k)\right)$ $-2+\frac{4}{3}+\frac{20}{3}k-\frac{1}{18}(1+10k+25k^2)=\frac{29}{6}k-\frac{5}{6}k^2$ $10k^2-23k+13=0$ $(10k-13)(k-1)=0$ $k=\frac{13}{10}$ or $1 \Rightarrow \alpha=1,\frac{5}{4}$ Another value of $\alpha=\frac{5}{4}$

7 **(a)** Empirical studies suggest that the surface area of a human body is related to its weight and height. If the weight is w kg and the height is h cm, then the surface area S, in square metres, is approximately given by

$$S = 0.0072w^{0.425}h^{0.725}.$$

(i) Find expressions in terms of w and h for $\frac{\partial S}{\partial w}$ and $\frac{\partial S}{\partial h}$. [2]

(ii) A boy weighs 17 kg and stands at 100 cm on his birthday. Given that he is growing at a rate of 2 kg per year and 9 cm per year, find the rate at which his surface area is increasing on his birthday. [2]

(b) Given that $\mathrm{f}(x, y) = \mathrm{e}^{xy} + y\sin x$ where $x, y \in \mathbb{R}$. Let P be the point $(0, -2)$ on the xy-plane.

(i) Find the directional derivative of $\mathrm{f}(x, y)$ at P in the direction of the vector $\mathbf{i} + \sqrt{3}\mathbf{j}$. [3]

(ii) Find the greatest possible rate of change of $\mathrm{f}(x, y)$ at P. [1]

(iii) Find a unit vector along which $\mathrm{f}(x, y)$ is stationary at P. [3]

(iv) Find the equation of the tangent plane to the surface $z = \mathrm{f}(x, y)$ at P. [2]

Qn	Solution
(a)(i)	$S = 0.0072w^{0.425}h^{0.725}$. $\frac{\partial S}{\partial w} = 0.0072 \times 0.425 w^{0.425-1}h^{0.725} = \frac{0.00306h^{0.725}}{w^{0.575}}$ $\frac{\partial S}{\partial h} = 0.0072 \times 0.725 w^{0.425}h^{0.725-1} = \frac{0.00522w^{0.425}}{h^{0.275}}$
(a)(ii)	$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right\|_{w=17,h=100} = \left.\left(\frac{\partial S}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{\partial S}{\partial h}\cdot\frac{\mathrm{d}h}{\mathrm{d}t}\right)\right\|_{w=17,h=100}$ $= \left.\left(\frac{0.00306h^{0.725}}{w^{0.575}}\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{0.00522w^{0.425}}{h^{0.275}}\frac{\mathrm{d}h}{\mathrm{d}t}\right)\right\|_{w=17,h=100}$ $= \frac{0.00306(100)^{0.725}}{17^{0.575}}(2) + \frac{0.00522(17)^{0.425}}{100^{0.275}}(9)$ ≈ 0.078 The boy's surface area is increasing at 0.078 m^2 per year on his birthday.
(b)(i)	$\mathrm{f}(x, y) = \mathrm{e}^{xy} + y\sin x \Rightarrow \mathrm{f}_x(x, y) = y\mathrm{e}^{xy} + y\cos x,\ \ \mathrm{f}_y(x, y) = x\mathrm{e}^{xy} + \sin x$.

	$\nabla \mathrm{f}(x,y)=\mathrm{f}_x(x,y)\mathbf{i}+\mathrm{f}_y(x,y)\mathbf{j}=\left(y\,\mathrm{e}^{xy}+y\cos x\right)\mathbf{i}+\left(x\,\mathrm{e}^{xy}+\sin x\right)\mathbf{j}$ $\mathbf{u}=\frac{1}{2}\left(\mathbf{i}+\sqrt{3}\mathbf{j}\right)$. $D_{\mathbf{u}}\,\mathrm{f}(0,-2)=\nabla \mathrm{f}(0,-2)\bullet\mathbf{u}$ $=-4\mathbf{i}\bullet\frac{1}{2}\left(\mathbf{i}+\sqrt{3}\mathbf{j}\right)$ $=-2$
(b)(ii)	Greatest rate of change of $\mathrm{f}(x,y)$ at P $=\left\lvert\nabla \mathrm{f}(0,-2)\right\rvert=4$.
(b)(iii)	Let $\mathbf{u}=a\mathbf{i}+b\mathbf{j}$ be a unit vector along which $\mathrm{f}(x,y)$ is stationary at P. $D_{\mathbf{u}}\,\mathrm{f}(0,-2)=0\Rightarrow\nabla \mathrm{f}(0,-2)\bullet\mathbf{u}=0$ $\Rightarrow -4\mathbf{i}\bullet(a\mathbf{i}+b\mathbf{j})=0$ $\Rightarrow -4a=0$ $\Rightarrow a=0$ Since $\mathbf{u}$ is a unit vector, $a^2+b^2=1\Rightarrow b=\pm 1$. Hence $\mathbf{u}=\pm\mathbf{j}$.
(b)(iv)	The tangent plane to the surface $z=\mathrm{f}(x,y)$ at P is given by $z-\underbrace{\mathrm{f}(0,-2)}_{1}=\underbrace{\mathrm{f}_x(0,-2)}_{-4}(x-0)+\underbrace{\mathrm{f}_y(0,-2)}_{0}\left(y-(-2)\right)$ $\Rightarrow 4x+z=1$

8 A factory produces microchips 24 hours a day in two shifts. In shift A (8am to 8pm), the average number of defective microchips per hour is 2.5.

(a) State the conditions in which the number of defective microchips per hour produced in shift A can be well modelled by a Poisson distribution. [1]

You should assume that these conditions hold.

(b) Find the probability that, in a randomly chosen 6 hour period in shift A, the total number of defective microchips is more than 12. [2]

In shift B (8pm to 8am), the number of defective microchips per hour follows a Poisson distribution with mean 4.5. Shift A and Shift B do not overlap in timing.

(c) Find the probability that, in a randomly chosen day, the total number of defective microchips is less than 80. [2]

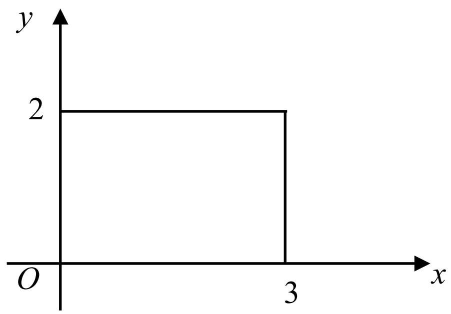
(d) Find the probability that, in a randomly chosen day, the total number of defective microchips is less than 80, given that there were 23 defective microchips in shift A. [2]

Qn	Solution
8(a)	Whether a microchip is defective or not is independent of whether any other microchips is defective. The average rate of microchips being defective within shift A is constant.
8(b)	Let A_i be the number of defects in a randomly chosen *i* hour period in shift A. $A_6 \sim \text{Po}(6\times 2.5)$, i.e. $A_6 \sim \text{Po}(15)$ $\text{P}(A_6 > 12) = 1 - \text{P}(A_6 \le 12)$ $= 0.73238$ $= 0.732$ (3 s.f.)
8(c)	Let B_i be the number of defects in a randomly chosen *i* hour period in shift B.

	$A_{12}+B_{12} \sim \text{Po}(12\times 2.5+12\times 4.5)$ i.e. $A_{12}+B_{12} \sim \text{Po}(84)$ Probability required $= \text{P}(A_{12}+B_{12} < 80)$ $= \text{P}(A_{12}+B_{12} \le 79)$ $= 0.31664$ $= 0.317$ (3 s.f.)
8(d)	$B_{12} \sim \text{Po}(12\times 4.5)$ i.e. $B_{12} \sim \text{Po}(54)$ Probability required $= \text{P}(A_{12}+B_{12} < 80 \mid A_{12}=23)$ $= \dfrac{\text{P}(A_{12}=23)\text{P}(B_{12} \le 56)}{\text{P}(A_{12}=23)}$ $= 0.64064$ $= 0.641$ (3 s.f.)

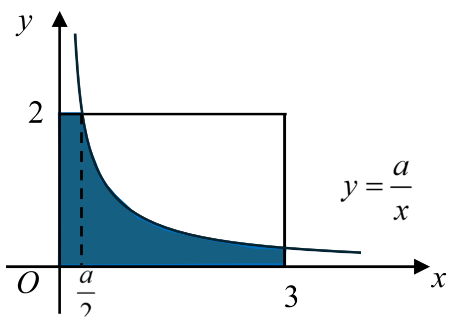
9 Two independent random variables $X \sim \mathrm{U}(0,3)$ and $Y \sim \mathrm{U}(0,2)$ are the length and width of a rectangle respectively. The area of this rectangle is denoted by A.

(a) The diagram below shows the region satisfying the inequalities $0 \le x \le 3$ and $0 \le y \le 2$. By considering a suitable curve in the diagram, find the probability that the area of the rectangle, A, is less than a, where $0 \le a \le 6$. [4]



(b) Hence find the probability density function of A. [2]

(c) Find the mean and variance of A. [2]

Qn	Solution
9(a)	 Area of rectangle $xy = a$ Consider the curve $y = \frac{a}{x}$ together with the diagram given in the question. (Note that the shaded region corresponds to the parts within the rectangle which satisfy the inequality $xy < a$)

	$\mathrm{P}(A<a)=\dfrac{\text{area of shaded region}}{2\times 3}$ $=\dfrac{2\times\frac{a}{2}+\int_{\frac{a}{2}}^{3}\frac{a}{x}\,\mathrm{d}x}{6}$ $=\dfrac{1}{6}\left(a+\left[a\ln x\right]_{\frac{a}{2}}^{3}\right)$ $=\dfrac{1}{6}\left(a+a\ln 3-a\ln\dfrac{a}{2}\right)$ $=\dfrac{a}{6}\left(1-\ln\dfrac{a}{6}\right)$
9(b)	For $0<a<6$, $\dfrac{\mathrm{d}}{\mathrm{d}a}\mathrm{P}(A<a)=\dfrac{1}{6}\left(1-\ln\dfrac{a}{6}\right)+\dfrac{a}{6}\left(-\dfrac{6}{a}\cdot\dfrac{1}{6}\right)$ $=-\dfrac{1}{6}\ln\dfrac{a}{6}$ $\therefore$ p.d.f of $A=\begin{cases}-\dfrac{1}{6}\ln\dfrac{a}{6}, & \text{for } 0<a<6\\ 0, & \text{otherwise}\end{cases}$
9(c)	$\mathrm{E}(A)=\int_0^6 a\left(-\dfrac{1}{6}\ln\dfrac{a}{6}\right)\mathrm{d}a=1.5000=1.50$ (3 s.f.) $\mathrm{E}(A^2)=\int_0^6 a^2\left(-\dfrac{1}{6}\ln\dfrac{a}{6}\right)\mathrm{d}a=3.9999$ $\mathrm{Var}(A)=3.9999-1.5000^2=1.7499=1.75$ (3 s.f.)

10 A member of the King's Air Rifle Club is practising target shooting. The member fires at a target until he hits the bullseye, then moves on to the next one. The probability of hitting the bullseye with any given shot is p, which remains constant for each target. $X(n)$ represents the number of shots required to hit the bullseye for the nth target.

(a) Write down the distribution of $X(n)$ and an assumption under this context. Hence find $\mathrm{P}\left(X(n) \leq n+1\right)$. [4]

The total number of shots required to hit n bullseyes is denoted by $S(n)$.

(b) Find the probability that the nth bullseye is hit with only one shot given that a total of at most $n + 1$ shots are required for the first n bullseyes. [4]

(c) It is given that the member has 60 bullets and $p = 0.8$. Find the probability that the first 50 bullseyes are hit before the member runs out of bullets.

[3]

Qn	**Solution**
10(a)	$X(n) \sim \mathrm{Geo}(p)$ His shots are independent of one another. $\mathrm{P}\left(X(n) \leq n+1\right)$ $= 1 - \mathrm{P}\left(X(n) \geq n+2\right)$ $= 1-(1-p)^{n+1}$

10(b)	$\mathrm{P}\left(X(n)=1 \mid S(n)\le n+1\right)$ $=\dfrac{\mathrm{P}(X(n)=1 \text{ and } S(n)\le n+1)}{\mathrm{P}(S(n)\le n+1)}$ $=\dfrac{\mathrm{P}(X(n)=1 \text{ and } S(n-1)\le n)}{\mathrm{P}\left(S(n)=n\right)+\mathrm{P}\left(S(n)=n+1\right)}$ $=\dfrac{\mathrm{P}(X(n)=1)\left[\mathrm{P}\left(S(n-1)=n-1\right)+\mathrm{P}\left(S(n-1)=n\right)\right]}{\mathrm{P}\left(S(n)=n\right)+\mathrm{P}\left(S(n)=n+1\right)}$ $=\dfrac{p(p^{n-1}+\binom{n-1}{1}(1-p)p^{n-1})}{p^{n}+n(1-p)p^{n}}$ $=\dfrac{1+(n-1)(1-p)}{1+n(1-p)}$ $=\dfrac{n+p(1-n)}{1+n(1-p)}$
10(c)	$X(i)\sim \mathrm{Geo}(p),\quad i=1,\ldots,50$ Since n = 50 is large, by Central Limit Theorem, $X(1)+X(2)+X(3)+\ldots+X(50)\sim \mathrm{N}\left(50\times\dfrac{1}{0.8},50\times\dfrac{0.2}{0.8^{2}}\right)$ $X(1)+X(2)+X(3)+\ldots+X(50)\sim \mathrm{N}\left(62.5,\ 15.625\right)$ $\mathrm{P}(50\le X(1)+X(2)+X(3)+\ldots+X(50)\le 60)$ $=0.263$

11 A study by the Nanyang Research Academy investigated the impact of alcohol on reaction time. Fifty randomly selected male participants, aged 35, consumed 178ml of wine. Thirty minutes later, their reaction times were measured using a standardized driving simulation. The participants' reaction times (in milliseconds) were recorded. The results are as follows:

$$\sum x = k \text{ and } \sum x^2 = 10\ 989\ 354.$$

(a) The typical reaction time of a 35-year-old male driver who did not consume alcohol is 400 milliseconds. If $k = 22\ 243$, is there sufficient evidence, at 5% significance level, to suggest that the consumption of 178ml of wine increase the reaction time of a 35-year-old male driver? [5]

(b) The Tippler Winery Association aims to challenge the findings of the study in (a). Given that the population variance is 20 514. What is the range of values of k that would be required to suggest that wine consumption does not significantly impact reaction time? [3]

Qn	Solution
11(a)	Let μ denote the population mean reaction time of 35-year-old male driver after consumption of 178ml of wine. $\bar{x} = \frac{\sum x}{50} = \frac{22243}{50} = 444.86$ $s^2 = \frac{1}{50-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right] = \frac{1}{49}\left[10{,}989{,}354 - \frac{22243^2}{50}\right] = 22333.32694 = 149.443^2$ Test: $H_0: \mu = 400\text{ms}$ $H_1: \mu > 400\text{ms}$ at 5% significance level Under H_0, since n is large, by Central Limit Theorem, $\overline{X} \sim N\left(400, \frac{149.443^2}{50}\right)$ approximately and test statistic $Z = \frac{\overline{X} - 400}{143.23/\sqrt{50}} \sim N(0,1)$ approximately. Reject H_0 if p-value ≤ 0.05 Using GC, p-value $= 0.017013$ Since p-value ≤ 0.05, we reject H_0, there is sufficient evidence at 5% level of significance that the consumption of 178ml of wine affects the reaction time of 35-year-old male driver.

11(b)	Since H_0 is not rejected, p-value > 0.05 $\mathrm{P}\left(Z > \dfrac{\bar{x}-400}{\sqrt{20514}/\sqrt{50}}\right) > 0.05$ $\dfrac{\bar{x}-400}{\sqrt{20514}/\sqrt{50}} < 1.64485$ $\bar{x} < 433.32$ $k < 21666$ Range required is $0 < k < 21,600$ (accept 21,700)