Qn	Solution
1	**Inequalities**
(i)	Using Cauchy-Schwarz Inequality, $\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n}(1)(x_i)\right)^2 \le \left(\sum_{i=1}^{n}1^2\right)\left(\sum_{i=1}^{n}x_i^{\ 2}\right) = n\left(\sum_{i=1}^{n}x_i^{\ 2}\right)$ Hence, $\sum_{i=1}^{n}x_i^{\ 2} \ge \frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2$ (shown)
(ii)	Using AM-GM inequality, $\sum_{i=1}^{n}(1+a)^{i-1} = 1+(1+a)+(1+a)^2+\ldots+(1+a)^{n-1}$ $\ge n\left[(1+a)^{1+2+\ldots+(n-1)}\right]^{\frac{1}{n}}$ $= n\left[(1+a)^{\frac{(n-1)n}{2}}\right]^{\frac{1}{n}}$ $= n(1+a)^{\frac{(n-1)}{2}}$ (shown) Equality holds iff $1=(1+a)=(1+a)^2=\ldots=(1+a)^{n-1}$ Therefore, $a = 0$
(iii)	From part (i), let $x_i = (1+a)^{i-1}$ Therefore, $\sum_{i=1}^{n}\left[(1+a)^{i-1}\right]^2 \ge \frac{1}{n}\left(\sum_{i=1}^{n}(1+a)^{i-1}\right)^2$ From part (ii), since $\sum_{i=1}^{n}(1+a)^{i-1} \ge n(1+a)^{\frac{n-1}{2}}$ $\sum_{i=1}^{n}\left[(1+a)^{i-1}\right]^2 \ge \frac{1}{n}\left(n(1+a)^{\frac{n-1}{2}}\right)^2$ $\sum_{i=1}^{n}(1+a)^{2i-2} \ge n(1+a)^{n-1}$ $\frac{1}{(1+a)^2}\sum_{i=1}^{n}(1+a)^{2i} \ge n(1+a)^{n-1}$ $\sum_{i=1}^{n}(1+a)^{2i} \ge n(1+a)^{n+1}$ (shown)

Qn	Solution
2	**Number Theory**
(i)	Let $n = 2k$ where $k \in \mathbb{Z}^+$ $n^4 + 4^n = (2k)^4 + 4^{2k}$ $= 16k^4 + 2^{4k}$ $= 2\left(8k^4 + 2^{4k-1}\right)$ $= 2m$ Where $m = 8k^4 + 2^{4k-1} \in \mathbb{Z}^+$ Therefore, $n^4 + 4^n$ is even.
(ii)	$\left(x^4 + 4y^4\right) = \left(x^2 + ay^2 + bxy\right)\left(x^2 + ay^2 - bxy\right)$ $= x^4 + 2ax^2y^2 - bx^3y + a^2y^4 - abxy^3 + bx^3y + abxy^3 - b^2x^2y^2$ $= x^4 + a^2y^4 + \left(2a - b^2\right)x^2y^2$ $a^2 = 4 \Rightarrow a = 2$ (since $a > 0$) $2a - b^2 = 0 \Rightarrow b^2 = 2a$ Since $a = 2$, $b^2 = 4 \Rightarrow b = 2$ (since $b > 0$)
(iii)	Case 1: When *n* is even, from part (i), $n^4 + 4^n$ is even. Since $n > 1$, $n^4 + 4^n > 2$ Hence, $n^4 + 4^n$ is not prime Case 2: When *n* is odd, Let $n = 2p + 1$ where $p \in \mathbb{Z}^+$ $n^4 + 4^n = (2p+1)^4 + 4^{2p+1}$ $= (2p+1)^4 + 4\left(\left(2^2\right)^{2p}\right)$ $= (2p+1)^4 + 4\left(2^p\right)^4$ Using the expression for (ii), $(2p+1)^4 + 4\left(2^p\right)^4$ $= \left[(2p+1)^2 + 2\left(2^p\right)^2 + 2(2p+1)\left(2^p\right)\right]\left[(2p+1)^2 + 2\left(2^p\right)^2 - 2(2p+1)\left(2^p\right)\right]$ $= \left[(2p+1)^2 + 2^{2p+1} + (2p+1)\left(2^{p+1}\right)\right]\left[(2p+1)^2 + 2^{2p+1} - (2p+1)\left(2^{p+1}\right)\right]$ Since $p \in \mathbb{Z}^+$, $(2p+1)^2 + 2^{2p+1} + (2p+1)\left(2^{p+1}\right) > 1$ and

	$\left(2p+1\right)^2+2^{2p+1}-\left(2p+1\right)\left(2^{p+1}\right)$ $=\left[\left(2p+1\right)-2^p\right]^2-\left(2^p\right)^2+2^{2p+1}$ $=\left[\left(2p+1\right)-2^p\right]^2-2^{2p}+2\left(2^{2p}\right)$ $=\left[\left(2p+1\right)-2^p\right]^2+2^{2p}>1$ Since n^4+4^n can be written as a product of two integers more than 1, n^4+4^n is not prime when n is odd. Therefore, for any integer $n>1$, n^4+4^n is not prime.

Qn	Solution
3	**Counting and Probability**
(a)(i)	By the Bijection Principle, it can be viewed as the set of 20-digit binary sequence with five '1's. Number of ways of distributing 15 identical marbles among 6 children is $\binom{15+6-1}{5}=\binom{20}{5}=15504$.
(ii)	To ensure that each child receives at least one marble, we give each child one marble first. We are then left with $15-6=9$ marbles to be distributed among 6 children. Number of ways of distributing 15 identical marbles among 6 children such that each child receives at least one marble is $\binom{9+6-1}{5}=\binom{14}{5}=2002$.
(iii)	To ensure that Tom, one of the children, receives exactly two marbles, we distribute the rest of the marbles among the 5 children. $\mathrm{P}\left(\text{Tom receives exactly two marbles}\right)$ $=\dfrac{\binom{13+5-1}{4}}{15504}=\dfrac{\binom{17}{4}}{15504}$ $=\dfrac{2380}{15504}$ $=\dfrac{35}{228}$
(b)(i)	Number of ways of distributing k distinct marbles among 6 children is 6^k.
(ii)	Let A_i denote the event that child i receive no marbles $i=1,2,3,4,5,6$. N = number of ways with at least one child not receiving any marbles $=\left\lvert\bigcup_{i=1}^{6} A_i\right\rvert$ $=\sum_{i=1}^{6}\lvert A_i\rvert-\sum_{1\le i<j\le 6}\lvert A_i\cap A_j\rvert+\sum_{1\le i<j<k\le 6}\lvert A_i\cap A_j\cap A_k\rvert-\ldots+(-1)^{6-1}\lvert A_1\cap A_2\cap\ldots\cap A_6\rvert$ $=\binom{6}{1}\cdot 5^k-\binom{6}{2}\cdot 4^k+\binom{6}{3}\cdot 3^k-\binom{6}{4}\cdot 2^k+\binom{6}{5}\cdot 1^k\ -1\cdot 0$

	Hence required number of ways is $\left\|\bigcap_{i=1}^{6}\overline{A_i}\right\|$ $=\|S\|-\left\|\bigcup_{i=1}^{6}A_i\right\|$ $=6^k-\binom{6}{1}\cdot 5^k+\binom{6}{2}\cdot 4^k-\binom{6}{3}\cdot 3^k+\binom{6}{4}\cdot 2^k-\binom{6}{5}\cdot 1^k+1\cdot 0$ $=\sum_{i=0}^{6}(-1)^i\binom{6}{i}(6-i)^k$

Qn	Solution
4	**Functions/Calculus**
(a)(i)	$y_n = x_n - 1 = \frac{x_{n-1}x_{n-2}+1}{x_{n-1}+x_{n-2}} - 1$ $= \frac{(x_{n-1}-1)(x_{n-2}-1)}{x_{n-1}+x_{n-2}}$ $= \frac{y_{n-1}y_{n-2}}{y_{n-1}+y_{n-2}+2}$ for $n \geq 3$.
(ii)	Let $\mathrm{P}(n)$ be the statement $0 < y_n < 1$ for $n \geq 1$. Since $0 < y_1 = y_2 = \frac{1}{2} < 1$, $\therefore P(1)$ and $P(2)$ are true. Assume that $P(k-2), P(k-1)$ are true for some $k \in \mathbb{Z}^+, k \geq 3$. $y_k = \frac{y_{k-1}y_{k-2}}{y_{k-1}+y_{k-2}+2}$ and since all the terms are positive on the RHS (by induction hypothesis), we have also $y_k > 0$. Also, $y_k = \frac{y_{k-1}y_{k-2}}{y_{k-1}+y_{k-2}+2} < \frac{1(1)}{0+0+2} = \frac{1}{2} < 1$ So, $P(k)$ is true. Since $P(1)$ and $P(2)$ are true, by mathematical induction, $P(n)$ is true for all $n \geq 1$.
(iii)	$y_n = \frac{y_{n-1}y_{n-2}}{y_{n-1}+y_{n-2}+2} < y_{n-1} \times \frac{1}{0+0+2} = \frac{1}{2} y_{n-1}$ since part (ii) gives $0 < y_n < 1$ for $n \geq 1$.
(iv)	We have $y_1 + y_2 + y_3 + ... + y_n < y_1 + y_2 + \frac{1}{2} y_2 + \frac{1}{4} y_2 + ... + \frac{1}{2^{n-2}} y_2$ $= \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^{n-2}} < \frac{1}{2} + \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{3}{2}$ Since all terms in the series are positive, and the sequence of partial sums is bounded, the series converges. Alternative method: $y_1 + y_2 + y_3 + ... + y_n < \frac{1}{2} + \lim_{N \to \infty} \sum_{n=1}^{N} \left(\frac{1}{2}\right)^n$ and since the limit on the RHS exists as it is a sum of Geometric series with ratio 0.5<1, by comparison test, the series on the LHS converges.
(b)(i)	$I_n - I_{n-1} = \int_0^{\frac{\pi}{4}} \frac{\sin(2n+1)x}{\sin x}\, \mathrm{d}x - \int_0^{\frac{\pi}{4}} \frac{\sin(2n-1)x}{\sin x}\, \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\, \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \frac{2\cos(2nx)\sin x}{\sin x}\, \mathrm{d}x$ $= \frac{2}{2n}\left[\sin 2nx\right]_0^{\frac{\pi}{4}} = \frac{1}{n}\sin\left(\frac{1}{2}n\pi\right)$

(ii)	$\int_0^{\frac{\pi}{4}} \frac{\sin(11x)}{\sin x}\,\mathrm{d}x = I_5$ Using the relation in part (i), $I_5 - I_4 = \frac{1}{5}\sin\left(\frac{5}{2}\pi\right)$ $I_4 - I_3 = \frac{1}{4}\sin\left(\frac{4}{2}\pi\right)$ $I_3 - I_2 = \frac{1}{3}\sin\left(\frac{3}{2}\pi\right)$ $I_2 - I_1 = \frac{1}{2}\sin\left(\frac{2}{2}\pi\right)$ $I_1 - I_0 = \frac{1}{1}\sin\left(\frac{1}{2}\pi\right)$ Using MOD, $I_5 - I_0 = \frac{1}{5}\sin\left(\frac{5}{2}\pi\right) + \frac{1}{4}\sin\left(\frac{4}{2}\pi\right) + \frac{1}{3}\sin\left(\frac{3}{2}\pi\right) + \frac{1}{2}\sin\left(\frac{2}{2}\pi\right) + \frac{1}{1}\sin\left(\frac{1}{2}\pi\right)$ $I_5 = \frac{1}{5}(1) + \frac{1}{4}(0) + \frac{1}{3}(-1) + \frac{1}{2}(0) + \frac{1}{1}(1) + \frac{\pi}{4}$ $= \frac{13}{15} + \frac{\pi}{4}$

Qn	Solution
5	**Sequences and Series / Calculus**
(a) (i)	Define $b_n = a_n - p^{-n+1}$ for $n \in \mathbb{Z}^+$ $b_{n+1} - b_n = a_{n+1} - p^{-n} - \left(a_n - p^{-n+1}\right)$ $= a_{n+1} - a_n - p^{-n} + p^{-n+1}$ $> -p^{-n} - p^{-n} + p^{-n+1}$ since $a_{n+1} + p^{-n} > a_n$ $= p^{-n}\left(p-2\right) \geq 0$ since $p \geq 2$ $\Rightarrow b_{n+1} > b_n \quad \forall n \in \mathbb{Z}^+$ $\Rightarrow \{b_n\}_{n=1,2,\ldots}$ is strictly increasing $\therefore \left\{a_n - p^{-n+1}\right\}_{n=1,2,\ldots}$ is strictly increasing
(ii)	By triangle inequality, $\left\|a_n - p^{-n+1}\right\| \leq \left\|a_n\right\| + \left\|p^{-n+1}\right\| \leq M+1 \quad \forall \text{n} \in \mathbb{Z}^+$ since p^{-n+1} is strictly decreasing and $p^{-n+1} < p^{-1+1} = 1 \quad \forall n \in \mathbb{Z}^+$. Thus $\{b_n\}_{n=1,2,\ldots}$ is bounded above by $M+1$.
	Since $\left\{a_n - p^{-n+1}\right\}_{n=1,2,\ldots}$ is strictly increasing and bounded above, it converges. That is, there exists a finite number l such that $l = \lim\limits_{n\to\infty}\left(a_n - p^{-n+1}\right) = \lim\limits_{n\to\infty}\left(a_n\right) - \lim\limits_{n\to\infty}\left(p^{-n+1}\right)$ $= \lim\limits_{n\to\infty}\left(a_n\right)$ since $\lim\limits_{n\to\infty}\left(p^{-n+1}\right) = \lim\limits_{n\to\infty}\left(\dfrac{1}{p^{n-1}}\right) = 0$ as $p \geq 2 > 1$
5(b) (i)	$z = 1 - \mathrm{e}^{\mathrm{i}\theta}\cos\theta = 1 - \left(\cos\theta + \mathrm{i}\sin\theta\right)\cos\theta$ $= 1 - \cos^2\theta - \mathrm{i}\sin\theta\cos\theta$ $= \sin^2\theta - \mathrm{i}\sin\theta\cos\theta$ $= \sin\theta\left(\sin\theta - \mathrm{i}\cos\theta\right)$ $= \sin\theta\left(\cos\left(\theta - \dfrac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta - \dfrac{\pi}{2}\right)\right)$ $\left\|z\right\| = \left\|\sin\theta\left(\cos\left(\theta - \dfrac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta - \dfrac{\pi}{2}\right)\right)\right\| = \left\|\sin\theta\right\|\left\|\cos\left(\theta - \dfrac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta - \dfrac{\pi}{2}\right)\right\|$ $= \sin\theta$ $\arg\left(z\right) = \arg\left(\sin^2\theta - \mathrm{i}\sin\theta\cos\theta\right)$ $= \arg\left(\sin\theta\right) + \arg\left(\sin\theta - \mathrm{i}\cos\theta\right)$ $= \arg\left(\cos\left(\theta - \dfrac{\pi}{2}\right) + \mathrm{i}\sin\left(\theta - \dfrac{\pi}{2}\right)\right)$ $= \theta - \dfrac{\pi}{2}$

5(b) (ii)	$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\cos^r\theta\cos r\theta$ $=1-\binom{n}{1}\cos\theta\cos\theta+\binom{n}{2}\cos^2\theta\cos 2\theta-...+(-1)^n\binom{n}{n}\cos^n\theta\cos n\theta$ $=\mathrm{Re}\begin{pmatrix}1-\binom{n}{1}\cos\theta(\cos\theta+\mathrm{i}\sin\theta)+\binom{n}{2}\cos^2\theta(\cos 2\theta+\mathrm{i}\sin 2\theta)-... \\ +(-1)^n\binom{n}{n}\cos^n\theta(\cos n\theta+\mathrm{i}\sin n\theta)\end{pmatrix}$ $=\mathrm{Re}\left[\left(1-\mathrm{e}^{\mathrm{i}\theta}\cos\theta\right)^n\right]=\mathrm{Re}\left(z^n\right)$ $z^n=\left(1-\mathrm{e}^{\mathrm{i}\theta}\cos\theta\right)^n$ $\Rightarrow z^n=\sin^n\theta\left(\cos\left(\theta-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(\theta-\frac{\pi}{2}\right)\right)^n=\sin^n\theta\mathrm{e}^{\mathrm{i}n\left(\theta-\frac{\pi}{2}\right)}$ $\mathrm{Re}\left(z^n\right)=\sin^n\theta\cos n\left(\theta-\frac{\pi}{2}\right)$

Qn	Solution
6	**Derangments**
(a)	Consider the derangement of the set: $\{1,2,3,\cdots,n\}$. Since the first digit cannot be '1', there are $n-1$ choices for the first digit. Suppose that the first digit is 'k' (let's say $k=2$), we have the following two cases. **Case 1:** If '1' is placed in the second position, the number of ways to arrange the remaining $n-2$ digits such that $b_i \neq i$ for each $i=3,4,\ldots,n$ is D_{n-2}. **Case 2:** If '1' is **not** placed in the second position, then the number of ways to arrange $1,3,4,\ldots,n$ such that $b_i \neq i$ for each $i=2,3,\ldots,n$ is D_{n-1}. Hence, if the first digit is '2', the total number of derangements is $D_{n-1}+D_{n-2}$. A similar argument holds for the remaining $n-2$ cases. By Multiplication Principle, $D_n=(n-1)(D_{n-1}+D_{n-2}), n\geq 3$, where $D_1=0$ and $D_2=1$.
(b)	Given that $D_n=(n-1)(D_{n-1}+D_{n-2})$, we have $D_n-nD_{n-1}=(n-1)(D_{n-1}+D_{n-2})-nD_{n-1}$ $D_n-nD_{n-1}=-(D_{n-1}-(n-1)D_{n-2})$ Sub $x_n=D_n-nD_{n-1}$, $D_n-nD_{n-1}=(n-1)(D_{n-1}+D_{n-2})-nD_{n-1}$ $x_n=-x_{n-1}$ Since $D_1=0$ and $D_2=1$, we have $x_2=D_2-2D_1=1-2(0)=1$ $x_3=-x_2=-1$ $x_4=-x_3=1$ Thus, $x_n=(-1)^n$ and we have $(-1)^n=D_n-nD_{n-1}$ $D_n=nD_{n-1}+(-1)^n$
(c)	$D_n=nD_{n-1}+(-1)^n$ $\dfrac{D_n}{n!}=\dfrac{D_{n-1}}{(n-1)!}+\dfrac{(-1)^n}{n!}$ Sub $y_n=\dfrac{D_n}{n!}$,

$$D_n = nD_{n-1} + (-1)^n$$

$$y_n = y_{n-1} + \frac{(-1)^n}{n!}$$

$$y_k = y_{k-1} + \frac{(-1)^k}{k!}$$

$$y_k - y_{k-1} = \frac{(-1)^k}{k!}$$

$$\sum_{k=2}^{n}(y_k - y_{k-1}) = \sum_{k=2}^{n}\frac{(-1)^k}{k!}$$

$$y_n - y_1 = \sum_{k=2}^{n}\frac{(-1)^k}{k!} \qquad \text{using method of differences}$$

$$y_n - 0 = \sum_{k=2}^{n}\frac{(-1)^k}{k!} \qquad \left(\because y_1 = \frac{D_1}{1!} = \frac{0}{1} = 0\right)$$

$$y_n = \sum_{k=2}^{n}\frac{(-1)^k}{k!}$$

$$= \sum_{k=0}^{n}\frac{(-1)^k}{k!} \qquad \left(\because \frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} = 0\right)$$

$$D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} \qquad \left(\because y_n = \frac{D_n}{n!}\right)$$

(d)

First, you take $n!$, which represents all arrangements of the whole sequence.

Then, you must subtract all arrangements in which each element appears in its original location, and now you have $n! - \binom{n}{1}(n-1)!$. Then, you must add back in permutations in which each set of two elements stay in their original positions, as we subtracted them twice. Now, we have $n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)!$. Principle of Inclusion and Exclusion continues to give us this pattern. We have to alternate between subtracting and adding.

Now, we can change the form of $\binom{n}{k}(n-k)!$. This is written as $\frac{n!}{(n-k)!k!} \times (n-k)! = \frac{n!}{k!}$.

We can now write our relation as

$$n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n\left(\frac{n!}{n!}\right).$$

We can factor out $n!$ and get

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!}\right) = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.$$

(e)	$D_7 = 7!\sum_{k=0}^{7}\frac{(-1)^k}{k!} = 1854$
(f)	To find the value of $\lim_{n\to\infty}\frac{D_n}{n!}$ Observe that $\frac{D_n}{n!} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+...+(-1)^n\frac{1}{n!} = \sum_{k=0}^{n}\frac{(-1)^k}{k!}$ And $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^r}{r!}+... = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ By letting $x=-1,$ we get $\lim_{n\to\infty}\frac{D_n}{n!} = \frac{1}{e}$