NANYANG JUNIOR COLLEGE

JC2 Common Test 2

Higher 2

FURTHER MATHEMATICS **9649/02**

Paper 2 **4th July 2023**

3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

Section A: Pure Mathematics [50 marks]

1 Solve the recurrence relation $y_{n+2} + y_{n+1} + y_n = 0$, $n \geq 0$, $y_0 = \sqrt{3}$, $y_1 = 0$, expressing y_n as a single trigonometric function. [4]

2 **Do not use a calculator in answering this question.**

The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a & 4 \\ a & 4 & 7 \\ -2 & 2 & 0 \end{pmatrix},$$

where $a > 0$.

(a) Determine the value of a such that the kernel of T is a line through the origin, and hence state the equation of this line. [4]

(b) Using the value of a in **(a)**, show that the range of T is a plane through the origin and state its cartesian equation. [2]

3 Express the complex number

$$1 + \mathrm{e}^{2\mathrm{i}\alpha} \text{ where } -\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$$

in exponential form, justifying your working. [3]

Use de Moivre's theorem to sum the series

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \sin(2r+1)\alpha \,. \quad [6]$$

4 Use differentiation to prove that for all $x \geq 0$,

$$\tan^{-1} x \geq \frac{x}{1+x^2}$$

and state the value of x at which equality is achieved. [4]

The region in the first quadrant bounded by the curves $y = \tan^{-1} x$, $y = \dfrac{x}{1+x^2}$ and the line $x = 1$ is rotated about the y-axis through 4 right angles.

Find the exact volume of the solid of revolution. [5]

5 In one model of population growth, it is assumed that the rate at which a species of fish is caught is directly proportional to the population N of the fish in the lake. With this assumption, the logistic model is replaced by the Schaefer model, given by the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{M}\right)N - EN$$

where r, E and M are positive constants.

(a) Give the meaning of the term EN in the above differential equation. [1]

(b) Show that if $E < r$, then there are two equilibrium populations given by

$$N = N_1 \text{ and } N = N_2$$

where $N_1 < N_2$, giving the values of N_1 and N_2 in terms of r, E and M where appropriate. [3]

(c) With the aid of a sketch, explain why $N = N_1$ is unstable and that $N = N_2$ is stable. [3]

The sustainable yield Y is defined to be the rate at which fish are caught when the fish population reaches stable equilibrium and E is defined to be the total effort to harvest the species of fish.

(d) Express Y as a function of E and sketch the graph of Y against E, called the yield-effort curve. [2]

(e) Determine E so as to maximise Y, and hence find the maximum sustainable yield Y_m. [2]

6 **(a)** The region R is bounded by the curve $y = \sin x$, the x-axis and the line $x = \frac{\pi}{2}$. Using shell method, find the exact volume of the solid generated when R is rotated through 2π radians about the y-axis. [2]

(b) A scientist is analysing a surface whose upper edge follows the curve

$$y = \sin x, \text{ where } 0 \le x \le \pi.$$

This surface is to be created by bending a rod along the curve and rotating it through 2π radians about the x-axis to form a 3-dimensional structure.

(i) Find the arc length of the upper edge, correct your answer to 4 decimal places. [2]

(ii) Let A be the surface area of revolution of this flexible surface. Show that A can be expressed as $k\pi\int_0^1 \sqrt{1+u^2}\ \mathrm{d}u$, where k is a constant and u is a trigonometric expression in terms of x to be determined. [3]

(iii) Hence by using the substitution $u = \tan\theta$, find the exact value of A. [4]

Section B: Probability and Statistics [48 marks]

7 There are two types of batteries in a bin. A battery is randomly chosen from the bin, its type noted and then placed back into the bin. The probability of choosing a type *i* battery with probability p_i , and $\sum_{i=1}^{2} p_i = 1$.

(a) Find the probability of drawing exactly three consecutive type 1 batteries in terms of p_1 in at most six draws. [3]

When in use, type *i* batteries last (in hours) an exponentially distributed time with rate $\lambda_i, i = 1,2$.

(b) If a randomly chosen battery is still operating after *t* hours of use, find the probability it will still be operating after an additional *s* hours. [3]

8 Independent variables X and Y are uniformly distributed over the interval from $a \le x, y \le b$, where $a, b \in \mathbb{R}^+$.

(a) State $\mathrm{P}(X < Y)$. [1]

(b) By considering the cumulative distribution of $\max(X,Y)$, show that the expected value of $\max(X,Y)$ is $\frac{1}{3}(2b+a)$. [6]

9 **Do not use a calculator in answering this question.**

The random variable X has probability density function given by

$$\mathrm{f}(x) = \begin{cases} \dfrac{2}{\pi(1+x^2)} & \text{for } -k \le x \le k, \\ 0 & \text{otherwise,} \end{cases}$$

where *k* is a constant.

Determine the value of *k* and prove that $P(|X| \le \tan\frac{\pi}{8}) = \frac{1}{2}$.

Write down the value of $\mathrm{E}(X)$ and find $\mathrm{Var}(X)$. [9]

10 Suppose that a bookstore has customers arriving at an average of 30 customers per hour and the time between arrivals is exponentially distributed.

(a) State two assumptions needed for the arrival times to be well modelled as an exponential distribution. [2]

(b) After a customer arrives, find the probability that it takes less than a minute for the next customer to arrive. [2]

(c) Let A be the number of customers arriving in a 40-minute period. State the distribution of A and hence find the probability such that A is between three standard deviations away from the mean. [3]

(d) Given that the number of customers arriving in the first *t* minutes, $N(t) = 1$, show that X_1, the arrival time of the first customer, is uniformly distributed in [0, *t*]. [4]

11 **(a)** In an investigation of the effects of colours on appetite, subjects were placed in either a red background themed restaurant or a blue background themed restaurant. Responses from subjects are taken and measured on some suitable scale of measurement. The researchers make the claim that "red background themed restaurants enhance the appetite in a restaurant". For the investigation of the effect of red background themed restaurant on appetite, a sample of 20 subjects was taken, and a 95% confidence interval on the average score of appetite of subjects from the data is $(22.28, 23.29)$. Find the value of the unbiased estimate of the population variance of the score of appetite. State an assumption made in the calculation. [5]

(b) Platelets are involved in the formation of blood clots, and it is known that smokers suffer more often from disorders involving blood clots than nonsmokers. To study the effect of cigarette smoking on platelet function, blood samples can be drawn from individuals before and after they smoke a cigarette and the extent to which the blood platelets come together can be measured.

Dr. Peter H. Levine performed this study and published a paper, where he presented data collected from 11 individuals. We show his data in the following table, which gives the maximum percentage of all the platelets that came together in the blood samples before and after smoking a cigarette.

Before	After
25	27
25	29
27	37
44	56
30	46
67	82
53	57
53	80
52	61
60	59
28	43

(ii) Carry out a suitable hypothesis test to determine if smoking affects platelet function at 1% level of significance, showing your working clearly. [5]

(iii) Andy claimed that the data provided evidence that the mean reading of maximum percentage of all platelets that came together after smoking a cigarette exceeds 50% since the sample mean is 52.45%. Do you agree with him? Explain your stand and support it with calculations. [5]

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