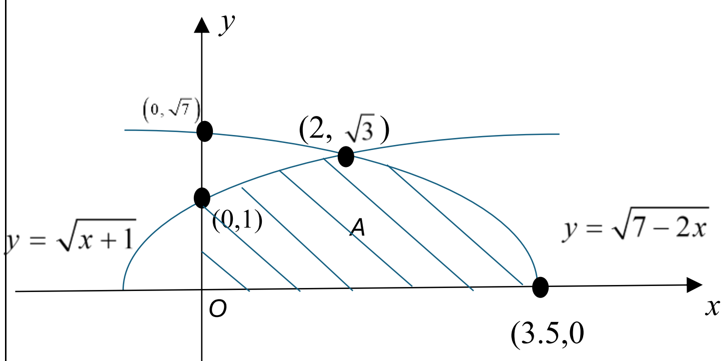
2025 C2 H2 Mathematics Preliminary Examination Paper 2 Suggested Solutions

Section A: Pure Mathematics (40 marks)

1 The region A is bounded by the curves $y=\sqrt{x+1}$, $y=\sqrt{7-2x}$, the x–axis and the y–axis.

(a) Find the exact area of A. [4]

(b) Find the volume of the solid obtained when A is rotated through 2π radians about the y–axis. [3]

<table>
<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td>(a)</td><td>


Coordinates of point of intersection: $\left(2,\sqrt{3}\right)$

<u>Method 1</u><u>: Using x-axis.</u>
$$\begin{aligned}\text{Area} &= \int_0^2 y\ \mathrm{d}x+\int_2^{3.5} y\ \mathrm{d}x\\ &=\int_0^2 \sqrt{x+1}\ \mathrm{d}x+\int_2^{3.5}\sqrt{7-2x}\ \mathrm{d}x\\ &=\left[\frac{2(x+1)^{\frac{3}{2}}}{3}\right]_0^2-\left[\frac{(7-2x)^{\frac{3}{2}}}{3}\right]_2^{3.5}\\ &=\left(\frac{2\sqrt{27}}{3}-\frac{2}{3}\right)-\left(-\frac{\sqrt{27}}{3}\right)\\ &=2\sqrt{3}-\frac{2}{3}+\sqrt{3}\\ &=3\sqrt{3}-\frac{2}{3}\ \text{ units}^2\end{aligned}$$
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	<u>Method 2:</u> <u>Using y-axis.</u> $\text{Area} = \int_0^{\sqrt{3}} x \, \mathrm{d}y - \int_1^{\sqrt{3}} x \, \mathrm{d}y$ $= \int_0^{\sqrt{3}} \frac{7-y^2}{2} \, \mathrm{d}y - \int_1^{\sqrt{3}} y^2 - 1 \, \mathrm{d}y$ $= \left[\frac{7y}{2} - \frac{y^3}{6} \right]_0^{\sqrt{3}} - \left[\frac{y^3}{3} - y \right]_1^{\sqrt{3}}$ $= \left(\frac{7\sqrt{3}}{2} - \frac{3\sqrt{3}}{6} \right) - \left[\left(\frac{3\sqrt{3}}{3} - \sqrt{3} \right) - \left(\frac{1}{3} - 1 \right) \right]$ $= 3\sqrt{3} - \frac{2}{3} \text{ units}^2$
(b)	$y = \sqrt{7-2x}$ $x = \frac{7-y^2}{2}$ $\text{Vol} = \pi \int_0^{\sqrt{3}} \left(\frac{7-y^2}{2} \right)^2 \mathrm{d}y - \pi \int_1^{\sqrt{3}} \left(y^2 - 1 \right)^2 \mathrm{d}y$ $= 47.38326$ $\approx 47.4 \text{ units}^3$ (3 s.f.)

2 It is given that $y=\ln\left[\sin\left(x+\frac{\pi}{4}\right)\right]$, where $-\frac{\pi}{4}<x<\frac{3\pi}{4}$.

(a) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1=0$. Hence find the first four non-zero terms of the Maclaurin expansion of y, leaving your answer in exact form. [6]

(b) Verify the result obtained in part **(a)** is obtained using standard series from the List of Formulae (MF27). [5]

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<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td>(a)</td><td>

$\mathrm{e}^y=\sin\left(x+\frac{\pi}{4}\right)$

$\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos\left(x+\frac{\pi}{4}\right)$

$\mathrm{e}^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\mathrm{e}^y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sin\left(x+\frac{\pi}{4}\right)$

$\mathrm{e}^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\mathrm{e}^y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\mathrm{e}^y$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+1=0$

<u>Alternative Solution</u>

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos\left(x+\frac{\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}=\cot\left(x+\frac{\pi}{4}\right)$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\mathrm{cosec}^2\left(x+\frac{\pi}{4}\right)$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1=\cot^2\left(x+\frac{\pi}{4}\right)-\mathrm{cosec}^2\left(x+\frac{\pi}{4}\right)+1=0$

Applying further implicit differentiation

$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$

When $x=0$, $y=\ln\left(\frac{1}{\sqrt{2}}\right),\frac{\mathrm{d}y}{\mathrm{d}x}=1,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2,\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=4$

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	Maclaurin expansion of y is $y=\ln\left(\frac{1}{\sqrt{2}}\right)+x-x^2+\frac{2}{3}x^3+\ldots$
(b)	$y=\ln\left[\sin\left(x+\frac{\pi}{4}\right)\right]$ $=\ln\left(\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x\right)$ $=\ln\left[\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)\right]$ $=\ln\left[\frac{1}{\sqrt{2}}(\sin x+\cos x)\right]$ $\approx\ln\frac{1}{\sqrt{2}}+\ln\left(1+x-\frac{x^2}{2}-\frac{x^3}{6}\right)$ $=\ln\frac{1}{\sqrt{2}}+\left(x-\frac{x^2}{2}-\frac{x^3}{6}\right)-\frac{1}{2}\left(x-\frac{x^2}{2}-\frac{x^3}{6}\right)^2$ $+\frac{1}{3}\left(x-\frac{x^2}{2}-\frac{x^3}{6}\right)^3+\ldots$ $=\ln\frac{1}{\sqrt{2}}+\left(x-\frac{x^2}{2}-\frac{x^3}{6}\right)-\frac{1}{2}\left(x-\frac{x^2}{2}\right)^2+\frac{1}{3}x^3+\ldots$ $=\ln\frac{1}{\sqrt{2}}+\left(x-\frac{x^2}{2}-\frac{x^3}{6}\right)-\frac{1}{2}\left(x^2-x^3\right)+\frac{1}{3}x^3+\ldots$ $=\ln\frac{1}{\sqrt{2}}+x-x^2+\frac{2}{3}x^3+\ldots$ (verified)

3 The parametric equations of the curve C are

$$x = 1 - 3\operatorname{cosec}\theta \quad \text{and} \quad y = 2\cot\theta - 3, \text{ where } 0 \le \theta \le \pi.$$

(a) Show $\dfrac{dy}{dx} = -\dfrac{2}{3}\sec\theta$. Hence find the equation of the normal to C at the point where $\theta = \dfrac{\pi}{4}$. Give the equation in the form $y = Ax + B$, where A and B are exact constants to be found. [4]

(b) Show that the normal found in part **(a)** will cut C again. [2]

(c) Find the Cartesian equation of C. [2]

(d) Sketch C, indicating clearly its key features. [3]

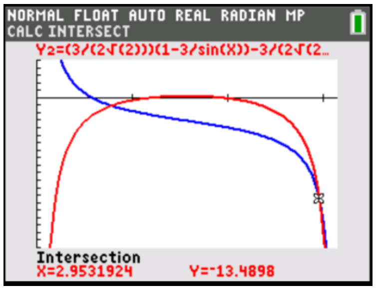
(e) Find the range of values of m such that there is no intersection between the line $y = m(x-1) - 3$ and C. [2]

Qn	Suggested Solutions
(a)	$\dfrac{dx}{d\theta} = -3(-\operatorname{cosec}\theta\cot\theta)$ $\quad \dfrac{dy}{d\theta} = -2\operatorname{cosec}^2\theta$ $\dfrac{dy}{dx} = -\dfrac{2\operatorname{cosec}^2\theta}{3\operatorname{cosec}\theta\cot\theta} = -\dfrac{2\operatorname{cosec}\theta}{3\cot\theta} = -\dfrac{2}{3}\sec\theta$ When $\theta = \dfrac{\pi}{4}$, $x = 1 - 3\sqrt{2}$ $y = -1$ $\dfrac{dy}{dx} = -\dfrac{2}{3}\left(\dfrac{1}{\cos\left(\pi/4\right)}\right) = -\dfrac{2\sqrt{2}}{3}$ Equation of normal: $y - (-1) = \dfrac{3}{2\sqrt{2}}\left(x - \left(1 - 3\sqrt{2}\right)\right)$ $y = \dfrac{3}{2\sqrt{2}}\left(x - 1 + 3\sqrt{2}\right) - 1$ $= \dfrac{3}{2\sqrt{2}}x - \dfrac{3}{2\sqrt{2}} + \dfrac{9}{2} - 1$ $\therefore y = \dfrac{3}{2\sqrt{2}}x - \dfrac{3}{2\sqrt{2}} + \dfrac{7}{2}$

(b) Substitute $x = 1 - 3\operatorname{cosec}\theta$ and $y = 2\cot\theta - 3$ into

$$2\cot\theta - 3 = \frac{3}{2\sqrt{2}}\left(1 - 3\operatorname{cosec}\theta\right) - \frac{3}{2\sqrt{2}} + \frac{7}{2}$$

By GC graph,



Enter $Y_1 = 2\cot\theta - 3$ and $Y_2 = \frac{3}{2\sqrt{2}}\left(1 - 3\operatorname{cosec}\theta\right) - \frac{3}{2\sqrt{2}} + \frac{7}{2}$,

and find points of intersection.

$\theta = 2.95319$ or $0.7853982 = \frac{\pi}{4}$

Since there are 2 values of θ, the normal line will cut the curve again.

(c) $x = 1 - 3\operatorname{cosec}\theta$ and $y = 2\cot\theta - 3$

$\operatorname{cosec}\theta = \frac{1-x}{3}$ and $\cot\theta = \frac{y+3}{2}$

For $0 \le \theta \le \pi$,

$0 \le \sin\theta \le 1$

$\operatorname{cosec}\theta \ge 1$

$-3\operatorname{cosec}\theta \le -3$

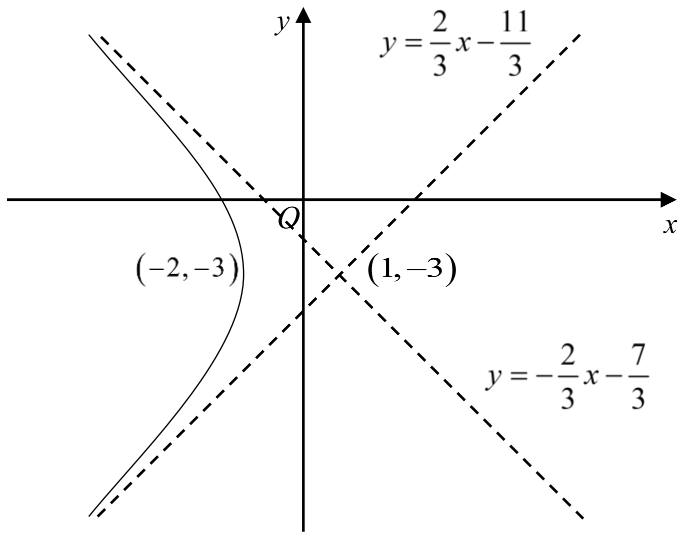
$1 - 3\operatorname{cosec}\theta \le -2$

$\Rightarrow x \le -2$

$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

$$\left(\frac{y+3}{2}\right)^2 + 1 = \left(\frac{1-x}{3}\right)^2$$

$$\left(\frac{x-1}{3}\right)^2 - \left(\frac{y+3}{2}\right)^2 = 1, \quad x \le -2$$

(d)	
(e)	When $x = 1$, $y = m(1-1)-3 = -3$ The line passes through the centre $(1,-3)$, of the hyperbola . For no intersection between the line and C, The range of values for m is $m \le -\frac{2}{3}$ or $m \ge \frac{2}{3}$.

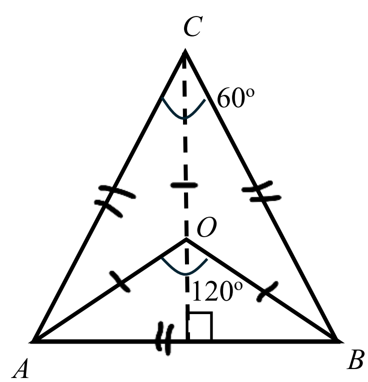
4 Let A, B and C be the points on the same plane with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors such that $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$.

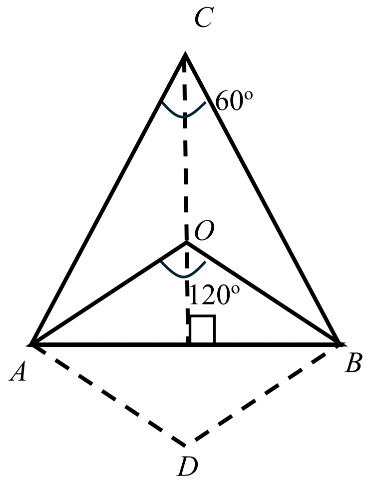
(a) **(i)** By considering $\mathbf{c}\bullet\mathbf{c}$, find the value of $\mathbf{a}\cdot\mathbf{b}$. [3]

(ii) Find the angle $\angle AOB$. [2]

(iii) Draw the position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ on a single diagram. Using your diagram, identify the type of triangle ABC. [2]

(b) The point D has position vector $\mathbf{a}$ + $\mathbf{b}$. Find the area of the quadrilateral $ACBD$. [2]

Qn	Suggested Solutions
(ai)	$\mathbf{c}=-\mathbf{a}-\mathbf{b}$ $\mathbf{c}\cdot\mathbf{c}=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$ $=\lvert\mathbf{a}\rvert^2+2\mathbf{a}\cdot\mathbf{b}+\lvert\mathbf{b}\rvert^2$ $1=1+2\mathbf{a}\cdot\mathbf{b}+1$ $\mathbf{a}\cdot\mathbf{b}=-\frac{1}{2}$
(aii)	$-\frac{1}{2}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\angle AOB$ $\cos\angle AOB=-\frac{1}{2}\Rightarrow\angle AOB=120°$
(aiii)	Since $\angle AOB=120°$, $\angle OAB=30°$ with $OA=OB=OC$. It can be shown that the triangles OAB, OAC and OBC are congruent triangles and $\angle ACB=\angle ABC=\angle BCA=60°$. Hence triangle ABC is an equilateral triangle.  Triangle ABC is an equilateral triangle.
(b)	Since $\overrightarrow{OD}=\mathbf{a}+\mathbf{b}=-\mathbf{c}$ and OD and AB are diagonals of the rhombus $OADB$, OD and AB are perpendicular to each other. Hence $ADBC$ is a kite.

	 **<u>Method 1</u>** Area of $ACBD = 2\times\frac{1}{2}\left\|\overrightarrow{AC}\times\overrightarrow{DC}\right\|$ $= \left\|(\mathbf{c}-\mathbf{a})\times 2\mathbf{c}\right\|$ $= 2\left\|(-\mathbf{b}-\mathbf{a})\times \mathbf{a}\right\|$ $= 2(\sin 120^\circ) = \sqrt{3}$ **<u>Method 2</u>** Area of $ACBD = \frac{1}{2}\times DC\times AB = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$

Section B: Probability and Statistics (60 marks)

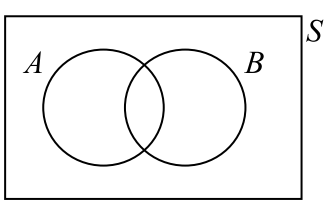
5 The eleven letters in the word INSPIRATION are each printed on separate, identical cards.

(a) Find the number of ways in which the cards can be arranged in a row if,

(i) there are no restrictions, [1]

(ii) the letters N are together or the letters I must all be separated, but not both. [3]

(b) Three of the eleven cards are removed at random. Find the probability that the letters on the eight cards left behind are all distinct. [2]

Qn	Suggested Solutions
(a)(i)	Number of ways $= \dfrac{11!}{2!3!} = 3326400$
(a)(ii)	Number of ways for N,N together $= \dfrac{10!}{3!}$ $= 604800$ Number of ways for I,I,I to be separated $= \dfrac{8!}{2!} \times {}^9C_3$ $= 1693440$ Number of ways for both NN together and III separated $= 7! \times {}^8C_3$ $= 282240$  A: ‘N’s Together B: ‘I’s separated Number of ways the letters N must be together or the letters I must be separated, but not both $= 604800 + 1693440 - 2(282240)$ $= 1733760$

(b)	Required probability $= \dfrac{{}^{6}C_{6} \times {}^{3}C_{1} \times {}^{2}C_{1}}{{}^{11}C_{8}}$ OR $= \dfrac{{}^{2}C_{1} \times {}^{3}C_{2}}{{}^{11}C_{3}}$ $= \dfrac{2}{55}$ **Method 2:** Probability $= \dfrac{3}{11}\dfrac{2}{10}\dfrac{2}{9} \times \dfrac{3!}{2!} = \dfrac{2}{55}$

6 A basketball free throw game involves a team of two players, Ben and John. The game consists of at most three throws and the moment a player makes a successful shot, the team wins, and the game will end.

The game uses the following rules.

- Only one player is selected for each game and the probability that Ben is selected in a game is 0.7.
- The probabilities that Ben and John make a successful shot in any single attempt are 0.1 and 0.07 respectively.
- The shots are independent of each other.

(a) Find the probability that the team wins the game. [3]

(b) The team did not win the game. Find the probability that John was the one who was selected to shoot. [3]

(c) The team attempts the game repeatedly until the first game is won. Find the least number of attempts required such that the probability of winning within *n* games is at least 0.95. [2]

Qn	**Suggested Solutions**
(a)	$\mathrm{P}(\text{wins})$ $= \mathrm{P}(\text{wins and Ben selected}) + \mathrm{P}(\text{wins and John selected})$ $= \mathrm{P}(\text{wins} \mid \text{Ben selected})\mathrm{P}(\text{Ben selected})$ $+\mathrm{P}(\text{wins} \mid \text{John selected})\mathrm{P}(\text{John selected})$ $= (0.1 + 0.9\times 0.1 + 0.9^2 \times 0.1)(0.7)$ $+(0.07 + 0.93\times 0.07 + 0.93^2 \times 0.07)(0.3)$ $= 0.2483929$ OR $\mathrm{P}(\text{wins})$ $= \mathrm{P}(\text{wins and Ben selected}) + \mathrm{P}(\text{wins and John selected})$ $= \mathrm{P}(\text{wins} \mid \text{Ben selected})\mathrm{P}(\text{Ben selected}) +$ $\mathrm{P}(\text{wins} \mid \text{John selected})\mathrm{P}(\text{John selected})$ $= (1-0.9^3)(0.7) + (1-0.93^3)(0.3)$ $= 0.2483929$

(b)

$\mathrm{P}\left(\text{John selected} \mid \text{did not win}\right)$

$$= \frac{\mathrm{P}\left(\text{John selected and did not win}\right)}{\mathrm{P}\left(\text{did not win}\right)}$$

$$= \frac{0.3 \times 0.93^3}{1 - 0.2483929}$$

$= 0.321 \text{ (3sf)}$

(c)

Required probability = 1 – P(did not win in all n games)

$$= 1 - \left(1 - 0.2483929\right)^n$$

OR

Required probability

= P(win in 1st game) + P(win in 2nd game)

+ P(win in 3rd game) + … + P(win in n^{th} game)

$$= 0.2483929 + \left(1 - 0.2483929\right)\left(0.2483929\right) + \left(1 - 0.2483929\right)^2\left(0.2483929\right) + \ldots + \left(1 - 0.2483929\right)^{n-1}\left(0.2483929\right)$$

$$= \frac{0.2483929\left(1 - \left(1 - 0.2483929\right)^n\right)}{1 - \left(1 - 0.2483929\right)}$$

$$= 1 - \left(1 - 0.2483929\right)^n$$

$$1 - \left(1 - 0.2483929\right)^n \geq 0.95$$

$$\left(1 - 0.2483929\right)^n \leq 0.05$$

n	$\left(1 - 0.2483929\right)^n$
10	$0.0575 > 0.05$
11	$0.0432 < 0.05$

Smallest n is 11.

7 An ice-cream seller records the monthly ice cream sales, s thousands dollars for different temperature, t degrees Celsius during the winter season. The recorded values are shown in the table below.

t	1	4	5	6	7	8	9
s	14	15	15	16	18	21	23

(a) It is given that the regression line of s on t is $s = 1.125t + 11$. Using this regression line, find the sum of the squares of the residuals. [1]

(b) State the coordinates of an additional data point such that, with all 8 data points, the regression line remains the same as in part **(a)**. [1]

(c) Sketch a scatter diagram of s against t for the data given in the table. [1]

The following three models are proposed, where a, b, c, d, f and h are positive constants.

(A) $s = at^2 + b$

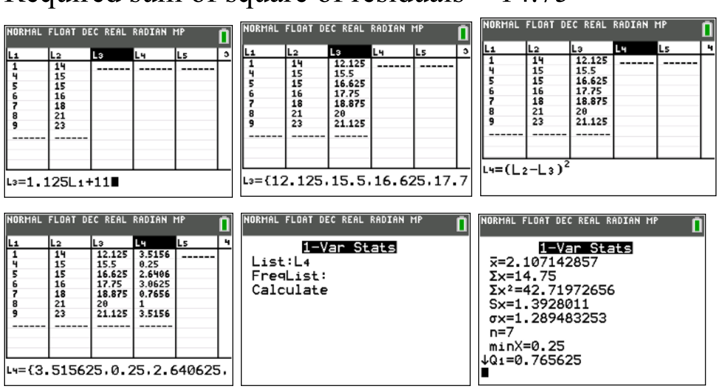
(B) $s = -ce^t + d$

(C) $s = f\ln(t+h)$

(d) Explain which of these models give the best fit to the data. State the values of the constants for the chosen model. [2]

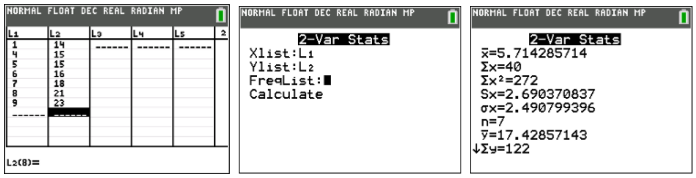
A temperature of F degrees Fahrenheit is equivalent to a temperature of C degrees Celsius, where $F = \frac{9}{5}C + 32$.

(e) Using the model you chose in part **(d)**, re-write the equation so that it can be used to estimate the monthly sales when the temperature, T, is given in degrees Fahrenheit. [2]

Qn	Suggested Solutions
(a)	Required sum of square of residuals = 14.75 

(b)

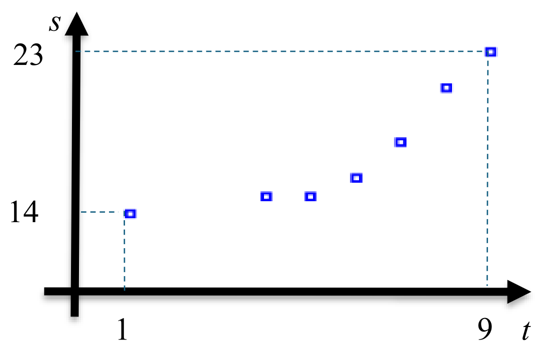
Required point $=(\bar{t},\bar{s})=(5.71,17.4)$



OR

Any points on the given regression line such as $(0,11)$.

(c)



(d)

Since the data points lie close to a curve with positive gradient and concave upwards, model (A) is most appropriate.

Or

The scatter diagram shows that as t increases, s increases at an increasing rate. Thus, model (A) is most appropriate.

$s=0.118586994t^2+12.82061957$

$\therefore\ a=0.119$ (to 3sf) and $b=12.8$ (to 3sf)

(e)

Let T be the temperature given in Fahrenheit

$$T=\frac{9}{5}t+32$$

$$t=\frac{5}{9}(T-32)$$

$$s=0.118586994\left[\frac{5}{9}(T-32)\right]^2+12.82061957$$

$$s\approx 0.0366(T-32)^2+12.8 \text{ (to 3 sf)}$$

8 A food producer claims that the mean mass of a can of beans it produces is 425 g. Following customer feedback, the production manager wishes to test if the mean mass of a can of beans is indeed 425 g.

The production manager took a random sample of size 50 and the mass of each can, in x g, is recorded and the results are shown below:

$$\sum x = 21209, \ \sum (x - 424.18)^2 = 522$$

(a) State what it means for a sample to be random in this context. [1]

(b) Find the unbiased estimates for the population mean and variance. [2]

(c) State the hypotheses for the manager's test, defining any parameters you use. Carry out the test at the 5% level of significance, giving your conclusion in the context of the question. [5]

The production manager wishes to test whether the mean mass of a can of beans has increased using the alternative manufacturing process. He finds that the mean mass of 55 randomly chosen cans is 426.5 g. He carries out a hypothesis test at 10% level of significance.

(d) Explain, with justification, how the population standard deviation of the mass of a can produced under the alternative process will affect the conclusion made by the production manager. [3]

Qn	Suggested Solutions
(a)	Every can in the population has an equal chance of being selected and is selected independently of the others.
(b)	Unbiased estimate for the population mean $\bar{x} = \frac{\sum x}{50} = \frac{21209}{50} = 424.18$ Unbiased estimate for the population variance $s^2 = \frac{50}{50-1} \frac{\sum (x-\bar{x})^2}{50}$ $= \frac{\sum (x - 424.18)^2}{49}$ $= \frac{522}{49}$ or 10.65306122

(c)	Let X be the mass (in grams) of a can of beans. Let μ and σ^2 be the population mean and variance of X. $H_0 : \mu = 425$ $H_1 : \mu \neq 425$ $n = 50,\ \bar{x} = 424.18,\ s^2 = \dfrac{522}{49}$ Under H_0, since $n = 50$ is large, by the Central Limit Theorem , $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ approximately. Test statistic $Z = \dfrac{\bar{X} - \mu}{\sqrt{\dfrac{S^2}{n}}} \sim N(0,1)$ approximately Level of significance: 5% Reject H_0 if p-value ≤ 0.05 Assuming H_0 is true, by G.C., p-value $= 0.0757$ (3 s.f.) Since p-value $= 0.0757 > 0.05$, we do not reject H_0 at 5% level of significance and conclude that there is insufficient evidence to say that the population mean mass of a can of beans is not 425 g.
(d)	Let Y be the mass (in grams) of a can of beans produced using the alternative process. Let μ and σ^2 be the population mean and variance of Y. $H_0 : \mu = 425$ $H_1 : \mu > 425$ $n = 55,\ \bar{y} = 426.5$ Under H_0, since $n = 55$ is large, by the Central Limit Theorem, $\bar{Y} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ approximately. Test statistic $Z = \dfrac{\bar{Y} - \mu}{\sqrt{\dfrac{\sigma^2}{n}}} \sim N(0,1)$ approximately

Level of significance: 10%
Reject H_0 if $z \geq 1.281551567$

$$\frac{426.5-425}{\sqrt{\frac{\sigma^2}{55}}} \geq 1.281551567$$

Since $\sigma \geq 0$, $\sqrt{\sigma^2} \leq \dfrac{1.5\sqrt{55}}{1.281551567}$

$$\Leftrightarrow \sigma \leq 8.680335632$$
$$\Leftrightarrow \sigma \leq 8.68 \ (3 \text{ s.f.})$$
$$\Leftrightarrow 0 < \sigma \leq 8.68 \ (3 \text{ s.f.})$$

At 10% level of significance, if $0 < \sigma \leq 8.68$, then H_0 is rejected and the production manager may conclude that mean mass of a can of beans produced using the alternative process has increased.

9 In each round of a treasure hunt game, a player randomly selects a spot from a large number of predefined treasure locations on an island. Each spot uncovers one outcome, and a score x is awarded based on the outcome. The table below shows all the possible outcomes in a round and the corresponding scores. Each round is independent and the game resets after every round, so that the probabilities remain unchanged.

Outcome	Cursed trap	Small trap	Mystery box	Gold chest
Score, x	-3	-0.3	p	5
$P(X=x)$	0.2	p	q	0.1

(a) Show that $E(X) = -0.1 + 0.4p - p^2$. [2]

(b) Hence find the maximum and minimum possible values of $E(X)$. [2]

(c) The treasure hunt game is played for 30 rounds. If $p = 0.4$, find the probability that the player's mean score exceeds 0. [3]

(d) The treasure hunt game is played for 10 rounds. Given that the probability of finding more than 3 small traps in these 10 rounds is 10%, find the value of p. [2]

(e) Over a long period of time, it is observed that the number of small traps found in 10 rounds of the game follows a bimodal distribution, with one of the modes being 3. Find the two exact possible values of p, showing your working clearly. [3]

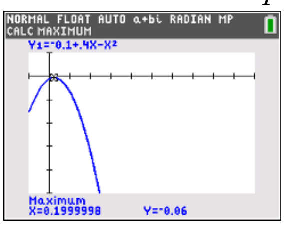
<table>
<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td>(a)</td><td>

$0.2+p+q+0.1=1 \Rightarrow q=0.7-p$

$$\begin{aligned}\mathrm{E}(X)&=(-3\times0.2)+(-0.3\times p)+(p\times q)+(5\times0.1)\\&=(-0.6)+(-0.3p)+(p(0.7-p))+(0.5)\\&=-0.1+0.4p-p^2\end{aligned}$$

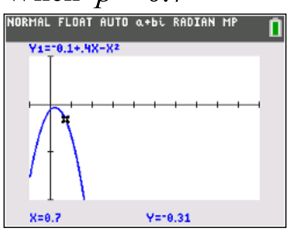
</td></tr>
<tr><td>(b)</td><td>

Enter $Y=-0.1+0.4p-p^2$ into GC to find maximum point



$p\ge0$ and $q\ge0\Rightarrow0.7-p\ge0\Rightarrow p\le0.7$

When $p=0.7$

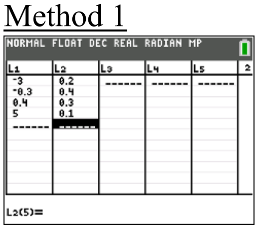


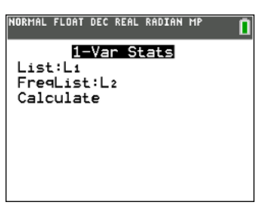
Max $\mathrm{E}(X)=-0.06$ when $p=0.2$

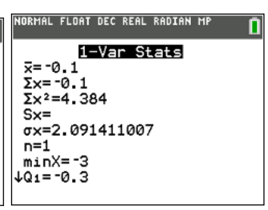
Min $\mathrm{E}(X)=-0.31$ when $p=0.7$

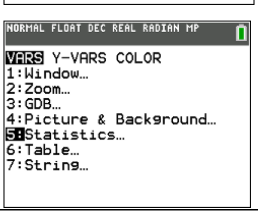
</td></tr>
<tr><td>(c)</td><td>

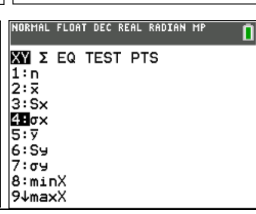
<u>Method 1</u>

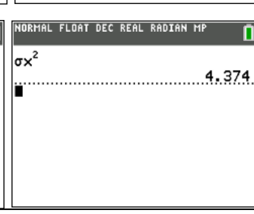












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$\mathrm{E}(X) = -0.1$

$\mathrm{Var}(X) = 4.374$

<u>Method 2</u>

$\mathrm{E}(X) = -0.1 + 0.4(0.4) - (0.4)^2 = -0.1$

$\mathrm{E}(X^2) = (3^2 \times 0.2) + (0.3^2 \times 0.4) + (0.4^2 \times 0.3) + (5^2 \times 0.1)$

$= 4.384$

$\mathrm{Var}(X) = 4.384 - (-0.1)^2 = 4.374$

$\overline{X} = \dfrac{X_1 + X_2 + \cdots + X_{30}}{30}$

Since n = 30 is large, by Central Limit Theorem,

$\overline{X} \sim \mathrm{N}\left(-0.1, \dfrac{4.374}{30}\right)$ approximately.

$\mathrm{P}(\overline{X} \geq 0) = 0.39670 \approx 0.397 \quad (\text{to 3 sf})$

</td></tr>
<tr><td>(d)</td><td>

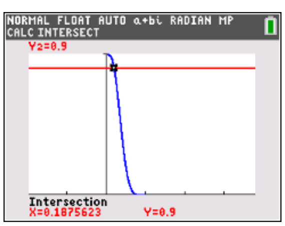
Let Y be the number of small traps, out of 10 rounds.

$Y \sim \mathrm{B}(10, p)$

$\mathrm{P}(Y > 3) = 0.10$

$1 - \mathrm{P}(Y \leq 3) = 0.10$

$\mathrm{P}(Y \leq 3) = 0.9$



$\therefore \; p = 0.188$ (to 3 s.f.)

</td></tr>
<tr><td>(e)</td><td>

$\mathrm{P}(X = 2) = \mathrm{P}(X = 3)$

${}^{10}C_2 p^2 (1-p)^8 = {}^{10}C_3 p^3 (1-p)^7$

Since $(1-p) > 0$ and $p > 0$,

$p = \dfrac{3}{11}$

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</table>

	$\mathrm{P}(X=4)=\mathrm{P}(X=3)$ ${}^{10}C_4 p^4(1-p)^6 = {}^{10}C_3 p^3(1-p)^7$ $p=\frac{4}{11}$ $\therefore p=\frac{3}{11}$ or $p=\frac{4}{11}$

10 **In this question you should state the parameters of any distributions you use.**

At a burger shop the wait time, W (in minutes), is defined to be the time from when a customer places an order at the counter until the food is collected. It was proposed that W is modelled by $\mathrm{N}(2,1.5^2)$.

(a) Give a reason why this model is not suitable. [1]

A new model for W is given by $\mathrm{N}(5,1^2)$.

(b) Find the range of values of k such that at least 90% of customers experience a wait time longer than k minutes. [2]

A customer bought burgers from the shop on three independent occasions, with wait times denoted by W_1, W_2 and W_3 respectively. Let $\bar{W}=\frac{W_1+W_2+W_3}{3}$.

(c) Find the values of $\mathrm{Var}(\bar{W}-W_1)$ and $\mathrm{Var}(\bar{W})+\mathrm{Var}(W_1)$ and hence, determine whether $\mathrm{Var}(\bar{W}-W_1)=\mathrm{Var}(\bar{W})+\mathrm{Var}(W_1)$. [2]

(d) Find the probability that the mean wait time is within one minute of the wait time on the first occasion. [3]

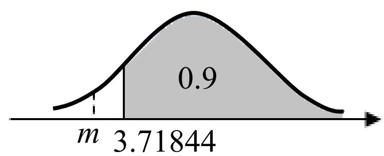
On the 4th occasion, the customer went to the restroom immediately after the order was placed. The time spent in the restroom, in T minutes, is modelled by $T \sim N(7,1.5^2)$.

Assume that the time taken to walk to the restroom is negligible and W and T follow independent normal distributions.

(e) The customer is in a rush and will not wait for more than 3 minutes after leaving the restroom. Given that the burger is not ready for collection after the customer leaves the restroom, find the probability that the customer will leave the shop without collecting the burger. [4]

To reduce the wait time for customers, tshe shop later installs self-order kiosks. The wait time from when a customer places an order at the kiosk until the food is collected is modelled as being reduced by 20% compared to that at the counter.

(f) 8 customers who ordered at the counter and 8 customers who ordered at the kiosk are randomly selected and their wait times are recorded. The wait time of each customer is independent of the others. Find the probability that exactly 2 of these 16 customers have a wait time of at least 5 minutes. [4]

Qn	Suggested Solutions
(a)	According to the model, $\mathrm{P}(W<0)=0.0912$, which means 9.12 % of the orders will have a negative wait time, which is not realistic.
(b)	$\mathrm{P}(W>k)\geq 0.9$ $0<k\leq 3.71$ OR $\mathrm{P}\left(Z>\dfrac{k-5}{1}\right)\geq 0.9$  $0<\dfrac{k-5}{1}\leq -1.28155$ $0<k\leq 3.71$
(c)	$\mathrm{Var}\left(\overline{W}-W_1\right)=\mathrm{Var}\left(\dfrac{W_1+W_2+W_3}{3}-W_1\right)$ $=\mathrm{Var}\left(\dfrac{-2W_1+W_2+W_3}{3}\right)$ $=\dfrac{1}{9}(4+1+1)$ $=\dfrac{2}{3}$ $\mathrm{Var}\left(\overline{W}\right)+\mathrm{Var}\left(W_1\right)=\dfrac{1}{3}+1$ $=\dfrac{4}{3}$ Hence $\mathrm{Var}\left(\overline{W}-W_1\right)\neq\mathrm{Var}\left(\overline{W}\right)+\mathrm{Var}\left(W_1\right)$

(d)	$\dfrac{-2W_1+W_2+W_3}{3} \sim \mathrm{N}\left(0,\dfrac{2}{3}\right)$ Required probability $=\mathrm{P}\left(\left\|\bar{W}-W_1\right\|\le 1\right)$ $=\mathrm{P}\left(-1\le \bar{W}-W_1\le 1\right)$ $=0.779\ (3\text{ s.f.})$
(e)	$W-T\sim \mathrm{N}\left(5-7,1^2+1.5^2\right)=\mathrm{N}\left(-2,3.25\right)$ $\mathrm{P}\left(W-T>3\mid W>T\right)$ $=\dfrac{\mathrm{P}\left(W-T>3 \text{ and } W>T\right)}{\mathrm{P}\left(W>T\right)}$ $=\dfrac{\mathrm{P}\left(W-T>3\right)}{\mathrm{P}\left(W-T>0\right)}$ $=0.0208\ (3\text{ s.f.})$
(f)	Let K be the wait time of a customer who placed order at the kiosk $K=0.8W\sim \mathrm{N}\left(0.8\times 5,0.8^2\times 1^2\right)$ $\mathrm{P}\left(K\ge 5\right)=0.10565$ Let X and Y be the number of customers who placed order at the counter and kiosk respectively, out of 8, with at least 5 minutes waiting time. $X\sim \mathrm{B}(8,0.5)$ $\qquad$ $Y\sim \mathrm{B}(8,0.10565)$ Required probability $=\mathrm{P}\left(X=2\cap Y=0\right)+\mathrm{P}\left(X=1\cap Y=1\right)+\mathrm{P}\left(X=0\cap Y=2\right)$ $=\mathrm{P}\left(X=2\right)\times \mathrm{P}(Y=0)+\mathrm{P}\left(X=1\right)\times \mathrm{P}(Y=1)+\mathrm{P}\left(X=0\right)\times \mathrm{P}(Y=2)$ $=0.0575$